PanAmerican Mathematical Journal
Volume 29 (2019), Number 2, 93 - 103

A Broyden-type Banach to Hilbert Space Scheme for Solving Equations

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Communicated by Ram U. Verma

(Received February 22, 2019; Revision Accepted March 10, 2019)

www.internationalpubls.com

Abstract

We present a new semi-local convergence analysis for an inverse free Broyden - type Banach to Hilbert space scheme (BTS) in order to approximate a locally unique solution of an equation. The analysis is based on a center-Lipschitz-type condition and our idea of the restricted convergence region. The operators involved have regularly continuous divided differences. This way we provide, weaker sufficient semi-local convergence conditions, tighter error bounds, and a more precise information on the location of the solution. Hence, our approach extends the applicability of BTS under the same hypotheses as before.

AMS Subject Classification: 65J15, 47H17, 47J05, 49M15

Key words: Broyden's method, Banach space, semi-local convergence, regularly continuous divided differences.

1 Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1}$$

where F is a continuous operator defined on a open convex subset Ω of a Banach space $\mathcal{B}$ with values in a Hilbert space H.

Broyden's method BM

$$x_+ = x - A\,F(x), \quad y = F(x_+) - F(x), \quad A_+ = A - \frac{A\,F(x_+)\,\langle A^* AF(x), .\rangle}{\langle A^* AF(x), F(x_+) - F(x)\rangle}, \tag{2}$$

where $\mathcal{L}(H,\mathcal{B}) := \{A : H \longrightarrow \mathcal{B},\ \text{bounded and linear}\}$, A^* is the adjoint of A, and $\langle\cdot,\cdot\rangle$ stands for the inner product in H.

A plethora of convergence results for this type of schemes have appeared in the literature [1, 3, 5, 8, 9, 10, 11] (see also, e.g. [4], and the references there in). BTS requires no inverse, so no linear subproblem needs to be solved at each iteration.

The convergence region for such methods is small in general [12, 13, 14, 15]. In the present study, we extend the convergence region for BTS. To achieve this goal, we first introduce the center-Lipschitz condition which determines a subset of the original region for the operator containing the iterates. The scalar functions are then related to the subset instead of the original region. This way, the scalar functions are more precise than if they were depending on the original region. The new technique leads to : weaker sufficient convergence conditions, tighter error bounds on the distances involved, and an at least as precise information on the location of the solution. These advantages are obtained under the same computational cost as in earlier studies [8, 9, 10, 11], since in practice the new functions are special cases of the old functions. This idea can be used to study other iterative methods [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

The rest of the study is structured as follows. Section 2 contains some preliminary results for regularly continuous dd. In Section 3, we provide the semi-local convergence analysis of BTS.

2 Preliminaries: regularly continuous dd

In order to make the paper as self–contained as possible, we reintroduce some definitions and some results on regularly continuous dd. The proofs are omitted, and can be found in [4, 11]. In this section, $\mathcal{B}$ and H are Banach spaces, equipped with the norm $\|\ .\ \|$. We denote by $U(z,R) = \{x \in \mathcal{B} : \| x - z \| < R, \}$ the open ball centered at z and of radius $R > 0$, whereas $\overline{U}(z,R)$ denotes its closure. For $x \in \mathcal{B}$, denote by $\mathcal{K}_x$ the subspace of operators vanishing at x $\mathcal{K}_x = \{A \in \mathcal{L}(\mathcal{B},H) : A\,x = 0\}$. Let $\mathcal{N}$ be the class of increasing concave functions $v : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, with $v(0) = 0$. Note that $\mathcal{N}$ contains the functions in the form $\varphi(t) = c\,t^p$, ($c \geq 0$, and $p \in (0,1]$).

Definition 2.1. *[11] An operator $[.,.;F]$ belonging in $\mathcal{L}(\mathcal{B},H)$ is called the first order divided difference (briefly dd) of F at the points x and y in $\mathcal{B}$ ($x \neq y$), if the following secant equation holds $[x,y;F]\,(y-x) = F(y) - F(x)$.*

If F is Fréchet differentiable at x, then $[x,x;F] = F'(x)$. Otherwise, the following limit (if it exists) $\lim_{t\searrow 0}[x, x + t\,h; F]\,h = \lim_{t\searrow 0} \dfrac{F(x + t\,h) - F(x)}{t}$ vary according to h, with $\| h \| = 1$, and this limit is the Fréchet derivative (or the directional derivative) $F'(x)\,h$ of F in the direction h (i.e., if we suppose that F is Fréchet differentiable at x, then the Fréchet derivative is characterized as a limit of dd in the uniform topology of the space of continuous linear mappings of $\mathcal{B}$ into H).

Remark 2.2. *(a) Let $(x, y) \in \mathcal{B} \times H$, the set $\{A \in \mathcal{L}(\mathcal{B}, H) : A\,x = y\}$ constitute an affine manifold in $\mathcal{L}(\mathcal{B}, H)$.*

(b) Let A and A_0 in $\mathcal{L}(\mathcal{B}, H)$, and $(x, y) \in \mathcal{B} \times H$, such that $A_0\,x = A\,x = y$. Then $(A - A_0)\,x = 0$, and $A \in A_0 + \mathcal{K}_x$.

The following result gives some properties of set–valued mapping $\Upsilon_{x,y} : \mathcal{C}(\mathcal{B}, H) \rightrightarrows \mathcal{L}(\mathcal{B}, H)$ given by $\Upsilon_{x,y}(F) = [x, y; F]$ for the pair $(x, y) \in \mathcal{B}^2$.

Proposition 2.3. *(a) $\Upsilon_{x,y}(F) = F$ if and only if F is linear.*

(b) $\Upsilon_{x,y}$ is linear, i.e., for F_1, F_2 in $\mathcal{C}(\mathcal{B}, H)$, and $(\alpha, \beta) \in \mathbb{K}^2$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), we have

$$\Upsilon_{x,y}(\alpha\,F_1 + \beta\,F_2) = \alpha\,\Upsilon_{x,y}(F_1) + \beta\,\Upsilon_{x,y}(F_2).$$

(c) If F is a composition of operators F_1 and F_2 (i.e., $F = F_1 \circ F_2$), then

$$\Upsilon_{x,y}(F) = \Upsilon_{F_2(x),F_2(y)}(F_1)\,\Upsilon_{x,y}(F_2).$$

Definition 2.4. *[11] The dd $[x, y; F]$ is said to be w_1–regularly continuous on $\Omega \subseteq \mathcal{B}$ for $w_1 \in \mathcal{N}$ (call it regularity modulus), if the following inequality holds for each $x, y, u, v \in \Omega$*

$$\begin{aligned} & w_1^{-1}\bigg(\min\{\| [x,y;F] \|, \| [u,v;F] \|\} + \| [x,y;F] - [u,v;F] \| \bigg) \\ & -w_1^{-1}\bigg(\min\{\| [x,y;F] \|, \| [u,v;F] \|\} \bigg) \le \| x - u \| + \| y - v \| . \end{aligned} \tag{3}$$

The dd $[x, y; F]$ is said to be regularly continuous on Ω, if it has there a regularity modulus.

We introduce a special notion (see also [5, 6, 7]).

Definition 2.5. *The dd $[x, y; F]$ is said to be w_0- center regularly continuous on $\Omega \subset X$ for $w_0 \in \mathcal{N}$ (call it center regularity modulus), if for fixed $x_{-1}, x_0 \in \Omega$ the following inequality holds for each x, y in Ω*

$$\begin{aligned} & w_0^{-1}\bigg(\min\{\| [x,y;F] \|, \| [x_0,x_{-1};F] \|\} + \| [x,y;F] - [x_0,x_{-1};F] \| \bigg) \\ & -w_0^{-1}\bigg(\min\{\| [x,y;F] \|, \| [x_0,x_{-1};F] \|\} \bigg) \le \| x - x_0 \| + \| y - x_{-1} \| . \end{aligned} \tag{4}$$

Clearly, we have that Definition 2.5 is a special case of Definition 2.4,

$$w_0(t) \leq w_1(t) \quad \text{for each} \quad t \in [0, \infty), \tag{5}$$

holds in general, and $\frac{w_1}{w_0}$ can be arbitrarily large [2, 4]. If w_0, w_1 are linear functions ($w_1(t) = c_1 t$ and $w_0(t) = c_0 t$), then (4), and (5) become Lipschitz, and center–Lipschitz continuous conditions, respectively, i.e., the following hold respectively for each $(x, y, u, v) \in \Omega^4$:

$$\| [x, y; F] - [u, v; F] \| \leq c_1 (\| x - u \| + \| y - v \|) \tag{6}$$

and

$$\| [x, y; F] - [x_0, x_{-1}; F] \| \leq c_0 (\| x - x_0 \| + \| y - x_{-1} \|). \tag{7}$$

Then, estimate (5) gives

$$c_0 \leq c_1. \tag{8}$$

We need the following auxiliary result.

Lemma 2.6. *[9] If dd $[x, y; F]$ is w–regularly continuous on Ω, then we have*

$$|w_1^{-1}(\| [x, y; F] \|) - w_1^{-1}(\| [u, v; F] \|)| \leq \| x - u \| + \| y - v \|, \quad \text{for each } (x, y, u, v) \in \Omega^4.$$

Then, the following holds for all $(x, y, u, v) \in \Omega^4$:

$$w_1^{-1}(\| [x, y; F] \|) \geq (w_1^{-1}(\| [u, v; F] \|) - \| x - u \| - \| y - v \|)^+, \tag{9}$$

where ρ^+ ($\rho \in \mathbb{R}$) denotes the nonnegative part of ρ: $\rho^+ = \max\{\rho, 0\}$.

In particular, if dd $[x, y; F]$ is w_0–regularly continuous on Ω (i.e., condition (4) holds), then, (9) holds, with w_0, x_0, and x_{-1} replacing w, u, and v, respectively.

Suppose that equation

$$w_0(t) = 1 \tag{10}$$

has at least one positive solution. Denote by r_0 the smallest such solution. Moreover, define

$$\Omega_0 = \Omega \cap U(x_0, r_0). \tag{11}$$

Notice also that we have a similar estimate for function w on Ω^4.

Definition 2.7. *The dd $[x, y; F]$ is said to be restricted $w-$regularly continuous on $\Omega_0 \subset \Omega$ for $w \in \mathcal{N}$, if the following inequality holds for each x, $y, u, v \in \Omega_0$*

$$\begin{aligned} & w^{-1}\Big(\min\{\| [x, y; F] \|, \| [u, v; F] \|\} + \| [x, y; F] - [u, v; F] \| \Big) \\ & - w^{-1}\Big(\min\{\| [x, y; F] \|, \| [u, v; F] \|\} \Big) \leq \| x - u \| + \| y - v \| . \end{aligned} \tag{12}$$

Notice that

$$w(t) \leq w_1(t) \text{ for each } \; t \in [0, r_0) \tag{13}$$

holds, since $\Omega_0 \subseteq \Omega$. Function w depends on function w_0. The construction of function w was not possible before in the earlier studies using only function w_1[11]. Clearly, in those studies w can simply replace w_1, since the iterates lie in Ω_0 related to w, which is a more precise location than Ω used in [11] related to w_1. This modification leads to the already stated advantages, if strict inequality holds in (5) or (13).

We suppose from now on that

$$w_0(t) \leq w(t) \text{ for each } \; t \in [0, r_0). \tag{14}$$

3 Semi-local convergence analysis of BTS

We present a semi-local convergence result for BTS. The proofs are the proper modifications of the ones in [11], where, we use the more precise (4), (12) instead of (3) and w instead of w_1.. First, we denote

$$A_0 = [x_0, x_{-1}; F]^{-1}. \tag{15}$$

As in [11], for the selected dd $[x, y; F]$, such that (3) holds with w modulus, we associate the current iteration (x, A), and we consider $q = (\overline{t}, \overline{\gamma}, \overline{\delta})$, where

$$\overline{t} = \| x - x_0 \|, \quad \overline{\gamma} = \| x - x_- \|, \quad \overline{\delta} = \| x_+ - x \| .$$

The next, auxiliary result relates $\bar{\delta}_+ := \|x_{++} - x_+\| = \|A_+ F(x_+)\|$ with the triple $(\bar{t}, \bar{\gamma}, \bar{\delta})$. For briefing we denote

$$\alpha_0 := w_0^{-1}(1 - \underline{h}), \alpha = w^{-1}(1 - \underline{h}), \bar{\gamma}_0 := \|x_0 - x_{-1}\|, a := \alpha - \bar{\gamma}_0.$$

Lemma 3.1. *Let the selected* $dd[x_1, x_2; F]$ *of* F *be restricted* $w-$*regularity continuous on* Ω_0. *If* $\bar{t}_+ + \bar{t} < a$, *then*

$$\bar{\delta}_+ \leq \bar{\delta} \left(\frac{w(a - \bar{t}_+ - \bar{t} + \bar{\delta} + \bar{\gamma}) - w(a - \bar{t}_+ - \bar{t})}{w_0(a - \bar{t}_+ - \bar{t})} \right) \leq \bar{\delta} \left(\frac{w(a - \bar{t}_+ - \bar{t} + \bar{\delta} + \bar{\gamma})}{w_0(a - \bar{t}_+ - \bar{t})} - 1 \right). \tag{16}$$

In view of the lemma we have the following majorant generator $g(t, \gamma, \delta) = (t_+, \gamma_+, \delta_+)$:

$$t_+ := t + \delta, \; \gamma_+ := \delta,$$

$$\bar{\delta}_+ = \bar{\delta} \left(\frac{w(a - t_+ - t + \delta + \gamma)}{w_0(a - t_+ - t)} - 1 \right) = \bar{\delta} \left(\frac{w(a - 2t + \gamma)}{w_0(a - 2t - \delta)} - 1 \right). \tag{17}$$

We say that the triple $q' = (t', \gamma', \delta')$ majorizes $q = (t, \gamma, \delta)$(i.e.$q \prec q'$), if

$$t \leq t' \text{ and } \gamma \leq \gamma' \text{ and } \delta \leq \delta'.$$

So by the above lemmas $\bar{q}_+ \prec f(\bar{q})$. Starting with the initial triple q_0, the generator iterates producing a majorant sequence as long as (17) remains defined and

$$2t_n + \delta_n < a.$$

Under the above condition, the sequence (x_n, A_n) generated by the method (2) starting with (x_0, A_0) converge to a solution of the system

$$F(x) = 0 \text{ and } A[x, x; F] = I. \tag{18}$$

Lemma 3.2. *If q_0 is such that $\bar{q}_0 \prec q_0$ and $2t_n + \delta_n < a$, then*

(i) $\bar{q}_n \prec q_n$;

(ii) $\gamma_\infty = \delta_\infty = 0$ *and* $t_n \leq \frac{1}{2}(a - \delta_n)$;

(iii) The sequence (x_n, A_n) remains in the ball $B((x_0, A_0), (t_\infty, r_A))$. where

$$r_A := \frac{w(a-\delta_0) - w(a-2t_\infty)}{w(a-\delta_0)w(a-2t_\infty)} + \frac{w(a+\gamma_0) - w(a-\delta_0)}{1 - w(a+\gamma_0) + w(a-\delta_0)},$$

and converges to a solution (x_∞, A_∞) of the system (18).

(iv) x_∞ is the only solution of the equation $F(x) = 0$ in the ball $B(x_0, a - t_\infty)$;

(v) For all $n = 0, 1, 2, \ldots$,

$$\|F(x_{n+1})\| \leq \delta_n(w(a - 2t_n + \gamma_n) - w(a - 2t_n - \delta_n)),$$

$$\Delta_n := \|x_\infty - x_n\| \leq t_\infty - t_n, \|$$

$$\frac{\Delta_{n+1}}{\Delta_n} \leq \frac{w(\Delta_{n-1}}{w(a - 2t_n + \gamma_n)};$$

(vi) All these inequalities are exact in the sense that they hold as equalities for a scalar quadratic polynomial.

We have seen in Lemma 3.2, that the convergence of x_n is guaranteed, if one chooses x_{-1}, x_0, A_0, q_0 such that $\bar{q}_0 \prec q_0$ and $2t_n + \delta_n < a$. So, our next task is to find the set of all q_0 which satisfies $2t_n + \delta_n < a$.

For linear $w(w(t) = ct))$, (17) takes the form

$$t_+ := t + \delta,\ \gamma_+ := \delta,\ \delta_+ := \delta\frac{\gamma + \delta}{a - 2t - \delta}, \tag{19}$$

where $a = c^{-1} - \|x_0 - x_{-1}\|$. This way, we have the following Proposition.

Proposition 3.3. *(i) The function $I(t, \gamma, \delta) := (a - 2t)^2 - 4\delta(a - 2t + \gamma)$ is an invariant of the generator (19).*

(ii) The sequence $(t_n, \gamma_n, \delta_n)$*, generated by (19) starting from*$(0, \gamma_0, \delta_0)$*, converges if and only if*

$$I_0 := I(0, \gamma_0, \delta_0) = (c^{-1} - \gamma_0)^2 - 4c^{-1}\delta_0) \geq 0,$$

and

$$2t_n + \delta_n < a \Leftrightarrow t_n = \frac{1}{2}(c^{-1} - \gamma_0) - \delta_n - \sqrt{\delta_n(\gamma_n + \delta_n) + 0.25 I_0} =: F(\gamma_n, \delta_n).$$

(iii) The function F *is a solution of the system*

$$x\left(\delta, \delta\frac{\gamma + \delta}{a - 2x(\gamma, \delta) - \delta}\right) = x(\gamma, \delta) + \delta \text{ and } x(0,0) = t_\infty. \qquad (20)$$

In view of the above Proposition, we have the following Theorem.

Theorem 3.4. *Let the selected* $dd[x_1, x_2; F]$ *of* F *be restricted* $w-$*regularity continuous on* Ω_0*. If the initial data* $x_0, A_0, \gamma_0, \delta_0$ *are such that*

$$\|x_0 - x_{-1}\| \leq \gamma_0 \text{ and } \|A_0 F(x_0)\| \leq \delta_0 \leq F_\infty(0, \gamma_0),$$

then

(i) $\gamma_\infty = \delta_\infty = 0$ *and*

$$(\|x_n - x_0\| \leq t_n \leq \frac{1}{2}(a - \delta_n) \text{ and } \|x_n - x_{n-1}\| \leq \gamma_n \text{ and } \|x_{n+1} - x_n\| \leq \delta_n);$$

(ii) The sequence (x_n, A_n) *from the initial data* (x_0, A_0) *converges to a solution* (x_∞, A_∞) *of the system (15);*

(iii) x_∞ *is the only solution of the equation* $F(x) = 0$ *in the ball* $B(x_0, a - t_\infty)$*;*

(iv) For all $n = 0, 1, 2, \ldots,$

$$\|F(x_{n+1})\| \leq \delta_n(w(a - 2t_n + \gamma_n) - w(a - 2t_n - \delta_n)),$$

$$\Delta_n := \|x_\infty - x_n\| \leq t_\infty - t_n, \|$$

$$\frac{\Delta_{n+1}}{\Delta_n} \leq \frac{\Delta_{n-1}}{w(a - 2t_n + \gamma_n)}.$$

Corollary 3.5. *Let the selected* $dd[x_1, x_2; F]$ *of* F *be restricted Lipschitz continuous on* Ω*. If the initial data* $x_0, x_{-1}, A_0, \gamma_0, \delta_0$ *are such that*

$$\|x_0 - x_{-1}\| \leq \gamma_0 \text{ and }, \|A_0 F(x_0)\| \leq \delta_0 \leq \frac{a^2}{4(a + \gamma_0)},$$

then

(i) $\gamma_\infty = \delta_\infty = 0, I_0 := (c^{-1} - \gamma_0)^2 - 4c^{-1}\delta_0 \geq 0$ *and for all* $n \geq 1$

$$(\|x_n - x_0\| \leq t_n \leq \frac{1}{2}(c^{-1} - \gamma_0 - \delta_0 - \sqrt{\delta_n^2 + \gamma_n\delta_n + 0.25I_0}),$$
$$\|x_n - x_{n-1}\| \leq \gamma_n,$$
$$\|F(x_{n+1})\| \leq \delta_n.$$

(ii) *The sequence* (x_n, A_n) *from the initial data* (x_0, A_0) *remains in the ball* $B((x_0, A_0), (t_\infty, r_A))$, *where*

$$t_\infty = \frac{1}{2}(c^{-1} - \gamma_0 - \sqrt{I_0}),\ r_A := \frac{c^{-1} - \gamma_0 - \delta_0 - \sqrt{I_0}}{c(c^{-1} - \gamma_0 - \delta_0)\sqrt{I_0}} + \frac{\gamma_0 + \delta_0}{1 - c(\gamma_0 + \delta_0)},$$

and converges to a solution (x_∞, A_∞) *of the system*

$$F(x) = 0 \text{ and } A[x, x; F] + I;$$

(iii) x_∞ *is the only solution of the equation* $F(x) = 0$ *in the ball* $B(x_0, 0.5(a - t_\infty))$;

(iv) *For each* $n = 0, 1, 2, \ldots$,

$$\|F(x_{n+1})\| \leq c\delta_n(\gamma_n + \delta_n),$$
$$\Delta_n := \|x_\infty - x_n\| \leq \delta_n + \sqrt{\delta_n^2 + \gamma_n\delta_n + 0.25I_0} - 0.5\sqrt{I_0},$$
$$\frac{\Delta_{n+1}}{\Delta_n} \leq \frac{\Delta_{n-1}}{\gamma_n + 2\sqrt{\gamma_n^2 + \gamma_n\gamma_{n-1} + 0.25I_0}}.$$

Remark 3.6. *(a) The results obtained in this study reduce to the corresponding ones in [11], if equality holds in (5) and (14), i.e.,*$w_0(t) = w(t) = w_1(t)$. *Otherwise, our results provide weaker sufficient convergence conditions, error bounds than in [11] (see also the definition of* a *and* a_0*). Moreover, the information on the uniqueness of the solution* $x^\star$ *is more precise, since* $a_0 - t_\infty > a - t_\infty$. *As an example (16) is given in [11] with* w_0, w *replacing* w_1 *leading to a less precise estimate. Similar comments can be made for the other estimates.*

Using (3), (4) and our idea of restricted convergence region, but not (12), we have already obtained weaker sufficient convergence conditions for many iterative methods such as Newton's, Secant, and Newton–type methods (under very general conditions [1, 2, 3, 4, 5, 6, 7]). In particular, our work using regularly continuous divided differences can be found in [5].

(b) If $w(t) \leq w_0(t)$ *for all* $t \in [0, r_0)$ *holds instead of (14), then clearly function* w_0 *(still at least as small as function* w_1*) can replace* w *in the preceding results.*

(c) If Ω_0 *is replaced by* $\Omega_0^* = \bigcup(x_1, r - \|A_0F(x_0)\|)$ *then in Definition 2.7 a function even tighter than* w *can be used, so, the results can be weakened even further, since* $\Omega_0^* \subseteq \Omega_0$, *and* x_1 *still depends on the initial data.*

Conclusion

We presented the convergence analysis of BTS in order to approximate a locally unique solution of a nonlinear equation. Using a combination of w–regular continuous and w_0–center–regular continuous conditions and our idea of restricted convergence region, we provided a tighter semi-local convergence analysis than before [5, 8, 9, 10, 11]. Special cases are also given in this study. It is worth noticing that the new advantages are obtained under the same computational effort as before, since in practice the computation of the old function w_1 requires the computation of new functions w_0 and w as special cases.

References

[1] I. K. Argyros, On the convergence of Broyden's method, *Comm. Appl. Nonlinear Anal.* **11** (2004), 77 - 86.

[2] I. K. Argyros, A unifying local–semilocal convergence analysis and applications for two - point Newton - like methods in Banach space, *J. Math. Anal. Appl.* **298** (2004), 374 - 397.

[3] I. K. Argyros, On the convergence of Broyden - like methods, *Acta Math. Sin. (Engl. Ser.)* **23** (2007), 2087 - 2096.

[4] I. K. Argyros and Á. A. Magreñán, A Contemporary Study of Iterative Methods Convergence, Dynamics and Applications, N.Y., 2018.

[5] I. K. Argyros and S. Hilout, On the convergence of some iterative procedures under regular smoothness, *PanAmerican Math. J.* **19** (2009), 17 - 34.

[6] I. K. Argyros and S. Hilout, A Kantorovich - type analysis of Broyden's method using recurrent functions, *J. Appl. Math. Comput.* **32** (2010), 353 - 368.

[7] I. K. Argyros and S. Hilout, A unifying theorem for Newton's method on spaces with a convergence structure, *J. Complexity* **27**(1) (2011), 39 - 54.

[8] C. G. Broyden, A class of methods for solving nonlinear simultaneous equations, *Math. Comp.* **19**(1965), 577 - 593.

[9] A. Galperin, Secant method with regularity continuous divided differences, *J. Comput. Appl. Math.* **193** (2006), 574 - 575.

[10] A. Galperin, Ulm's method without derivatives, *Nonlinear Anal.* **71** (2009), 2094 - 2113.

[11] A. Galperin, Broyden's method for operators with regularly continuous divided differences, *J. Korean Math. Soc.* **52**(1) (2015), 43 - 65.

[12] M. A. Hernández, M. J. Rubio and J. A. Ezquerro, Secant - like methods for slving nonlinear integral equations of the Hammerstein type, *J. Comput. Appl. Math.* **115** (2000), 245 - 254.

[13] L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed spaces, Pergamon Press, Oxford, 1982.

[14] C. T. Kelley, Iterative Methods for Linear and Nonlinear Equations, (with separately available software), Frontiers in Applied Mathematics, 16, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.

[15] G. G. Magaril-Il'yaev and V. M. Tikhomirov, Convex Analysis: Theory and Applications, Translated from the 2000 Russian edition by Dmitry Chibisov and revised by the authors, Translations of Mathematical Monographs, 222, American Mathematical Society, Providence, RI, 2003.