Adaptive methods for periodic initial value problems of second order differential equations

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ABSTRACT

In this paper numerical methods involving higher order derivatives for the solution of periodic initial value problems of second order differential equations are derived. The methods depend upon a parameter $p > 0$ and reduce to their classical counter parts as $p \to 0$. The methods are periodically stable when the parameter p is chosen as the square of the frequency of the linear homogeneous equation. The numerical methods involving derivatives of order up to 2q are of polynomial order 2q and trigonometric order one. Numerical results are presented for both the linear and nonlinear problems. The applicability of implicit adaptive methods to linear systems is illustrated.

1. INTRODUCTION

The development of numerical integration formulae for the direct integration of the periodic initial value problem

$$y'' = f(t,y),\ y(t_0) = y_0,\ y'(t_0) = y_0' \qquad (1.1)$$

has created considerable interest in the recent years. Lambert and Watson [10] have originated the concept of the interval of periodicity of a numerical method and connected it with the symmetry property of linear multistep methods. They have presented a symmetric linear multistep method of order two with periodicity interval $(0, \infty)$ and defined it to be a P-stable method. Dahlquist [3] has shown that the order of an unconditionally stable linear multistep method cannot exceed two and the method must be implicit. However, higher order P-stable methods have been discussed by Hairer [5] and Jain et al [8]. The numerical methods adaptive to oscillating phenomena have been derived and applied to several important physical problems of the form (1.1) by Stiefel and Bettis [12], Bettis [2], Jain et al [6,7,8] and Fatunla [4]. In this paper a class of higher order explicit methods depending upon a parameter $p > 0$ are derived. The numerical method involving 2q th derivative of the function f(t,y) of (1.1) is shown to be of polynomial order 2q and of trigonometric order one for arbitrary choice of p. As $p \to 0$, it reduces to the classical method of order 2q+2. Also an implicit method of polynomial order four and trigonometric order one is presented. Both the explicit and the implicit methods are periodically stable when p is chosen as the square of the frequency of the periodic solution of the homogeneous problem. The applicability of the implicit adaptive methods to systems of linear equations is illustrated with reference to a numerical example. The numerical results presented are comparable in accuracy to the existing periodically stable higher order methods of Hairer [5] and Fatunla [4].

2.1. Derivation of the explicit methods

We write (1.1) in the form

$$y'' + py = \phi(t,y) \qquad (2.1)$$

where

$$\phi(t,y) = f(t,y) + py \qquad (2.2)$$

and $p > 0$, is an arbitrary parameter to be determined. The equation (2.1) can be written as (see [1])

$$y'' + py = g(t) \qquad (2.3)$$

where g(t) is an approximation to $\phi(t,y)$. The general solution of (2.3) is

$$y(t) = A\cos\sqrt{p}t + B\sin\sqrt{p}t + \frac{1}{\sqrt{p}}\int_{t_n}^{t} g(\tau)\sin\sqrt{p}(t-\tau)\,d\tau \qquad (2.4)$$

where A and B are arbitrary constants.
Substituting $t = t_{n+1}, t_n$ and t_{n-1} in (2.4) and eliminating A and B from the resulting equations we obtain

$$y(t_{n+1}) - 2\cos\sqrt{p}h.y(t_n) + y(t_{n-1}) = \frac{1}{\sqrt{p}}\int_{t_n}^{t_{n+1}} [g(\tau) + g(2t_n-\tau)]\sin\sqrt{p}(t_{n+1}-\tau)\,d\tau \qquad (2.5)$$

We use (2.5), to construct higher order explicit methods involving higher order derivatives of the solution y(t) of (1.1).
Approximating $g(\tau)$ and $g(2t_n-\tau)$ by Taylor interpolating polynomials of order 2q and omitting truncation error terms we have

$$g(\tau) = \sum_{m=0}^{2q} \frac{(\tau-t_n)^m}{(m)!} g^{(m)}(t_n) \qquad (2.6a)$$

$$g(2t_n-\tau) = \sum_{m=0}^{2q} (-1)^m . \frac{(\tau-t_n)^m}{(m)!} g^{(m)}(t_n) \qquad (2.6b)$$

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Substituting (2.6) in (2.5) we obtain

$$y_{n+1}-2[\cos\omega]y_n+y_{n-1} = 2\sum_{m=0}^{q} h^{2m+2}F_{2m+2}g_n^{(2m)}$$

where $\omega = \sqrt{p}\,h$ and (2.7)

$$F_{2m+2} = \frac{1}{\sqrt{p}}\frac{1}{(2m)!}\frac{1}{h^{2m+2}}\int_{t_n}^{t_{n+1}}(\tau-t_n)^{2m}$$

$$\sin[\sqrt{p}\,(t_{n+1}-\tau)]\,d\tau \tag{2.8}$$

Integrating (2.8) by parts we obtain the recurrence relation

$$F_{2m+2} = \frac{1}{\omega^2}\left(\frac{1}{(2m)!}-F_{2m}\right),\ m=0,1,2,\ldots \tag{2.9}$$

with $F_0 = \cos\omega$

It can be easily verified from (2.8) that

$$F_{2m+2}\to\frac{1}{(2m+2)!}\quad \text{as } \omega\to 0 \tag{2.10}$$

$$F_{2m+2}\to 0\quad \text{as } \omega\to\infty \tag{2.11}$$

and from (2.9)

$$\lim_{\omega\to\infty}\ \omega^2F_{2m+2} = \frac{1}{(2m)!},\ \text{for } m = 1,2,\ldots \tag{2.12}$$

We note that the method (2.7) can be rearranged and written as

$$\delta^2y_n = 2\sum_{m=0}^{q-1}\frac{h^{2m+2}}{(2m+2)!}f_n^{(2m)}+2h^{2q+2}F_{2q+2}f_n^{(2q)} \tag{2.13}$$

Taking $q = 1$, we obtain

$$\delta^2y_n = h^2f_n+2h^4F_4f_n'' \tag{2.14}$$

where

$$F_4 = \frac{1}{\omega^2}\left(\frac{1}{2}-\frac{1-\cos\omega}{\omega^2}\right)$$

2.2. An implicit adaptive method

The implicit two step method

$$\delta^2y_n = h^2\{\lambda(\sigma)f_{n+1}+[1-2\lambda(\sigma)]f_n+\lambda(\sigma)f_{n-1}\} \tag{2.15}$$

which is presented in [7] is of polynomial order two and trigonometric order one. By using the method of undetermined coefficients, we can easily obtain the implicit method

$$\delta^2y_n = h^2\{\lambda(\sigma)f_{n+1}+[1-2\lambda(\sigma)]f_n+\lambda(\sigma)f_{n-1}\}$$

$$+h^4\eta(\sigma)\,[f_{n+1}''-2(\cos 2\sigma)f_n''+f_{n-1}''] \tag{2.16}$$

where $\sigma = \frac{\sqrt{p}h}{2}$ and

$$\lambda(\sigma) = \frac{1}{4}\left(\frac{1}{\sin^2\sigma}-\frac{1}{\sigma^2}\right)$$

$$\eta(\sigma) = \left[\frac{1}{12}-\lambda(\sigma)\right]/4\sin^2\sigma.$$

The method (2.16) is of polynomial order four and trigonometric order one.

3. ORDER OF THE METHODS

Substituting $y(t) = e^{i\sqrt{p}t}$ in (2.7), (2.15) and (2.16) we find that the methods are of trigonometric order one (see [1]). Also substituting $y(t) = t^m$, $m = 0,1,2,\ldots$ we note that the method (2.7) is of polynomial order 2q, the method (2.15) is of polynomial order 2 and the method (2.16) is of polynomial order 4 for arbitrary choice of p. The methods (2.7) solve (2.3), where g(t) is a polynomial of degree atmost 2q exactly independent of the chosen step size h. As $p\to 0$, the methods (2.7), (2.15) and (2.16) become

$$\delta^2y_n = 2\sum_{m=0}^{q}\frac{h^{2m+2}}{(2m+2)!}f_n^{(2m)} \tag{3.1}$$

$$\delta^2y_n = \frac{h^2}{12}(f_{n+1}+10f_n+f_{n-1}) \tag{3.2}$$

and

$$\delta^2y_n = \frac{h^2}{12}(f_{n+1}+10f_n+f_{n-1})-\frac{h^4}{240}(f_{n+1}''-2f_n''+f_{n-1}'') \tag{3.3}$$

which are of polynomial orders 2q+2, 4 and 6 respectively.

4. STABILITY OF HIGHER ORDER METHODS

Definition 4.1

An adaptive method is said to be periodically stable if, when the method is applied to the test equation

$$y'' = -\lambda^2y \tag{4.1}$$

with the exact initial conditions, it gives rise to the solution which is identical to that of the differential equation for an arbitrary h and the free parameter is chosen as the square of the frequency (see [8]).

Applying the methods (2.7), (2.15) and (2.16) to the test equation (4.1), we obtain

$$y_{n+1}-2\cos\omega\,y_n+y_{n-1} = 0 \tag{4.2}$$

whose characteristic equation

$$\xi^2-2\cos\omega\xi+1 = 0 \tag{4.3}$$

has complex roots of modulus unity. Hence the methods (2.7), (2.15) and (2.16) produce periodically stable results for any arbitrary h.

5. NUMERICAL RESULTS

We adopt the methods (2.14), (2.15) and (2.16) to solve the following linear and nonlinear problems. In implementing the methods, y and y′ are computed using the exact solution of the corresponding problem.

Problem 1

We consider the nearly periodic initial value problem (see [4])

$$y''+y = 0.001e^{it}, y(0) = 1, y'(0) = 0.9995\,i \qquad (5.1)$$

whose theoretical solution is

$$y(t) = u(t) + i\,v(t)$$

$$u(t) = \cos t + 0.0005\, t \sin t$$

$$v(t) = \sin t - 0.0005\, t \cos t \qquad (5.2)$$

Equation (5.1) represents the motion on perturbation of a circular orbit in the complex plane in which the point $y(t)$ spirals outwards such that its distance from the origin at any time t is given by $\tau(t) = \sqrt{u^2(t) + v^2(t)}$. Using the method (2.14), the absolute errors $|\tau(t_n) - \tau_n|$ and $\sqrt{[u(t_n)-u_n]^2 + [\bar{v}(t_n) - v_n]^2}$ are presented in tables 1 and 2 respectively for different step sizes h and at $t = 40\pi$. The results presented in Fatunla [4] are also tabulated for the comparison purposes.

Problem 2

We consider the stiff highly oscillatory problem (see [11])

$$x''+\lambda^2 x = \lambda^2 \sin t \qquad (5.3)$$

with $x(0) = 0,\ x'(0) = \frac{\lambda}{2} + \frac{1}{1-\frac{1}{\lambda^2}}$

The problem is solved using the methods (2.14), (2.15) and (2.16) for $\lambda = 10$ and $h = 0.25, 0.5$. In table 3, the absolute errors $|x(t_n) - x_n|$ obtained at t=100 are tabulated. The results obtained by using Hairer's P-stable method of polynomial order four given by

$$\delta^2 y_n = \frac{h^2}{12}(f_{n+1}+10f_n+f_{n-1}) - \frac{h^4}{144}(f''_{n+1}-2f''_n+f''_{n-1}) \qquad (5.4)$$

are also presented for comparison purposes.

Problem 3

We consider the undamped Duffing equation

$$y''+y+y^3 = B\cos\Omega t \qquad (5.5)$$

forced by a harmonic function $B = 0.002$ and $\Omega = 1.01$. The exact solution computed by Galerkin's approximation method with a precision of 10^{-12} of the coefficients is given by

$$y(t) = A_1 \cos\Omega t + A_3 \cos 3\Omega t + A_5 \cos 5\Omega t + A_7 \cos 7\Omega t + A_9 \cos 9\Omega t \qquad (5.6)$$

where

$A_1 = 0.200179477536$

$A_3 = 0.000246946143$

$A_5 = 0.000000304014$

$A_7 = 0.000000000374$

$A_9 = 0.000000000000.$

The problem is solved using the methods (2.14), (2.15), (2.16) and (5.4) for the step lengths $h = \frac{\pi}{18}, \frac{\pi}{15}$ and $\frac{\pi}{10}$. We assume the initial conditions y_0, y_1 and y'_0, y'_1, from the exact solution (5.6) while implementing the methods (2.14), (2.15), (2.16) and (5.4). Also an initial approximation to y_2 is assumed from the exact solution (5.6) while implementing the implicit methods (2.15), (2.16) and (5.4). To find $y'(t)$ which occurs in implementing the above methods we use the differentiation formula of $0(h^2)$:

$$y'_{n+1} = \frac{1}{2h}(3y_{n+1} - 4y_n + y_{n-1}) + 0(h^2) \qquad (5.7)$$

An extensive study for the determination of the parameter p for the above problem is made by the author in his thesis [1] in implementing the singlestep, the multistep and hybrid adaptive methods. However, the methods (2.14), (2.15) and (2.16) are used with $p = 1$ and the absolute errors $|y(t_n) - y_n|$ at $t=40\pi$ are tabulated in table 4.

Problem 4

We consider the linear system (see [9]).

$y'' = 2498y + 4998z$

$z'' = -2499y - 4999z$

with the exact solution $y=2\cos x$, $z=-\cos x$. The problem is solved using the implicit methods (2.15), (2.16) and (5.4) with $p=1$ and $h= 0.5$. The absolute errors in $y(t)$ and $z(t)$ at $t=5$ are tabulated in table 5. We noted that the explicit adaptive method (2.14) becomes unstable for $h= 0.5$ and $p=1$.

6. CONCLUSIONS

The numerical results presented in tables 1 and 2 show that the method (2.14) gives more accurate results when compared to Fatunla's adaptive method. Tables 3, 4 and 5 show the superiority of the method (2.16) over (1) the explicit method (2.14),
(2) the implicit linear multistep method (2.15)
and (3) the P-stable Hairer's method (5.4) of polynomial order four.

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TABLE 1

Absolute errors in $|\tau(t_n)-\tau_n|$ at $t = 40\pi$

h	Method (2.14) One eval. of p	Fatunla's method	
		One evaluation of p	Repeated eval. of p
$\frac{\pi}{4}$	4.52 (-06)	3.39 (-04)	2.04 (-04)
$\frac{\pi}{5}$	1.80 (-06)	2.33 (-04)	6.60 (-05)
$\frac{\pi}{6}$	8.51 (-07)	1.67 (-04)	2.60 (-05)
$\frac{\pi}{9}$	1.64 (-07)	7.80 (-05)	3.00 (-06)
$\frac{\pi}{12}$	5.04 (-08)	4.40 (-05)	0.00 (-06)

TABLE 2

Absolute errors in $\sqrt{[u(t_n)-u_n]^2+[v(t_n)-v_n]^2}$ at $t=40\pi$

h	Method (2.14) One eval. of p	Fatunla's method	
		One evaluation of p.	Repeated eval. of p
$\frac{\pi}{4}$	7.22 (-05)	3.89 (-04)	3.84 (-04)
$\frac{\pi}{5}$	2.87 (-05)	2.52 (-04)	1.59 (-04)
$\frac{\pi}{6}$	1.36 (-05)	1.76 (-04)	7.70 (-05)
$\frac{\pi}{9}$	2.63 (-06)	7.90 (-05)	1.50 (-05)
$\frac{\pi}{12}$	8.27 (-07)	4.50 (-05)	5.00 (-06)

TABLE 3

Absolute errors at $t = 100$

h	Method (2.14) $p = 100$	Method (2.15) $p = 100$	Method (2.16) $p = 100$	Hairer's method
0.25	1.467 (-05)	1.858 (-05)	1.516 (-06)	1.732 (-01)
0.5	2.211 (-04)	1.595 (-04)	1.888 (-06)	2.827 (-01)

TABLE 4

Absolute errors in $y(t)$ at $t = 40\pi$

h	Method (2.14) $p=1$	Method (2.15) $p=1$	Method (2.16) $p=1$	Hairer's method
$\frac{\pi}{18}$	2.514 (-05)	6.116 (-07)	7.669 (-08)	6.417 (-05)
$\frac{\pi}{15}$	4.087 (-05)	1.268 (-06)	1.069 (-07)	7.926 (-05)
$\frac{\pi}{10}$	1.568 (-04)	6.418 (-06)	2.488 (-08)	8.261 (-05)

Table 5

Absolute errors in $y(t)$ and $z(t)$ at $t=5$ for $h=0.5$

Methods	$\lvert y(t_n)-y_n\rvert$	$\lvert z(t_n)-z(t)\rvert$
Method (2.15)	4.400 (-04)	2.200 (-04)
Method (2.16)	1.441 (-05)	7.179 (-06)
Method (5.4)	7.002 (-04)	3.501 (-04)