A STUDY ON SOLUTIONS OF SOME CONVECTION DIFFUSION EQUATIONS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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To my family and my teachers

DECLARATION

By the Ph.D. Research Scholar

I hereby **declare** that the thesis entitled **"A STUDY ON SOLUTIONS OF SOME CONVECTION DIFFUSION EQUATIONS"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to **certify** that the thesis entitled **"A STUDY ON SOLUTIONS OF SOME CONVECTION DIFFUSION EQUATIONS"** submitted by **VENKATRAMANA P B**, (Reg. No. 177061 MA006) as the record of the research work carried out by him, is *accepted as the thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

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ABSTRACT

The Burgers' equation $u_t + uu_x = \nu u_{xx}$ is a nonlinear partial differential equation, named after the great physicist *Johannes Martinus Burgers* (1895-1981). Our study mainly focuses on (global) weak solutions of Cauchy problem for non-homogeneous Burgers' equation with a time dependent reaction term involving Dirac delta measure and their large time asymptotic analysis. Using Cole-Hopf transformation, we consider the associated two initial-boundary value problems by assuming a common boundary along positive t-axis. The existence and uniqueness of the boundary function along that boundary are established with the help of Abel's integral equation of first kind. Explicit representation of the boundary function is derived. The solutions of associated initial boundary value problems converge uniformly to a nonzero constant on compact sets as t approaches ∞. Also, using this results, the asymptotic behavior of Burgers' equation is discussed.

Secondly, In chapter 3, we consider a Riemann problem for a de-coupled system with locally integrable general source term and obtain explicit solutions. We find the weak solutions by the method of characteristics. Then we find the shock waves involving delta measures. Also, rarefaction wave solution is derived.

In chapter 4, the heat equation with Heaviside function in the source term equipped with Neumann boundary conditions and cosine function as the initial data is considered. In the first part of the article, we are focused to study corresponding two initial-boundary value problems and existence of the derivative of boundary function introduced along positive t-axis due to unit step function. The existence and uniqueness of the same is shown with the help of Volterra's integral equation of first kind. Also, we are concerned with large time behavior of the solutions to associated initial-boundary value problems.

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Chapter 1

General Introduction

Differential equations usually describe the change in the behavior of every material object in the nature with respect to time and space variables. It can be change in single variable for which one can use the concept of Ordinary Differential Equation(ODE). Otherwise one can describe the change in the behavior of the object for several variables through Partial Differential Equation(PDE).

A PDE is an equation involving two or more independent variables, an unknown function and its partial derivatives with respect to the independent variables up to certain order.

1.1 Preliminaries

We use the following notations for spaces in the thesis.

1. $C(\mathbb{R})$ denotes the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.

2. $C^\infty(\mathbb{R})$ denotes the space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$.

3. $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$, denotes the space of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$\|f\|_{L^p(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

4. $L^\infty(\mathbb{R})$ denotes the space of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$\|f\|_{L^\infty(\mathbb{R})} := \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| < \infty.$$

Definition 1.1.1. *(Evans (2010)).* ***Big-oh notation:*** *We write $f = O(g)$ as $x \to x_0$, provided that there exists a constant C such that*

$$|f(x)| \leqslant C|g(x)|,$$

for all x sufficiently close to x_0.

Definition 1.1.2. *(Evans (2010)).* ***Little-oh notation:*** *We write $f = o(g)$ as $x \to x_0$, provided*

$$\lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} = 0.$$

Definition 1.1.3. *(Kesavan (1989)).* ***Support of a function:*** *Let ϕ be a real (or complex) valued continuous function defined on an open set in $\mathbb{R}^n$. The support of ϕ, written as supp(ϕ), is defined as the closure of the set on which ϕ is non-zero.*

Definition 1.1.4. *(Kesavan (1989)).* ***Test functions:*** *The set of all infinitely differentiable functions defined on $\mathbb{R}^n$ with compact support is called test functions.*

Definition 1.1.5. ***Weak Solution:*** *(Stavroulakis and Tersian (2004)). Assume that $u_0(x) \in L^1_{loc}(\mathbb{R})$. A function $u(x,t) \in L^2_{loc}(\mathbb{R} \times [0,\infty))$ is a weak solution of*

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \tag{1.1.2}$$

if and only if

$$\int_0^\infty \int_{-\infty}^\infty \left(u\rho_t + \frac{u^2}{2}\rho_x \right) dxdt + \int_{-\infty}^\infty u_0(x)\rho(x,0)dx = 0, \tag{1.1.3}$$

for every test function $\rho \in C_c^\infty(\mathbb{R} \times [0,\infty))$.

Definition 1.1.6. *(Evans (2010)).* ***Minkowski's Inequality:***
Assume $1 \leq p \leq \infty$ and $u, v \in L^p(U)$, U is open in $\mathbb{R}^n$. Then

$$\| u + v \|_{L^p(U)} \leq \| u \|_{L^p(U)} + \| v \|_{L^p(U)} .$$

Definition 1.1.7 (Stavroulakis and Tersian (2004))**.** ***Weak derivative:*** *A function $v \in L^2_{loc}(\Omega)$ is said to be the weak $\dfrac{\partial}{\partial x_j}$ derivative of a given function $u \in L^2_{loc}(\Omega)$ if and only if*

$$\int_\Omega v(x)\,\rho(x)\,dx = -\int_\Omega u(x)\,\frac{\partial \rho}{\partial x_j}(x)\,dx,$$

for every test function $\rho(x) \in C_0^\infty(\Omega)$.

Definition 1.1.8 (Evans (2010)). ***Sobolev space,*** $W^{k,p}(\Omega)$***:*** *The Sobolev space* $W^{k,p}(\Omega)$ *consists of all locally summable functions* $f : \Omega \longrightarrow \mathbb{R}$ *such that for each multiindex* α *with* $|\alpha| \leq k$, $D^\alpha f$ *exists in the weak sense and belongs to* $L^p(\Omega)$. *i.e.,*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \ : \ D^\alpha f \in L^p(\Omega) \quad \textit{for all} \quad |\alpha| \leq k\},$$

where

$$D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_1^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}} f$$

with multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \geq 0$.

Definition 1.1.9. ***error function****: The error function is defined by*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} \, dz.$$

Definition 1.1.10. ***Complementary error function****: The complementary error function is defined by*

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} \, dz.$$

Definition 1.1.11 (Evans (2010)). ***Gronwall's inequality:*** *Let* $\xi(t)$ *be a nonnegative, summable function on* $[0, T]$ *which satisfies for a.e.* t *the integral inequality*

$$\xi(t) \leq C_1 \int_0^t \xi(s) \, ds + C_2$$

for constants C_1, $C_2 \geq 0$. *Then*

$$\xi(t) \leq C_2(1 + C_1 \, t \, e^{C_1 \, t})$$

for a.e. $0 \leq t \leq T$. *In particular, if*

$$\xi(t) \leq C_1 \int_0^t \xi(s) \, ds$$

for a.e. $0 \leq t \leq T$, *then* $\xi(t) = 0$ *a.e.*

Definition 1.1.12 (Gorenflo and Vessella (1991)). ***Euler's beta function:*** *Euler's beta function,* $B(x, y)$, *is a special function which is closely related to gamma function* $\Gamma(x)$ *defined by*

$$B(r, s) = \int_0^1 (1 - \lambda)^{r-1} \, \lambda^{s-1} \, d\lambda = \frac{\Gamma(r) \, \Gamma(s)}{\Gamma(r + s)}, \quad \textit{where} \quad r > 0, \quad s > 0.$$

By change of variables, we obtain following formulas:

$$\int_0^x (x-y)^{r-1}\, y^{s-1}\, dy = x^{r+s-1}\, B(r,s), \quad \textit{for} \quad r>0, \quad s>0, \quad x>0.$$

$$\int_p^q (q-y)^{r-1}\,(y-p)^{-r}\,dy = \int_0^1 (1-\lambda)^{r-1}\,\lambda^{-r}\,d\lambda$$
$$= \Gamma(r)\,\Gamma(1-r), \quad \textit{for} \quad r>0, \quad -\infty<p<q<\infty.$$

Definition 1.1.13 (Evans (2010))**.** ***Dominated Convergence Theorem:*** *Assume the functions $\{f_k\}_{k=1}^{\infty}$ are integrable and $f_k \longrightarrow f$ a.e. Suppose also $|f_k| \leq g$ a.e., for some summable function g. Then*

$$\int_{\mathbb{R}^n} f_k\, dx \longrightarrow \int_{\mathbb{R}^n} f\, dx.$$

We start our discussion with the heat equation.

Heat Equation

As we know, one of the application of the heat equation is to study the heat conduction. In this case, the heat equation provides an information about temperature at a given location in a metal bar as time changes. To determine the temperature in the bar at any given time, we need to solve the heat equation subject to the initial and boundary conditions.
The one dimensional heat equation is given by

$$u_t = k u_{xx}, \tag{1.1.4}$$

where k represents the thermal conductivity. Then the following function

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{\frac{-x^2}{4kt}}$$

is called the fundamental solution of the heat equation and is known as heat kernel.

We consider the initial value problem for heat equation on the whole real line as following:

$$\begin{cases} u_t = k u_{xx}, \quad x \in \mathbb{R}, \quad t>0, \\ u(x,0) = \phi(x), \quad x \in \mathbb{R}. \end{cases} \tag{1.1.5}$$

Then for integrable function $\phi(x)$ or $\phi \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$,

$$u(x,t) = \int_{\mathbb{R}} \frac{\phi(y)}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} dy \tag{1.1.6}$$

is the solution for (1.1.5) such that

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = \phi(x_0).$$

One can observe that, the heat equation is a second order linear parabolic partial differential equation. The solution (1.1.6) is the convolution of heat kernel with the initial function.

Also, we consider the heat equation in $0 \leq x < \infty$ equipped with initial data and first boundary condition as following:

$$\begin{cases} u_t = ku_{xx}, \quad 0 \leq x < \infty, \quad t > 0, \\ u(x,0) = \phi(x), \quad 0 \leq x < \infty, \\ u(0,t) = g(t), \quad t > 0. \end{cases} \tag{1.1.7}$$

Then the solution is given by

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_0^\infty \phi(\xi) \left[e^{\frac{-(\xi-x)^2}{4kt}} - e^{\frac{-(\xi+x)^2}{4kt}} \right] d\xi + \frac{x}{2\sqrt{\pi k}} \int_0^t \frac{g(\tau)}{(t-\tau)^{\frac{3}{2}}} e^{\frac{-x^2}{4k(t-\tau)}} d\tau. \tag{1.1.8}$$

Our principal work in the thesis is on the study of inhomogeneous Burgers' equation solutions and their asymptotic behavior via heat equation using Hopf-Cole transformation. Hereby we begin addressing about the Burgers' equation.

1.1.1 Viscous Burgers' Equation

Burgers' equation is a second order non-linear parabolic partial differential equation, which is of the form

$$u_t + uu_x = \epsilon u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1.9}$$

where ϵ is a coefficient of viscosity and $u = u(x,t)$ is the velocity of fluid. This equation consists of both non-linearity convection and diffusion terms. It is one of the PDE occurring in various field of applied mathematics such as fluid mechanics, traffic flow etc.

This equation was first discussed by Bateman (1915), due to extensive work of Burgers (1948), it is known as Burgers' equation. Later, this equation has got much attention and studied by Hopf (1950), and many others beginning from 1948.

If $\epsilon = 0$, then (1.1.9) becomes

$$u_t + uu_x = 0, \; x \in \mathbb{R}, \; t > 0. \tag{1.1.10}$$

The equation (1.1.10) is called a inviscid Burgers' equation and it is a hyperbolic partial differential equation. As an application, the Burgers' equation (1.1.9) can be derived from Navier Strokes equations.

1.1.2 Solution to viscous Burgers' equation

Consider the initial value problem for viscous Burgers' equation.

$$\begin{cases} u_t + uu_x = \epsilon u_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1.11}$$

Hopf (1950) and Cole (1951) considered a method to solve the above initial value problem by introducing a transformation, later known as Hopf-Cole transformation by reducing (1.1.11) to a linear problem. This transformation is given by

$$\phi(x,t) = \exp\left\{ -\frac{1}{2\epsilon} \int_{-\infty}^{x} u(y,t)dy \right\}.$$

Note that, the lower limit for the integral above can be any real number also. In fact, Hopf (1950) studied with lower limit zero.

Then the initial value problem (1.1.11) was reduced to

$$\begin{cases} \phi_t = \epsilon \phi_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ \phi(x,0) =: \phi_0(x) = \exp\{-\frac{1}{2\epsilon} \int_{-\infty}^{x} u_0(y)dy\}, & x \in \mathbb{R}. \end{cases} \tag{1.1.12}$$

The solution to the above initial value problem is given by

$$\phi(x,t) = \int_{\mathbb{R}} \frac{\phi_0(y)}{\sqrt{4\pi\epsilon t}} e^{\frac{-(x-y)^2}{4\epsilon t}} dy.$$

From this Hopf (1950) derived an explicit solution for the initial value problem (1.1.11), which is given by

$$\begin{aligned} u(x,t) &= -2\epsilon\frac{\phi_x}{\phi} \\ &= -2\epsilon\frac{\int_{-\infty}^{\infty}\frac{x-y}{t}\exp\{-\frac{(x-y)^2}{4\epsilon t}\}\phi_0(y)dy}{\int_{-\infty}^{\infty}\exp\{-\frac{(x-y)^2}{4\epsilon t}\}\phi_0(y)dy}. \end{aligned}$$

Though the explicit solution exists, it is very cumbersome to evaluate both the integrals in the numerator and denominator for various initial data. This is the primary motivation to study the behavior of solution to Burgers' equation via asymptotic analysis. This complication can be resolved by finding a simple approximate solution or by estimating the integrals using numerical methods.

Initially, Hopf (1950) studied the behavior of the solution of Burgers' equation to obtain the higher order asymptotic in the following cases.

- The behavior of the solution as $t \to \infty$ while keeping viscosity constant.
- The behavior of the solution as $\epsilon \to 0$ while x and t are fixed.

Ablowitz and De Lillo (1991) considered the initial value problem for forced Burgers' equation with Dirac delta function and time dependent continuous function in forcing term:

$$u_t = (u_x + u^2)_x + \delta(x)F(t), \tag{1.1.13}$$

equipped with initial data, specifically, $u(x,0) := u_0(x) = 0$. By introducing the generalized Hopf-Cole transformation, equation (1.1.13) reduces to homogeneous heat equation for which the solution is evaluated using Laplace transformation. The presence of Dirac delta function makes to see (1.1.13) as two initial boundary value problems on two upper quarter planes. Introducing the same boundary condition for the associated problems, the unique continuous solution is established via Volterra's integral equation with the assumption on $F(t)$ to be continuous and bounded.

Ablowitz and De Lillo (1993) analyzed (1.1.13) with zero initial data and forcing term involving rapidly varying function in time variable. Applying Hopf-Cole transformation, (1.1.13) reduces to Heat equation with Heaviside function in the source term. To obtain the continuous solution, it is assumed that both solutions and their spatial derivatives of heat equation with source term in two upper quadrants exists and are continuous at $x = 0$. Using the Laplace transformation on associated heat equations with assumed conditions on solutions, one can obtain Volterra's integral equation in t which admits unique continuous solution. In the second part of the article, authors considered rapidly varying forcing term, $F(t)$, and achieved the expression for large time behavior of solution to (1.1.13) which involves powers of small parameter ϵ even of semi-integers due to Dirac delta function.

Ablowitz and De Lillo (1996) inspected the initial value problem for forced Burgers' equation with time dependent random function and derivative of Dirac delta function in forcing term:

$$u_t = (u_x + u^2)_x + \delta'(x)F(t), \tag{1.1.14}$$

with zero initial data. The above equation is linearized using Hopf-Cole transformation with initial data and continuity condition. By introducing Fourier transformation, the solution of linearized problem is calculated. Then by inverse transformation, solution of (1.1.14) is obtained in terms of Volterra's integral type with weakly singular kernel. In the next part, for different choices of $F(t)$, authors considered the cases where $F(t) = F_0 = a\ constant$ and $F(t) = F_0\, e^{-\mu t}$, $\mu > 0$. In the first case, the solution is obtained by Abel's equation and completely solvable by Laplace transformation. Also, for large t, solution of (1.1.14) approaching to a constant is observed. For the exponentially decaying function case, one can see that asymptotic expansion for solution of associated heat equation is determined in terms of the boundary condition imposed. Also, the asymptotic expansion for solution of Burgers' equation is depending on large time expansion of corresponding heat equation. Further, for $F(t)$ authors substituted and analyzed random Gaussian noise of the Ornstein-Unlenbeck type. Explicit expressions are obtained for the statistical average of the solution and for some correlation functions.

Chung et al. (2014) studied a Cauchy problem posed for a nonhomogeneous viscous Burgers' equation with time independent point source, namely,

$$u_t + uu_x - u_{xx} = \delta(x), \quad x \in \mathbb{R},\ t > 0, \tag{1.1.15}$$

subject to the initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.1.16}$$

where $\delta(x)$ is the time independent Dirac delta measure and $u_0 \in W^{1,\infty}(\mathbb{R}) \cap C^1(\mathbb{R}) \cap L^1(\mathbb{R})$. They investigated the existence, uniqueness and smoothness of weak solution for the initial value problem (1.1.15)-(1.1.16) via inverse Hopf-Cole transformation. For the same they modified the problem to an initial value problem posed for linear heat like equation, namely,

$$\theta_t - \theta_{xx} = \frac{-H(x)\,\theta}{2}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1.17}$$

$$\theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}, \tag{1.1.18}$$

where $H(x)$ is the Heaviside function. This modification is derived with the aid of Hopf transformation given by

$$\theta(x,t) = exp\Big(-\frac{1}{2}\int_{-\infty}^{x} u(y,t)dy \Big).$$

In view of Heaviside function present in the source term of (1.1.17), the solution of the Cauchy problem (1.1.17)-(1.1.18) is established by deducing the solutions of the following two initial boundary value problems:

$$\begin{aligned} L_t - L_{xx} &= 0, \quad x < 0,\ t > 0, \\ L(x,0) &= \theta_0(x), \quad x < 0, \\ L(0,t) &= g(t), \quad t > 0 \end{aligned}$$

and

$$\begin{aligned} R_t - R_{xx} &= -\frac{R}{2}, \quad x \geq 0,\ t > 0, \\ R(x,0) &= \theta_0(x), \quad x \geq 0, \\ R(0,t) &= g(t), \quad t > 0, \end{aligned}$$

It is to be noted that same boundary condition $g(t)$ is picked at $x = 0$ in both the associated problems given above to have the solution $\theta(x,t)$ is continuous. The existence of $g(t)$ is shown by asserting $\theta_x(0^+,t) = \theta_x(0^-,t)$. For $t > 0$, we obtain $R_x(0^+,t) = L_x(0^-,t)$. This in turn gives the following integral equation:

$$\frac{1}{2}\int_o^\infty \xi(\theta_0(\xi)e^{\frac{-t}{2}} + \theta_0(-\xi))\frac{e^{\frac{-\xi^2}{4t}}}{t^{\frac{3}{2}}}d\xi - g(0)\left(\frac{e^{\frac{-t}{2}}+1}{\sqrt{t}}\right)$$
$$= \int_0^t (\frac{g(\tau)}{2} + g'(\tau))\frac{e^{\frac{-(t-\tau)}{2}}}{\sqrt{t-\tau}}d\tau + \int_0^t g'(t)\frac{1}{\sqrt{t-\tau}}d\tau. \qquad (1.1.19)$$

By rearranging, the above equation is expressed in Abel's integral equation of first kind. Using Gorenflo and Vessella (1991), the existence, uniqueness and continuously differentiability of the boundary data $g(t)$ is shown. Further, they imposed an additional condition on initial data, u_0, namely,

$$x\left[exp\left(-\frac{1}{2}\int_{-\infty}^x u_0(y)dy\right) - 1\right] \longrightarrow c, \quad \text{as} \quad x \longrightarrow -\infty. \qquad (1.1.20)$$

where $c \in \mathbb{R}$ and then obtained the large time behavior of boundary condition as follows:

$$\sqrt{t}\,g(t) \rightarrow \sqrt{\frac{2}{\pi}} \quad \text{and} \quad t^{\frac{3}{2}}g'(t) \longrightarrow \frac{-1}{\sqrt{2\pi}} \quad \text{as} \quad t \rightarrow \infty. \qquad (1.1.21)$$

This, in turn, the above estimates helps to find the asymptotic behavior of solutions to the initial value problems associated with (1.1.17)-(1.1.18). It is interesting to see the L^p-norm convergence of the weak solutions of nonhomogeneous Burgers' equation to its steady state solution uniformly on compact sets. Also, the problem illustrates that compactness of the solution trajectory is decided by first order term rather than second order term.

1.2 Organization of the thesis

This thesis is organized as follows:

Chapter 2 deals with understanding the viscous Burgers' equation with time dependent point source involving Dirac function. We consider an initial value problem for the non homogeneous viscous Burgers' equation where δ is the Dirac delta function concentrated at $x = 0$. We use the Hopf transformation for linearization. Linearized partial differential equation consists of Heaviside function and so one needs to study the problem on two upper quarter planes separately with common boundary along the positive t-axis. With the help of Abel's integral equation, we intend to establish the existence and uniqueness of the common boundary data of the linearized partial differential equations. We then look for the explicit representation of the boundary function. In view of the integrals involved in representation of the boundary function, we seek the asymptotic behavior of it for large time t. Using this asymptotic behavior, asymptotic analysis of the solutions to the linearized partial differential equation is established. Making use of inverse Hopf-Cole transformation, existence, uniqueness and regularity of the solutions to the non homogeneous viscous Burgers' equation is discussed. Eventually, the convergence of the solutions to zero on compact intervals is obtained.

In Chapter 3, we consider a Riemann problem for a de-coupled system with locally integrable general source term and obtain explicit solutions. We find the weak solutions by the method of characteristics. Then we find the shock waves involving delta measures. Also, rarefaction wave solution is derived.

In Chapter 4, we consider the Heat equation with Heaviside function in source term in a strip of length $2l$ centered at origin. The problem is equipped with cosine function in initial data and Neumann boundary conditions. The presence of unit step function splits the problem to associated two initial boundary problems in first and second quadrant strips of length l each. It is worth to note that common function, derivative of boundary function, is introduced at $x = 0$ to establish the continuous solution. The existence, uniqueness and continuous differentiability of the common function is shown using volterra's integral equation of first kind with difference kernel. It is interesting to observe that the kernel can be expressed in terms of Jacobi theta function. Further, the large time behavior and rate of convergence of common function is obtained employing Laplace transformation and Final Value theorem. Similarly, asymptotic behavior of the solutions to the associated initial boundary value problems is obtained.

Finally, Chapter 5 sets forth the conclusions of the thesis and future work.

Chapter 2

On the solutions of a viscous Burgers' equation with time dependent source term

2.1 Introduction

This chapter concerns with the existence, uniqueness and smoothness of solutions to the Burgers' equation with time dependent point source given by

$$u_t + uu_x - u_{xx} \;=\; \frac{2}{1+t}\,\delta(x), \quad x \in \mathbb{R}, \quad t > 0, \tag{2.1.1}$$

subject to the initial condition

$$u(x,0) \;=\; u_0(x), \quad x \in \mathbb{R}, \tag{2.1.2}$$

where $u_0 \in W^{1,\infty}(\mathbb{R}) \cap C^1(\mathbb{R}) \cap L^1(\mathbb{R})$.
In the literature, the study on viscous Burgers' equation with source terms

$$u_t + u\,u_x = u_{xx} + f(x,t), \quad x \in \mathbb{R}, \quad t > 0, \tag{2.1.3}$$

has obtained much recognition because of extended importance in numerous fields of science, technology and biology Balogh et al. (2001); Xu et al. (2007). Construction of explicit solutions for viscous Burgers' equation with inhomogeneous terms and large time behavior of these solutions are discussed by several authors. Eule and Friedrich (2006) discussed the solutions of externally forced Burgers' equation

$$u_t + u\,u_x = u_{xx} + xG(t), \tag{2.1.4}$$

by considering $G(t)$ to be constant in the first case and stochastic white noise force in the other case. They examined the problem in relation with stretched vortices in hydrodynamics flows. Salas (2010) examined a specific case of (2.1.4) and derived the n-shock wave solutions with the help of traveling wave method via generalized Hopf-Cole transformation. He connected the problem of solving (2.1.4) with the Riccati and heat equations. Also, several explicit solutions of (2.1.4) were listed in the paper.

Kloosterziel (1990) investigated the solutions for linear heat equation

$$v_t = v_{xx},\ \ x \in \mathbb{R},\ \ t > 0, \tag{2.1.5}$$

subject to the initial data

$$v(x,0) = v_0(x),\ \ x \in \mathbb{R}, \tag{2.1.6}$$

where v_0 is any square integrable function with respect to the exponentially growing weight function $e^{\frac{x^2}{2}}$. Using similarity transformation, the author constructed the following self-similar solutions of (2.1.5);

$$v_n(x,t) = \frac{1}{(2\pi)^{\frac{1}{4}}(2^n\, n!)^{\frac{1}{2}}(1+2t)^{\frac{1+n}{2}}}\; e^{\frac{x^2}{2(1+2t)}} H_n\left(\frac{x}{\sqrt{2(1+2t)}}\right), \tag{2.1.7}$$

where H_n is Hermite polynomial of order n. An interesting feature of these solutions (2.1.7) is that the set of functions $\{v_n(x,0)\}$ is a complete orthonormal system for the Hilbert space $L^2(\mathbb{R},\, e^{\frac{x^2}{2}})$ and hence any function $v_0 \in L^2(\mathbb{R},\, e^{\frac{x^2}{2}})$ can be expanded as an infinite series in terms of $\{v_n(x,0)\}$. Hence, the solution of (2.1.5)-(2.1.6) is represented as an infinite series of self-similar solutions (2.1.7). Another feature of the constructed self similar solutions (2.1.7) is that the decay rate is obtained easily which, in turn, gives the large time asymptotes to the solution of (2.1.5)-(2.1.6). Ding et al. (2001) constructed the explicit solutions of non-homogeneous Burgers' equation

$$u_t + u\,u_x = \mu u_{xx} + kx, \quad x \in \mathbb{R},\, t > 0, \tag{2.1.8}$$

subject to the initial data

$$u(x,0) = u_0(x), \ x \in \mathbb{R}, \tag{2.1.9}$$

where $\mu > 0$ and k is constant. The authors imposed two conditions on the initial function u_0 that it is locally integrable and $\int_0^x u_0(y)\,dy = o(x^2)$ as $|x| \to \infty$. They applied Hopf transformation to reduce (2.1.8)-(2.1.9) to the linear differential equation and then represented the solution of resulting linear differential equation in terms of Fourier-Hermite series. They proved that the solution $u(x,t)$ of the initial value problem (2.1.8)-(2.1.9) behaves like $\sqrt{k}x$ for large time t. However, Chidella and Yadav (2010) noticed that bounded and compactly supported initial functions u_0 do not satisfy the conditions imposed by Ding et al. (2001) and so considered the nonhomogeneous Burgers' equation (2.1.8)-(2.1.9) with an assumption on u_0 that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}-\int^x u_0(r)dr} dx < \infty.$$

Making use of Hopf transformation and standard transformations of Polyanin and Nazaikinskii (2015), they reduced the initial value problem (2.1.8)-(2.1.9) to an initial value problem for heat equation and then used the results of Kloosterziel (1990) to express the solution of the heat equation in terms of the self-similar solutions of the heat equation. Buyukasik and Pashaev (2013) discussed the shock wave solutions, triangular wave solutions, N-wave solutions and rational function solutions for the Burgers' equation (2.1.8). Rao and Yadav (2010b) considered a non-homogeneous Burgers' equation

$$u_t + u\,u_x = u_{xx} + \frac{kx}{(2\beta t+1)^2}, \quad x \in \mathbb{R},\, t > 0, \tag{2.1.10}$$

subject to the unbounded initial data and expressed the solutions in terms of the self similar solutions of a linear partial differential equation with variable coefficients. They obtained the large time behavior of the solution of the nonhomogeneous Burgers' equation. Rao and Yadav (2010a) investigated solutions for (2.1.10) by assuming that the initial data is compactly supported and bounded. Engu et al. (2017) proved the existence of a solution for the initial value problem of a nonhomogeneous Burgers' equation and expressed the solution in terms of

Hermite polynomials. Their analysis mainly depends on Hopf-Cole transformation and method of variation of parameters. The authors have also given the rates of convergence of an approximate solution to the true solution of the initial value problem.

However, investigating the solutions for viscous Burgers' equation with source term involving the Dirac delta measure becomes complicated as the linearization process of the viscous Burgers' equation with the source term leads to two different linear partial differential equations on the two upper quarter planes. Further, considering the nontrivial initial condition with the nonhomogeneous viscous Burgers' equation increases the complexity more. Chung et al. (2014) studied the existence, uniqueness and asymptotic behavior of solutions to a Cauchy problem for the viscous Burgers' equation with Dirac delta measure as source term.

The rest of the chapter is organized as follows. Section (2.2) deals with the linearization of the Cauchy problem and then the existence, uniqueness of the common boundary data of the resulting two initial-boundary value problems. Section (2.3) discusses the explicit representation of the common boundary data via Laplace transformation and Classical Abel's integral equation. Section (2.4), (2.5) examines the strict positiveness of the solution to corresponding heat like equation with source term and asymptotic behavior of the solutions and boundary data respectively. Further, Section (2.6) analyzes existence and uniqueness of the global weak solutions for the Cauchy problem.

2.2 Burgers' equation with point source

Consider the Burgers' equation with time dependent point source given by

$$u_t + u\,u_x - u_{xx} = \frac{2}{1+t}\,\delta(x)\,, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.2.11}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{2.2.12}$$

where $u_0 \in W^{1,\infty}(\mathbb{R}) \cap C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\delta(x)$ is the Dirac delta function.

The Hopf-Cole transformation Hopf (1950); Cole (1951), is given by

$$\theta(x,t) = exp\left\{-\frac{1}{2}\int_{-\infty}^{x} u(y,t)dy\right\}. \tag{2.2.13}$$

From the above transformation we calculate

$$u_t = -2\left(\frac{\theta_x}{\theta}\right)_t = -2\left(\frac{\theta_t}{\theta}\right)_x, \quad u_x = -2\left(\frac{\theta_x}{\theta}\right)_x \quad \text{and} \quad u_{xx} = -2\left(\frac{\theta_x}{\theta}\right)_{xx}.$$

On rearrangement, (2.2.11) can be written as

$$u_t = u_{xx} - uu_x + \frac{2}{1+t}\,\delta(x) = \left(u_x - \frac{u^2}{2}\right)_x + \frac{2}{1+t}\,\delta(x).$$

Substituting u_t, u_x and u_{xx} in the above equation and integrating with respect to space variable x leads to get

$$\frac{\theta_t}{\theta} = \frac{\theta\theta_{xx} - \theta_x^2}{\theta^2} + \frac{\theta_x^2}{\theta^2} - \frac{1}{(1+t)}\,H(x) + C.$$

Multiplying by θ^2 leads to obtain

$$\theta_t\,\theta = \theta\theta_{xx} - \theta_x^2 + \theta_x^2 - \frac{H(x)}{(1+t)}\,\theta^2 + C\,\theta^2 = \theta\theta_{xx} - \frac{H(x)}{(1+t)}\,\theta^2 + C\,\theta^2$$

where $H(x)$ is the Heaviside function and C is the integrating constant. In particularly assuming $C = 0$ and then dividing by $\theta(x,t)$, the above equation reduces to a heat equation with Heaviside function in the source term:

$$\theta_t - \theta_{xx} = -\frac{H(x)\,\theta}{(1+t)}.$$

Also, from (2.2.13) the corresponding initial data reduces to

$$\theta(x,0)=\exp\left\{-\frac{1}{2}\int_{-\infty}^{x}u(y,0)dy\right\}=\exp\left\{-\frac{1}{2}\int_{-\infty}^{x}u_0(y)dy\right\}.$$

In view of our assessment on $u_0 \in W^{1,\infty}(\mathbb{R})\cap C^1(\mathbb{R})\cap L^1(\mathbb{R})$, we get $\theta(x,0)=:$ $\theta_0(x)\in W^{2,\infty}(\mathbb{R})\cap C^2(\mathbb{R})$ and the above Cauchy problem can be rewritten as

$$\theta_t-\theta_{xx}+\frac{H(x)}{(1+t)}\theta = 0, \qquad x\in\mathbb{R},\quad t>0, \tag{2.2.14}$$

$$\theta(x,0) = \theta_0(x), \quad x\in\mathbb{R}. \tag{2.2.15}$$

Due to the presence of the Heaviside function, $H(x)$, the above Cauchy problem (2.2.14)-(2.2.15) is split into two problems on upper quarter planes, namely, right side domain $\{0\le x<\infty;\, t>0\}$ and left side domain $\{-\infty<x\le 0;\, t>0\}$.
Let $R(x,t)$ be the solution of initial-boundary value problem corresponding to (2.2.14)-(2.2.15) on right side domain $\{0\le x<\infty;\, t>0\}$ with $H(x)=1$ satisfying

$$\begin{aligned} R_t-R_{xx} &= -\frac{R}{(1+t)}, \quad 0\le x, \quad 0<t,\\ R(x,0) &= \theta_0(x), \quad 0\le x,\\ R(0,t) &= g(t), \quad 0<t, \end{aligned} \tag{2.2.16}$$

where $g(t)$ is the boundary condition introduced. To reduce the the above heat equation with source term into homogeneous heat equation, we consider the following transformation given by

$$w(x,t)=(1+t)R(x,t). \tag{2.2.17}$$

The initial-boundary value problem (2.2.16) subjected to (2.2.17) reduces to the associated initial-boundary value problem for homogeneous heat equation;

$$\begin{aligned} w_t-w_{xx} &= 0, \quad x\ge 0,\, t>0,\\ w(x,0) &= \theta_0(x), \quad x\ge 0,\\ w(0,t) &= (1+t)\, g(t), \quad t>0. \end{aligned} \tag{2.2.18}$$

From Polyanin and Nazaikinskii (2015) we see that the solution of above initial-boundary value problem is given by

$$w(x,t) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty \theta_0(\xi) \left[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}} \right] d\xi + \frac{x}{\sqrt{4\pi}} \int_0^t \frac{g(\tau)(1+\tau)}{(t-\tau)^{\frac{3}{2}}}\, e^{\frac{-x^2}{4(t-\tau)}}\, d\tau. \tag{2.2.19}$$

To simplify the second term in the right side of the above equation we consider the following integral and apply integration by parts which yields

$$\int_0^t \big(g(\tau)(1+\tau)\big)' \operatorname{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau = -g(0) \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) + \frac{x}{\sqrt{4\pi}} \int_0^t \frac{g(\tau)(1+\tau)}{(t-\tau)^{\frac{3}{2}}}\, e^{\frac{-x^2}{4(t-\tau)}}\, d\tau.$$

On rearrangement to the above equation and then substitution in (2.2.19), the solution of (2.2.18) can be expressed as

$$w(x,t) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty \theta_0(\xi) \left[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}} \right] d\xi + \int_0^t \big(g(\tau)(1+\tau)\big)' \operatorname{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau + g(0) \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$

Finally, retracing $R(x,t)$ via (2.2.17), the solution in right side domain is given by

$$R(x,t) = \frac{1}{1+t}\left[\frac{1}{2\sqrt{\pi t}} \int_0^\infty \theta_0(\xi) \left[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}} \right] d\xi + \int_0^t \big(g(\tau)(1+\tau)\big)' \operatorname{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau + g(0) \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) \right], \tag{2.2.20}$$

where $\operatorname{erfc}(x)$ is the complementary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2}\, dy.$$

Similarly, let $L(x,t)$ be the solution of the initial-boundary value problem on the left side domain $\{-\infty < x \leq 0;\, t > 0\}$ which satisfies

$$\begin{aligned} L_t - L_{xx} &= 0, \quad x < 0,\, t > 0, \\ L(x,0) &= \theta_0(x), \quad x < 0, \\ L(0,t) &= g(t), \quad t > 0. \end{aligned} \tag{2.2.21}$$

Then for the above system, the solution is given by

$$L(x,t) = \frac{-1}{2\sqrt{\pi t}} \int_0^\infty \theta_0(-\xi)\big[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}}\big]d\xi - \frac{x}{\sqrt{4\pi}} \int_0^t \frac{g(\tau)(1+\tau)}{(t-\tau)^{\frac{3}{2}}}\, e^{\frac{-x^2}{4(t-\tau)}}\, d\tau,$$

which can be simplified with second term as before and expressed as

$$L(x,t) = \frac{-1}{2\sqrt{\pi t}} \int_0^\infty \theta_0(-\xi)\big[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}}\big]d\xi \\ + \int_0^t g'(\tau)\ \text{erfc}\left(\frac{-x}{2\sqrt{t-\tau}}\right)d\tau + g(0)\ \text{erfc}\left(\frac{-x}{2\sqrt{t}}\right). \quad (2.2.22)$$

It is to be noted that the same boundary function $g(t)$ is taken for both the initial-boundary value problems (2.2.16) and (2.2.21) to assume that the solution of (2.2.14)-(2.2.15), $\theta(x,t)$, is continuous on the positive $t-$axis. Further, we assume temporarily that $g(t)$ is continuously differentiable on $[0,\infty)$ and will be calculated after showing the existence of the same.

To establish the existence of $g'($ or $g)$, we impose a constraint on $\theta(x,t)$ that space derivatives exists along the positive t-axis and are equal. That is,

$$R_x(0^+,t) = L_x(0^-,t), \quad t > 0. \quad (2.2.23)$$

Now, we calculate the spatial derivative of the right side domain solution, $R(x,t)$, given in (2.2.20):

$$R_x(x,t) = \frac{1}{4\sqrt{\pi}(1+t)t^{\frac{3}{2}}} \int_0^\infty \theta_0(\xi)\big[(\xi-x)e^{\frac{-(\xi-x)^2}{4t}} + (\xi+x)e^{\frac{-(\xi+x)^2}{4t}}\big]d\xi \\ + \frac{1}{\sqrt{\pi}(1+t)} \int_0^t \frac{\big(g(\tau)(1+\tau)\big)'}{\sqrt{(t-\tau)}} e^{\frac{-x^2}{4(t-\tau)}} d\tau + \frac{1}{(1+t)} \frac{g(0)}{\sqrt{\pi\, t}} e^{\frac{-x^2}{4t}}. \quad (2.2.24)$$

Also, differentiating the left side domain solution $L(x,t)$ with respect to space variable x, we obtain

$$L_x(x,t) = \frac{-1}{4\sqrt{\pi}t^{\frac{3}{2}}} \int_0^\infty \theta_0(-\xi)\Big((\xi-x)e^{\frac{-(\xi-x)^2}{4t}} + (\xi+x)e^{\frac{-(\xi+x)^2}{4t}}\Big)d\xi \\ + \frac{1}{\sqrt{\pi}} \int_0^t \frac{g'(\tau)}{\sqrt{t-\tau}}\, e^{\frac{-x)^2}{4(t-\tau)}}\, d\tau + \frac{g(0)}{\sqrt{\pi\, t}} e^{\frac{-x^2}{4t}}. \quad (2.2.25)$$

Substituting $x = 0$ in (2.2.24) and (2.2.25) we obtain $R_x(0,t)$ and $L_x(0,t)$ respectively. i.e.,

$$R_x(0,t) = \frac{1}{(1+t)2\sqrt{\pi t^3}} \int_0^\infty \xi\theta_0(\xi)e^{\frac{-\xi^2}{4t}}d\xi - \frac{1}{(1+t)\sqrt{\pi}} \int_0^t \big(g(\tau)(1+\tau)\big)' \frac{1}{\sqrt{t-\tau}}d\tau - \frac{g(0)}{(1+t)\sqrt{\pi t}},$$

$$L_x(0,t) = \frac{-1}{2\sqrt{\pi t^3}} \int_0^\infty \xi\theta_0(-\xi)e^{\frac{-\xi^2}{4t}}d\xi + \frac{1}{\sqrt{\pi}} \int_0^t \frac{g'(\tau)}{\sqrt{t-\tau}}d\tau + \frac{g(0)}{\sqrt{\pi t}}.$$

Then by above expressions of $R_x(0,t)$ and $L_x(0,t)$ we get

$$\frac{1}{2\sqrt{\pi}} \int_0^\infty \xi\left[\frac{\theta_0(\xi)}{1+t} + \theta_0(-\xi)\right]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - \frac{g(0)}{\sqrt{\pi t}}\left(\frac{2+t}{1+t}\right) = \frac{1}{\sqrt{\pi}(1+t)}\left[\int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}d\tau + \int_0^t \frac{(2+t+\tau)g'(\tau)}{\sqrt{t-\tau}}d\tau\right].$$

Applying integration by parts for the first integral in the right side and then simplifying the terms involving $g(0)$ in the above equation, we obtain

$$\frac{1}{4\sqrt{\pi}} \int_0^\infty \xi\left[\frac{\theta_0(\xi)}{1+t} + \theta_0(-\xi)\right]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - \frac{g(0)(3t+2)}{2\sqrt{\pi t}(1+t)} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{(3t-\tau+2)}{2\,(1+t)}\frac{g'(\tau)}{\sqrt{t-\tau}}d\tau. \tag{2.2.26}$$

Theorem 2.2.1. *For $\theta_0 \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R})$ such that $g(0) = \theta_0(0)$, there exists unique continuous bounded function $g(t)$ satisfying (2.2.26).*

Proof. It is to be noted that the equation (2.2.26) is of the Abel's integral equation form;

$$\frac{1}{\sqrt{\pi}} \int_0^t \frac{K(t,\tau)}{\sqrt{t-\tau}}h(\tau)d\tau = F(t), \quad \text{for all} \quad t > 0. \tag{2.2.27}$$

where

$$h(\tau) = g'(\tau), \quad K(t,\tau) = \frac{(3t-\tau+2)}{2(1+t)}$$

and

$$F(t) = \frac{1}{4\sqrt{\pi}} \int_0^\infty \xi\left[\frac{\theta_0(\xi)}{1+t} + \theta_0(-\xi)\right]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - \frac{g(0)(3t+2)}{2\sqrt{\pi t}(1+t)}. \tag{2.2.28}$$

It can be observed easily that $K(t,\tau)$ is continuous and $K(t,t) = 1$, for all $0 \leq \tau \leq t < \infty$. In fact, for $\tau = t$, we have

$$K(t,t) = \frac{3t - t + 2}{2(1+t)} = 1, \quad \forall\, t \in (0,\infty).$$

Further, $\dfrac{\partial K}{\partial t}$ is bounded for $0 \leq \tau \leq t \leq \infty$. i.e.,

$$\frac{\partial K}{\partial t} = \frac{1}{2}\left[\frac{3(1+t) - (3t - \tau + 2)}{(1+t)^2}\right] = \frac{1}{2}\left[\frac{\tau - 1}{(1+t)^2}\right] = -\frac{1}{2}\left[\frac{1-\tau}{(1+t)^2}\right]$$

Using the facts, for $\tau \leq t$ we have $1 + \tau \leq 1 + t$ and $1 \leq 1 + t$ implying $\dfrac{1}{1+t} \leq 1$, we obtain

$$\left|\frac{\partial K}{\partial t}\right| \leq \frac{1}{2}\frac{1+\tau}{(1+t)^2} \leq \frac{1}{2}\frac{1+t}{(1+t)^2} = \frac{1}{2(1+t)} \leq \frac{1}{2}.$$

Since $K(t,t) \neq 0$, the necessary condition for the existence of the solution to (2.2.27) is to be $F(0) = 0$. Note that, from the expression of $F(t)$ given in (2.2.28), $F(0)$ cannot be obtained directly. For the same we apply change of variable. i.e., put $\eta = \frac{\xi}{2\sqrt{t}}$. Then (2.2.28) simplifies to

$$F(t) = \frac{-1}{2\sqrt{\pi}}\int_0^\infty \left[\frac{\theta_0(2\sqrt{t}\,\eta)}{1+t} + \theta_0(-2\sqrt{t}\,\eta)\right]\frac{d(e^{-\eta^2})}{\sqrt{t}}\,d\eta - \frac{g(0)(3t+2)}{2\sqrt{\pi t}(1+t)}.$$

Applying integration by parts for the above equation, we obtain

$$F(t) = \frac{\theta_0(0)}{2\sqrt{\pi\, t}}\left(\frac{2+t}{1+t}\right) + \frac{2}{2\sqrt{\pi}}\int_0^\infty e^{-\eta^2}\left(\frac{\theta_0'(2\sqrt{t}\,\eta)}{1+t} + \theta_0'(-2\sqrt{t}\,\eta)\right)d\eta - \frac{g(0)(3t+2)}{2\sqrt{\pi t}(1+t)}.$$

For the choice of $\theta_0(0) = g(0)$ the above equation can be simplified as

$$F(t) = \frac{1}{2\sqrt{\pi}}\left[2\int_0^\infty e^{-\eta^2}\left(\frac{\theta_0'(2\sqrt{t}\eta)}{1+t} - \theta_0'(-2\sqrt{t}\eta)\right)d\eta - \frac{2\sqrt{t}}{1+t}\theta_0(0)\right].$$

Put $t = 0$ in the above expression from which we obtain $F(0) = 0$. Also, differentiating the above expression with respect to the time variable t, we arrive at

$$F'(t) = \frac{2}{\sqrt{\pi t}}\left[2\int_0^\infty e^{-\eta^2}\left(\frac{\eta\;\theta_0''(2\sqrt{t}\eta)}{(1+t)} - \frac{\sqrt{t}\;\theta_0'(2\sqrt{t}\eta)}{(1+t)^2} + \eta\;\theta_0''(2\sqrt{t}\eta)\right)d\eta - \frac{(1-t)\;\theta_0}{(1+t)^2}\right].$$

Since $\theta_0(x) \in W^{2,\infty}(\mathbb{R})$, we have

$$\left|\theta_0^{(i)}(x)\right| \leq m_i, \quad \text{for} \quad i = 0, 1, 2 \quad \text{and} \quad \forall\, x.$$

Let $m = \max\{m_i \mid i = 0, 1, 2\}$. But then

$$|F'(t)| \leq \frac{m}{2\sqrt{\pi t}}\left[\frac{2}{(1+t)} + \frac{\sqrt{\pi t}}{(1+t)^2} + 1\right].$$

To simplify the above inequality, we see that for all $t > 0$,

$$\sqrt{t} \leq (1+t)^2 \quad \Rightarrow \quad \frac{1}{(1+t)^2} \leq \frac{1}{\sqrt{t}} \quad \Rightarrow \quad \frac{\sqrt{t}}{(1+t)^2} \leq 1.$$

Hence, we get

$$|F'(t)| \leq \frac{m}{2\sqrt{\pi t}}\left[2 + \sqrt{\pi} + 1\right] = \frac{1}{\sqrt{t}}\left[\frac{m}{2\sqrt{\pi}}(3 + \sqrt{\pi})\right].$$

i.e,

$$|F'(t)| \leq \frac{C}{\sqrt{t}}, \quad \text{where } C = \frac{m}{2\sqrt{\pi}}(3 + \sqrt{\pi}) \text{ depending only on } \theta_0.$$

Define

$$(Tf)(t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{F(\tau)}{\sqrt{t-\tau}}d\tau = \frac{2}{\sqrt{\pi}}\frac{d}{dt}\int_0^t F(\tau)\sqrt{t-\tau}\; d\tau = \frac{1}{\sqrt{\pi}}\int_0^t \frac{F'(\tau)}{\sqrt{t-\tau}}d\tau.$$

But then

$$|(Tf)(t)| \leq \frac{C}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}\sqrt{t-\tau}}d\tau = C\sqrt{\pi}.$$

i.e.,

$$|(Tf)(t)| \leq C\sqrt{\pi}.$$

i.e., $|(Tf)(t)|$ is bounded for all $t > 0$. Also, $|(Tf)(t)|$ has no singularity which concludes the continuity for all $t > 0$. By using standard results Gorenflo and Vessella (1991), there exist a unique continuous solution $h(t)$ for Abel's equation satisfying

$$|h(t)| \leq e^{2Mt} \parallel (Tf)(t) \parallel_{L^\infty(0,t)}.$$

i.e.,

$$|h(t)| \leq e^{2Mt}C\sqrt{\pi} \quad \text{where} \quad M = \sup_{t\leq\tau}\left|\frac{\partial K}{\partial t}(t,\tau)\right|.$$

By defining $g(t) := \theta_0(0) + \int_0^t h(\tau)d\tau$, we conclude that $g(t)$ satisfies all the desired properties in the statement. Therefore, the solution of (2.2.14)-(2.2.15) is well established and given by

$$\theta(x,t) = \begin{cases} R(x,t), & x \geq 0, \\ L(x,t), & x < 0, \end{cases} \tag{2.2.29}$$

where $R(x,t)$ and $L(x,t)$ are given in (2.2.20) and (2.2.22) respectively. □

2.3 Explicit representation of boundary function

In this section, we establish three explicit expressions for the unique boundary condition $g(t)$. Making use of Laplace transformation on (2.2.26), we deduce one of the formula to evaluate $g(t)$. Secondly, rearranging the Able's integral equation into classical Abel's integral equation and then by evaluating the solution we obtain expressions which can be utilized to find the boundary data $g(t)$. In each case, we find the expression for boundary function for $u_0 \equiv 0$ or $\theta_0 \equiv 1$.

2.3.1 Expression for g(t) via Laplace transformation

Considering the below mentioned integral and then expressing it in the convolution form, we have

$$\begin{aligned}
\int_0^t \frac{3t-\tau+2}{\sqrt{t-\tau}} h(\tau)d\tau &= \int_0^t \frac{3(t-\tau)+2[(1+t)-(t-\tau)]}{\sqrt{t-\tau}} h(\tau)d\tau \\
&= \int_0^t \frac{(t-\tau)+2(1+t)}{\sqrt{t-\tau}} h(\tau)d\tau \\
&= \int_0^t \frac{t-\tau}{\sqrt{t-\tau}} h(\tau)d\tau + \int_0^t \frac{2(1+t)}{\sqrt{t-\tau}} h(\tau)d\tau \\
&= \int_0^t \sqrt{t-\tau}\, h(\tau)d\tau + 2(1+t)\int_0^t \frac{h(\tau)}{\sqrt{t-\tau}} d\tau \\
&= (\sqrt{t} * h) + 2(1+t)(\frac{1}{\sqrt{t}} * h) \\
&= (\sqrt{t} * h) + 2(\frac{1}{\sqrt{t}} * h) + 2t(\frac{1}{\sqrt{t}} * h).
\end{aligned}$$

Invoking the Laplace transform for the above convolution integral, we simplify the above expression as follows:

$$\begin{aligned}
\mathcal{L}\Big\{\int_0^t \frac{(3t-\tau+2)}{\sqrt{t-\tau}}h(\tau)d\tau\Big\} &= \mathcal{L}\{\sqrt{t}*h\}+2\mathcal{L}\{\frac{1}{\sqrt{t}}*h\}+2\mathcal{L}\{t(\frac{1}{\sqrt{t}}*h)\} \\
&= \mathcal{L}\{\sqrt{t}\}\mathcal{L}\{h\}+2\mathcal{L}\{\frac{1}{\sqrt{t}}\}\mathcal{L}\{h\}+2\big(-\frac{d}{ds}\mathcal{L}\{\sqrt{t}*h\}\big) \\
&= \frac{\sqrt{\pi}}{2s\sqrt{s}}H(s)+2\frac{\sqrt{\pi}}{\sqrt{s}}H(s)-2\frac{d}{ds}(\frac{\sqrt{\pi}}{\sqrt{s}}H(s)) \\
&= \sqrt{\pi}\Bigg[\Big(\frac{1}{2s\sqrt{s}}+\frac{2}{\sqrt{s}}\Big)H(s) \\
&\qquad -2\Big(-\frac{1}{2s\sqrt{s}}H(s)+\frac{1}{\sqrt{s}}H'(s)\Big)\Bigg] \\
&= \sqrt{\pi}\Bigg[\Big(\frac{1}{2s\sqrt{s}}+\frac{2}{\sqrt{s}}+\frac{1}{s\sqrt{s}}\Big)H(s)-\frac{2}{\sqrt{s}}H'(s)\Bigg] \\
&= \sqrt{\pi}\Bigg[\Big(\frac{3}{2s\sqrt{s}}+\frac{2}{\sqrt{s}}\Big)H(s)-\frac{2}{\sqrt{s}}H'(s)\Bigg].
\end{aligned}$$

i.e.,

$$\mathcal{L}\Big\{\int_0^t \frac{(3t-\tau+2)}{\sqrt{t-\tau}}h(\tau)d\tau\Big\} = \sqrt{\pi}\Bigg[\Big(\frac{3}{2s\sqrt{s}}+\frac{2}{\sqrt{s}}\Big)H(s)-\frac{2}{\sqrt{s}}H'(s)\Bigg]. \quad (2.3.30)$$

Multiply (2.2.26) by $2(1+t)$, we get

$$\frac{1}{2}\int_0^\infty \xi\Big[\theta_0(\xi)+(1+t)\theta_0(-\xi)\Big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi-g(0)\Big(3\sqrt{t}+\frac{2}{\sqrt{t}}\Big) = \int_0^t \frac{(3t-\tau+2)}{\sqrt{t-\tau}}h(\tau)d\tau.$$

Applying Laplace Transform and using (2.3.30), we obtain

$$\sqrt{\pi}\Bigg[\Big(\frac{3}{2s\sqrt{s}}+\frac{2}{\sqrt{s}}\Big)H(s)-\frac{2}{\sqrt{s}}H'(s)\Bigg] = M(s)-g(0)\Big(\frac{3\sqrt{\pi}}{2s\sqrt{s}}+\frac{2\sqrt{\pi}}{\sqrt{s}}\Big)$$

where

$$M(s)=\mathcal{L}\Big\{\frac{1}{2}\int_0^\infty \xi\Big[\theta_0(\xi)+(1+t)\theta_0(-\xi)\Big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi\Big\}.$$

Multiply by $\frac{2s\sqrt{s}}{\sqrt{\pi}}$ to the above equation, we get

$$(3+4s)H(s)-4sH'(s)=\frac{2s\sqrt{s}}{\sqrt{\pi}}M(s)-g(0)(3+4s).$$

Rearrangement of the above equation for $H'(s)$ will leads to get

$$H'(s)=\Big(\frac{3+4s}{4s}\Big)H(s)-\frac{\sqrt{s}}{2\sqrt{\pi}}M(s)+g(0)\Big(\frac{3+4s}{4s}\Big),$$

which also can be expressed as

$$H'(s) = \frac{3}{4}\frac{H(s)}{s} + H(s) - \frac{\sqrt{s}}{2\sqrt{\pi}}M(s) + g(0)\left(\frac{3}{4s} + 1\right). \tag{2.3.31}$$

Further, to simplify the above equation, we evaluate $M(s)$ given by

$$M(s) = \mathcal{L}\left\{\frac{1}{2}\int_0^\infty \xi\Big[\theta_0(\xi) + (1+t)\theta_0(-\xi)\Big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi\right\}.$$

Multiply and divide by $\sqrt{\pi}$ and then using the linearity of the transformation we can write

$$M(s) = \sqrt{\pi}\left[\mathcal{L}\left\{\int_0^\infty \frac{\xi\theta_0(\xi)e^{-\frac{\xi^2}{4t}}}{2\sqrt{\pi t^3}}d\xi + \int_0^\infty \frac{\xi(1+t)\theta_0(-\xi)e^{-\frac{\xi^2}{4t}}}{2\sqrt{\pi t^3}}d\xi\right\}\right].$$

Using the identities on Laplace transformation, in particularly, $\mathcal{L}\left\{\frac{\xi e^{-\frac{\xi^2}{4t}}}{2\sqrt{\pi t^3}}\right\} = e^{-\xi\sqrt{s}}$ and $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$ we see that

$$\begin{aligned} M(s) &= \sqrt{\pi}\left[\int_0^\infty \theta_0(\xi)e^{-\xi\sqrt{s}}d\xi + \int_0^\infty \theta_0(-\xi)e^{-\xi\sqrt{s}}d\xi + \int_0^\infty \theta_0(-\xi)\left(-\frac{d}{ds}e^{-\xi\sqrt{s}}\right)d\xi\right] \\ &= \sqrt{\pi}\left[\int_0^\infty (\theta_0(\xi)+\theta_0(-\xi))e^{-\xi\sqrt{s}}d\xi + \int_0^\infty \theta_0(-\xi)\frac{\xi}{2\sqrt{s}}e^{-\xi\sqrt{s}}d\xi\right]. \end{aligned}$$

Hence we arrive at

$$M(s) = \sqrt{\pi}\left[\int_0^\infty \big[\theta_0(\xi) + \left(1 + \frac{\xi}{2\sqrt{s}}\right)\theta_0(-\xi)\big]e^{-\xi\sqrt{s}}d\xi\right].$$

In order to simplify the calculation complexity we write the above equation to be

$$M(s) = \sqrt{\pi}I(s), \quad \text{where} \quad I(s) = \int_0^\infty \big[\theta_0(\xi) + \left(1 + \frac{\xi}{2\sqrt{s}}\right)\theta_0(-\xi)\big]e^{-\xi\sqrt{s}}d\xi.$$

Hence, substituting in (2.3.31) and then dividing by s, we get

$$\frac{H'(s)}{s} = \frac{3}{4}\frac{H(s)}{s^2} + \frac{H(s)}{s} - \frac{1}{2}\frac{I(s)}{\sqrt{s}} + g(0)\left(\frac{3}{4s^2} + \frac{1}{s}\right). \tag{2.3.32}$$

Note that the terms of above equation can be replaced with the use of identities on Laplace transformation as given below:

$$\begin{aligned} \frac{H(s)}{s} &= \mathcal{L}\left\{\int_0^t g'(r)dr\right\}, \qquad & \frac{H'(s)}{s} &= \mathcal{L}\left\{\int_0^t(-rg'(r))dr\right\}, \\ \mathcal{L}\{tf(t)\} &= -\frac{d}{ds}\mathcal{L}\{f(t)\}, \qquad & \frac{H(s)}{s^2} &= \mathcal{L}\left\{\int_0^t(g(r)-g(0))dr\right\}. \end{aligned}$$

Hence, the above identities reduce (2.3.32) into

$$\mathcal{L}\left\{\int_0^t(-rg'(r))dr\right\} = \frac{3}{4}\mathcal{L}\left\{\int_0^t(g(r)-g(0))dr\right\} + \mathcal{L}\left\{\int_0^t g'(r)dr\right\} + \mathcal{L}\{N\} + g(0)\left(\frac{3}{4}\mathcal{L}\{t\} + \mathcal{L}\{1\}\right)$$

where

$$N = \mathcal{L}^{-1}\{\frac{-I(s)}{2\sqrt{s}}\} = \frac{-1}{2}\left[\int_0^\infty(\theta_0(\xi)+\theta_0(-\xi))\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{\pi t}}d\xi + \frac{1}{2}\int_0^\infty \xi\theta_0(-\xi)\,\mathrm{erfc}(\frac{\xi}{2\sqrt{t}})d\xi\right].$$

Applying the Inverse Laplace transform one can obtain

$$\int_0^t(-rg'(r))dr = \frac{3}{4}\left[\int_0^t(g(r)-g(0))dr\right] + \int_0^t g'(r)dr + N + g(0)\left(\frac{3}{4}t+1\right)$$

and reducing the integrals involved in the above equation leads to

$$\int_0^t g(r)dr - tg(t) = \frac{3}{4}\left[\int_0^t g(r)dr - tg(0)\right] + g(t) - g(0) + N + g(0)\left(\frac{3}{4}t+1\right)$$

$$\int_0^t g(r)dr - (t+1)g(t) = \frac{3}{4}\int_0^t g(r)dr + N$$

$$\frac{1}{4}\int_0^t g(r)dr - (t+1)g(t) = N.$$

Substituting the expression of N:

$$\frac{1}{4}\int_0^t g(r)dr - (t+1)g(t) = \frac{-1}{2}\left[\int_0^\infty(\theta_0(\xi)+\theta_0(-\xi))\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{\pi t}}d\xi + \frac{1}{2}\int_0^\infty \xi\theta_0(-\xi)\,\mathrm{erfc}(\frac{\xi}{2\sqrt{t}})d\xi\right]. \tag{2.3.33}$$

Using integration by parts for the first integral in the right side N, we obtain

$$\frac{-1}{2}\int_0^\infty(\theta_0(\xi)+\theta_0(-\xi))\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{\pi t}}d\xi = -\theta_0(0) - \frac{1}{2}\int_0^\infty(\theta_0'(\xi)-\theta_0'(-\xi))\,\mathrm{erfc}(\frac{\xi}{2\sqrt{t}})d\xi.$$

Consequently, N can be represented as

$$\begin{aligned} N &= -\theta_0(0) - \frac{1}{2}\int_0^\infty(\theta_0'(\xi)-\theta_0'(-\xi))\,\mathrm{erfc}(\frac{\xi}{2\sqrt{t}})d\xi + \frac{1}{4}\int_0^\infty \xi\theta_0(-\xi)\,\mathrm{erfc}(\frac{\xi}{2\sqrt{t}})d\xi \\ &= -\theta_0(0) - \frac{1}{2}\left[\int_0^\infty\left(\theta_0'(\xi)-\theta_0'(-\xi)+\frac{\xi\theta_0'(-\xi)}{2}\right)\mathrm{erfc}(\frac{\xi}{2\sqrt{t}})d\xi\right]. \end{aligned}$$

In turn (2.3.33) can be rewritten as

$$\frac{1}{4}\int_0^t g(r)dr-(t+1)g(t) = -\theta_0(0)-\frac{1}{2}\Bigg[\int_0^\infty \Bigg(\theta_0'(\xi)-\theta_0'(-\xi)+\frac{\xi\theta_0'(-\xi)}{2}\Bigg)\operatorname{erfc}(\frac{\xi}{2\sqrt{t}})d\xi\Bigg].$$

On multiplication by $\frac{-1}{(t+1)^{\frac{5}{4}}}$ to the above equation

$$\frac{1}{4(t+1)^{\frac{5}{4}}}\int_0^t g(r)dr + \frac{1}{(t+1)^{\frac{1}{4}}}g(t) \\ = \frac{\theta_0(0)}{(t+1)^{\frac{5}{4}}} + \frac{1}{2(t+1)^{\frac{5}{4}}}\Bigg[\int_0^\infty \Bigg(\theta_0'(\xi)-\theta_0'(-\xi)+\frac{\xi\theta_0'(-\xi)}{2}\Bigg)\operatorname{erfc}(\frac{\xi}{2\sqrt{t}})d\xi\Bigg].$$

Combining the left side terms of the above equation leads to get

$$\frac{\partial}{\partial t}\Bigg(\frac{1}{(t+1)^{\frac{1}{4}}}\int_0^t g(r)dr\Bigg) = \frac{\theta_0(0)}{(t+1)^{\frac{5}{4}}} + \\ \frac{1}{2(t+1)^{\frac{5}{4}}}\Bigg[\int_0^\infty \Bigg(\theta_0'(\xi)-\theta_0'(-\xi)+\frac{\xi\theta_0'(-\xi)}{2}\Bigg)\operatorname{erfc}(\frac{\xi}{2\sqrt{t}})d\xi\Bigg].$$

Integrating with respect to t:

$$\frac{1}{(t+1)^{\frac{1}{4}}}\int_0^t g(r)dr = \theta_0(0)\int_0^t \frac{1}{(y+1)^{\frac{5}{4}}}\,dy + \\ \frac{1}{2}\int_0^t \frac{1}{(y+1)^{\frac{5}{4}}}\int_0^\infty \Bigg(\theta_0'(\xi)-\theta_0'(-\xi)+\frac{\xi\theta_0'(-\xi)}{2}\Bigg)\operatorname{erfc}(\frac{\xi}{2\sqrt{y}})\,d\xi\,dy.$$

Simplifying the first term in the right side of above equation

$$\frac{1}{(t+1)^{\frac{1}{4}}}\int_0^t g(r)dr = 4\theta_0(0)\Bigg(1-\frac{1}{(t+1)^{\frac{1}{4}}}\Bigg) + \\ \frac{1}{2}\int_0^t \frac{1}{(y+1)^{\frac{5}{4}}}\int_0^\infty \Bigg(\theta_0'(\xi)-\theta_0'(-\xi)+\frac{\xi\theta_0'(-\xi)}{2}\Bigg)\operatorname{erfc}(\frac{\xi}{2\sqrt{y}})d\xi dy.$$

Multiply by $(t+1)^{\frac{1}{4}}$:

$$\int_0^t g(r)dr = 4\theta_0(0)\big((t+1)^{\frac{1}{4}}-1\big) + \\ \frac{(t+1)^{\frac{1}{4}}}{2}\int_0^t \frac{1}{(y+1)^{\frac{5}{4}}}\int_0^\infty \Bigg(\theta_0'(\xi)-\theta_0'(-\xi)+\frac{\xi\theta_0'(-\xi)}{2}\Bigg)\operatorname{erfc}(\frac{\xi}{2\sqrt{y}})d\xi dy.$$

Differentiating w r to $'t'$

$$g(t) = \frac{\theta_0(0)}{(t+1)^{\frac{3}{4}}} \\ + \frac{1}{8(t+1)^{\frac{3}{4}}}\int_0^t \frac{1}{(y+1)^{\frac{5}{4}}}\int_0^\infty \Bigg(\theta_0'(\xi)-\theta_0'(-\xi)+\frac{\xi\theta_0'(-\xi)}{2}\Bigg)\operatorname{erfc}(\frac{\xi}{2\sqrt{y}})d\xi dy \\ + \frac{1}{2(t+1)}\int_0^\infty \Bigg(\theta_0'(\xi)-\theta_0'(-\xi)+\frac{\xi\theta_0'(-\xi)}{2}\Bigg)\operatorname{erfc}(\frac{\xi}{2\sqrt{t}})d\xi.$$

i.e.,

$$g(t) = \frac{\theta_0(0)}{(t+1)^{\frac{3}{4}}} + \frac{\varphi(t)}{2(t+1)} + \frac{1}{8(t+1)^{\frac{3}{4}}} \int_0^t \frac{\varphi(y)}{(y+1)^{\frac{5}{4}}} dy, \tag{2.3.34}$$

with

$$\varphi(y) = \int_0^\infty \left(\theta_0'(\xi) - \theta_0'(-\xi) + \frac{\xi\theta_0'(-\xi)}{2} \right) \operatorname{erfc}(\frac{\xi}{2\sqrt{t}}) d\xi \tag{2.3.35}$$

is the expression for the unique boundary function $g(t)$ obtained from the method of Laplace transformation.

Note that, from the definition of $\theta(x,t)$, for $u_0(x) = 0$ we have $\theta_0(x) \equiv 1$ and with simple calculation we see that (2.3.35) turns out to be

$$\varphi(y) = \int_0^\infty \frac{\xi}{2} \operatorname{erfc}(\frac{\xi}{2\sqrt{y}}) d\xi dy = y.$$

Hence from (2.3.34) we get

$$\begin{aligned} g(t) &= (t+1)^{\frac{-3}{4}} + \frac{(t+1)^{\frac{-3}{4}}}{8} \int_0^t \frac{y}{(y+1)^{\frac{5}{4}}} dy + \frac{t}{4(t+1)} \\ &= (t+1)^{\frac{-3}{4}} + \frac{(t+1)^{\frac{-3}{4}}}{16} \left[\frac{4(t+1)^{\frac{3}{4}}}{3} + \frac{4}{(t+1)^{\frac{1}{4}}} - \frac{16}{3} \right] + \frac{t}{4(t+1)} \\ &= \frac{2}{3(t+1)^{\frac{3}{4}}} + \frac{1}{3}. \end{aligned}$$

2.3.2 Expression for g(t) with integrating factor method

On rearrangement of (2.2.27), we have the Abel's integral equation of first kind:

$$\frac{1}{\sqrt{\pi}} \int_0^t \frac{3t - \tau + 2}{\sqrt{t-\tau}} h(\tau) d\tau = 2(1+t)F(t), \quad \text{for all} \quad t > 0,$$

where

$$h(\tau) = g'(\tau) \quad \text{and} \quad F(t) = \frac{1}{4\sqrt{\pi}} \int_0^\infty \xi \left[\frac{\theta_0(\xi)}{1+t} + \theta_0(-\xi) \right] \frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}} d\xi - \frac{g(0)(3t+2)}{2\sqrt{\pi t}(1+t)}.$$

Rearranging the kernel we can write

$$\frac{1}{\sqrt{\pi}} \left[3 \int_0^t \sqrt{t-\tau} h(\tau) d\tau + \int_0^t \frac{2\tau + 2}{\sqrt{t-\tau}} h(\tau) d\tau \right] = 2(1+t)F(t),$$

which can also be expressed as

$$3 \int_0^t \sqrt{t-\tau} \frac{d}{d\tau} \left(\int_0^\tau h(s) ds \right) d\tau + \int_0^t \frac{2\tau + 2}{\sqrt{t-\tau}} h(\tau) d\tau = 2\sqrt{\pi}(1+t)F(t).$$

Integration by parts leads to get

$$\frac{3}{2}\int_0^t \frac{\int_0^\tau h(s)ds}{\sqrt{t-\tau}}d\tau + \int_0^t \frac{2\tau+2}{\sqrt{t-\tau}}h(\tau)d\tau = 2\sqrt{\pi}(1+t)F(t).$$

On simplification we get the classical Abel's integral equation:

$$\frac{1}{\sqrt{\pi}}\int_0^t \left[\frac{3}{2}\int_0^\tau h(s)ds + (2\tau+2)h(\tau)\right]\frac{1}{\sqrt{t-\tau}}d\tau = 2(1+t)F(t).$$

Multiplying both sides of above equation by $\frac{1}{\sqrt{\pi}\sqrt{y-t}}$, where $0 < t < y < \infty$ and then integrating from 0 to y, we obtain

$$\frac{1}{\pi}\int_0^y \frac{1}{\sqrt{y-t}}\int_0^t \left[\frac{3}{2}\int_0^\tau h(s)ds + (2\tau+2)h(\tau)\right]\frac{1}{\sqrt{t-\tau}}d\tau\, dt = \int_0^y \frac{2(1+t)F(t)}{\sqrt{\pi}\sqrt{y-t}}dt.$$

Changing the order of integration and rearranging the terms yields

$$\frac{1}{\pi}\int_0^y \left(\int_\tau^y \frac{1}{\sqrt{y-t}}\frac{1}{\sqrt{t-\tau}}dt\right)\left[\frac{3}{2}\int_0^\tau h(s)ds + (2\tau+2)h(\tau)\right]d\tau$$
$$= \frac{1}{\sqrt{\pi}}\int_0^y \frac{2(1+t)F(t)}{\sqrt{y-t}}dt.$$

Simplifying the integral in the left side of the above equation directs to

$$\int_0^y \left[\frac{3}{2}\int_0^\tau h(s)ds + (2\tau+2)h(\tau)\right]d\tau = \frac{1}{\sqrt{\pi}}\int_0^y \frac{2(1+t)F(t)}{\sqrt{y-t}}dt.$$

Differentiating with respect to t, the solution for the above classical Abel's integral equation is given by

$$\frac{3}{2}\int_0^t h(s)ds + (2t+2)h(t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t (t-\tau)^{-\frac{1}{2}}2(1+\tau)F(\tau)d\tau.$$

Substituting $h(\tau) = g'(\tau)$ we get

$$\frac{3}{2}g(t) - \frac{3}{2}g(0) + (2t+2)g'(t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{2(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau.$$

Dividing the above equation by $2(1+t)$ we get

$$g'(t) + \frac{3}{4(1+t)}g(t) = \frac{1}{2(1+t)}\left[\frac{3}{2}g(0) + \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{2(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau\right]. \quad (2.3.36)$$

Since $F(0) = 0$, we see that

$$\frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{2(1+t)F(t)}{\sqrt{t-\tau}}d\tau = \frac{2}{\sqrt{\pi}}\int_0^t \frac{\frac{d}{d\tau}((1+\tau)F(\tau))}{\sqrt{t-\tau}}d\tau.$$

Hence (2.3.36) leads to get

$$g'(t) + \frac{3}{4(1+t)}g(t) = \frac{1}{2(1+t)}\left[\frac{3}{2}g(0) + \frac{2}{\sqrt{\pi}}\int_0^t \frac{\frac{d}{d\tau}((1+\tau)F(\tau))}{\sqrt{t-\tau}}d\tau\right].$$

By Integrating Factor method, we get

$$\frac{d}{dt}\left((t+1)^{\frac{3}{4}}g(t)\right) = \frac{(t+1)^{\frac{3}{4}}}{2(t+1)}\left[\frac{3}{2}g(0) + \frac{2}{\sqrt{\pi}}\int_0^t \frac{\frac{d}{d\tau}((1+\tau)F(\tau))}{\sqrt{t-\tau}}d\tau\right]$$

Integrating from 0 to t, we obtain

$$(t+1)^{\frac{3}{4}}g(t) - g(0) = \frac{3g(0)}{4}\int_0^t \frac{ds}{(s+1)^{\frac{1}{4}}} + \int_0^t \frac{1}{\sqrt{\pi}(s+1)^{\frac{1}{4}}}\int_0^s \frac{\frac{d}{d\tau}((1+\tau)F(\tau))}{\sqrt{t-\tau}}d\tau ds$$

Simplifying the first term in the right side of the above equation, we obtain

$$g(t) = g(0) + \frac{1}{\sqrt{\pi}(t+1)^{\frac{3}{4}}}\int_0^t \frac{1}{(s+1)^{\frac{1}{4}}}\int_0^s \frac{\frac{d}{d\tau}((1+\tau)F(\tau))}{\sqrt{t-\tau}}\, d\tau\, ds$$

which is an expression for the unique boundary condition $g(t)$ satisfying (2.2.27). For $u_0(x) = 0$, we have $\theta_0(x) = 1$. Then the function $F(t)$ (2.2.28) is given by

$$F(t) = -\frac{\sqrt{t}}{\sqrt{\pi}(t+1)^{\frac{3}{4}}}.$$

On substitution in the above equation to obtain $g(t)$, we get

$$g(t) = \frac{2}{3(t+1)^{\frac{3}{4}}} + \frac{1}{3}.$$

2.3.3 Boundary function g(t) via Abel's integral equation

From (2.3.36) we have

$$g'(t) + \frac{3}{4(1+t)}g(t) = \frac{1}{2(1+t)}\left[\frac{3}{2}g(0) + \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{2(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau\right].$$

By Integrating Factor method, we get

$$\frac{d}{dt}\left((t+1)^{\frac{3}{4}}g(t)\right) = \frac{3\ g(0)}{4\ (t+1)^{\frac{1}{4}}} + \frac{1}{\sqrt{\pi}(t+1)^{\frac{1}{4}}}\frac{d}{dt}\int_0^t \frac{(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau. \qquad (2.3.37)$$

To simplify the above, we evaluate the second term in the right side of the equation. For the same, consider the expression of $F(t)$ given in (2.2.28):

$$F(\tau) = \frac{1}{4\sqrt{\pi}}\int_0^\infty \xi\left[\frac{\theta_0(\xi)}{1+\tau} + \theta_0(-\xi)\right]\frac{e^{\frac{-\xi^2}{4\tau}}}{\sqrt{\tau^3}}d\xi - \frac{g(0)(3\tau+2)}{2\sqrt{\pi\tau}(1+\tau)}.$$

On rearrangement of the above expression we obtain

$$\sqrt{\pi}(1+\tau)F(\tau) = \frac{1}{4}\int_0^\infty \xi\Big[\theta_0(\xi) + (1+\tau)\theta_0(-\xi)\Big]\frac{e^{\frac{-\xi^2}{4\tau}}}{\sqrt{\tau^3}}d\xi - \frac{g(0)(3\tau+2)}{2\sqrt{\tau}}.$$

Change of variable as $\eta = \frac{\xi}{2\sqrt{\tau}}$ the above equation changes to

$$\sqrt{\pi}(1+\tau)F(\tau) = \frac{1}{\sqrt{\tau}}\int_0^\infty \eta\Big[\theta_0(2\sqrt{\tau}\eta) + (1+\tau)\theta_0(-2\sqrt{\tau}\eta)\Big]e^{-\eta^2}d\eta - \frac{g(0)(3\tau+2)}{2\sqrt{\tau}}.$$

Multiplying by the term $\dfrac{1}{\sqrt{t-\tau}}$;

$$\frac{\sqrt{\pi}(1+\tau)F(\tau)}{\sqrt{t-\tau}} = \frac{1}{\sqrt{\tau}\sqrt{t-\tau}}\int_0^\infty \eta\Big[\theta_0(2\sqrt{\tau}\eta) + (1+\tau)\theta_0(-2\sqrt{\tau}\eta)\Big]e^{-\eta^2}d\eta - \frac{g(0)(3\tau+2)}{2\sqrt{\tau}\sqrt{t-\tau}}.$$

Integrating from 0 to t,

$$\sqrt{\pi}\int_0^t \frac{(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau = \int_0^t\int_0^\infty \frac{\eta\big[\theta_0(2\sqrt{\tau}\eta) + (1+\tau)\theta_0(-2\sqrt{\tau}\eta)\big]e^{-\eta^2}}{\sqrt{\tau}\sqrt{t-\tau}}d\eta d\tau - \frac{g(0)}{2}\left[3\int_0^t \frac{\tau}{\sqrt{\tau}\sqrt{t-\tau}}d\tau + 2\int_0^t \frac{1}{\sqrt{\tau}\sqrt{t-\tau}}d\tau\right].$$

$$\sqrt{\pi}\int_0^t \frac{(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau = \int_0^t\int_0^\infty \frac{\eta\big[\theta_0(2\sqrt{\tau}\eta) + (1+\tau)\theta_0(-2\sqrt{\tau}\eta)\big]e^{-\eta^2}}{\sqrt{\tau}\sqrt{t-\tau}}d\eta d\tau - \frac{g(0)}{2}\left[3\frac{\pi}{2}t + 2\pi\right].$$

For simplification purposes, let us represent the first term on the right side of the above equation as $B(t)$. That is, the above equation can be rewritten as

$$\sqrt{\pi}\int_0^t \frac{(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau = B(t) - \frac{3}{4}g(0)\,\pi\,t - g(0)\,\pi$$

where

$$B(t) = \int_0^t\int_0^\infty \frac{\eta\big[\theta_0(2\sqrt{\tau}\eta) + (1+\tau)\theta_0(-2\sqrt{\tau}\eta)\big]e^{-\eta^2}}{\sqrt{\tau}\sqrt{t-\tau}}d\eta\, d\tau.$$

To obtain the desired, differentiate the above equation with respect to the time variable t which reduce the equation into

$$\sqrt{\pi}\frac{d}{dt}\int_0^t \frac{(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau = B'(t) - \frac{3}{4}g(0)\pi.$$

Finally, to get the simplified expression for second term of (2.3.37), multiply by $\dfrac{1}{\sqrt{\pi}(t+1)^{\frac{1}{4}}}$ which in turn leads to get

$$\frac{\sqrt{\pi}}{\sqrt{\pi}(t+1)^{\frac{1}{4}}}\frac{d}{dt}\int_0^t \frac{(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau = \frac{B'(t)}{\sqrt{\pi}(t+1)^{\frac{1}{4}}} - \frac{3\ g(0)\pi}{4\sqrt{\pi}(t+1)^{\frac{1}{4}}}$$

$$\frac{1}{\sqrt{\pi}(t+1)^{\frac{1}{4}}}\frac{d}{dt}\int_0^t \frac{(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau = \frac{B'(t)}{\pi(t+1)^{\frac{1}{4}}} - \frac{3\ g(0)}{4(t+1)^{\frac{1}{4}}}. \tag{2.3.38}$$

Substituting (2.3.38) in (2.3.37) simplifies to

$$\frac{d}{dt}\left((t+1)^{\frac{3}{4}}g(t)\right) = \frac{3\ g(0)}{4\ (t+1)^{\frac{1}{4}}} + \frac{B'(t)}{\pi(t+1)^{\frac{1}{4}}} - \frac{3\ g(0)}{4(t+1)^{\frac{1}{4}}}.$$

i.e.,

$$\frac{d}{dt}\left((t+1)^{\frac{3}{4}}g(t)\right) = \frac{B'(t)}{\pi(t+1)^{\frac{1}{4}}}.$$

To obtain the equation representing desired, $g(t)$, integrate the above expression from *zero* to t followed by simplification using Fundamental theorem of calculus;

$$\begin{aligned}(t+1)^{\frac{3}{4}}g(t) - g(0) &= \frac{1}{\pi}\int_0^t \frac{B'(r)}{(r+1)^{\frac{1}{4}}}dr \\ &= \frac{1}{\pi}\left[\frac{B(t)}{(t+1)^{\frac{1}{4}}} - B(0) + \int_0^t \frac{B(r)}{4(r+1)^{\frac{5}{4}}}dr\right]\end{aligned}$$

Simplifying and reorganizing the above leads to get

$$g(t) = \frac{1}{(t+1)^{\frac{3}{4}}}\left[g(0) - \frac{B(0)}{\pi}\right] + \frac{B(t)}{\pi(t+1)} + \frac{1}{4\pi(t+1)^{\frac{3}{4}}}\int_0^t \frac{B(r)}{(r+1)^{\frac{5}{4}}}dr, \tag{2.3.39}$$

which is the explicit expression of the unique boundary function $g(t)$ where $B(t)$ is given by

$$B(t) = \int_0^t\int_0^\infty \frac{\eta\big[\theta_0(2\sqrt{\tau}\eta) + (1+\tau)\theta_0(-2\sqrt{\tau}\eta)\big]e^{-\eta^2}}{\sqrt{\tau}\sqrt{t-\tau}}d\eta\, d\tau.$$

For $u_0(x) = 0$, we have $\theta_0(x) = 1$. For this case, the function $B(t)$ is given by

$$\begin{aligned}B(t) &= \int_0^t\int_0^\infty \frac{\eta\big[1 + (1+\tau)\big]e^{-\eta^2}}{\sqrt{\tau}\sqrt{t-\tau}}d\eta\ d\tau \\ &= \int_0^t \frac{2+\tau}{\sqrt{\tau}\sqrt{t-\tau}}\int_0^\infty \eta e^{-\eta^2}d\eta\ d\tau \\ &= \frac{1}{2}\left[\int_0^t \frac{2}{\sqrt{\tau}\sqrt{t-\tau}}d\tau + \int_0^t \frac{\tau}{\sqrt{\tau}\sqrt{t-\tau}}d\tau\right] = \frac{\pi}{4}(t+4).\end{aligned}$$

Put $t = 0$ in above expression, we get the value $B(0) = \pi$. Substituting in (2.3.39) and for the choice of $\theta_0(0) = g(0)$ we get

$$\begin{aligned}
g(t) &= \frac{1}{(t+1)^{\frac{3}{4}}}\left[1 - \frac{\pi}{\pi}\right] + \frac{\pi(t+4)}{4\pi(t+1)} + \frac{1}{4\pi(t+1)^{\frac{3}{4}}}\int_0^t \frac{\pi(t+4)}{4(r+1)^{\frac{5}{4}}}dr \\
&= \frac{(t+4)}{4(t+1)} + \frac{1}{16\ (t+1)^{\frac{3}{4}}}\left[\frac{4\left[t + 8(t+1)^{\frac{1}{4}} - 8\right]}{3(t+1)^{\frac{1}{4}}}\right] \\
&= \frac{1}{3} + \frac{2}{3(t+1)^{\frac{3}{4}}}.
\end{aligned}$$

i.e,

$$g(t) = \frac{2}{3(t+1)^{\frac{3}{4}}} + \frac{1}{3}.$$

2.4 Positiveness of the solution $\theta(x, t)$

In this section, we study the strict positiveness of the solution $\theta(x, t)$ for the heat equation with discontinuous unit step function in the source term (2.2.14)-(2.2.15). This is done to validate the inverse Hopf-Cole transformation which leads to get the solution for non-homogeneous Burgers' equation with time dependent point source (2.2.11)-(2.2.12).

Theorem 2.4.1. *For the initial data $\theta_0 \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R})$, the Cauchy problem (2.2.14)-(2.2.15) admits a positive solution. i.e., $\theta(x, t) > 0$ for all $x \in \mathbb{R}$ and $t > 0$.*

Proof. From (2.2.26) we have

$$\frac{1}{4}\int_0^\infty \xi\left[\frac{\theta_0(\xi)}{1+t} + \theta_0(-\xi)\right]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - \frac{g(0)(3t+2)}{2\sqrt{t}(1+t)} = \int_0^t \frac{(3t-\tau+2)}{2(1+t)}\frac{g'(\tau)}{\sqrt{t-\tau}}d\tau.$$

Multiplying by $2(t + 1)$ and rearranging the right side of the above equation we possess

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \xi\left[\theta_0(\xi) + (1+t)\theta_0(-\xi)\right]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - g(0)\left[3\sqrt{t} + \frac{2}{\sqrt{t}}\right] \\
&\qquad = \int_0^t \sqrt{(t-\tau)}g'(\tau)d\tau + 2(t+1)\int_0^t \frac{g'(\tau)}{\sqrt{t-\tau}}d\tau. \quad (2.4.40)
\end{aligned}$$

In view of the fact that $u_0 \in L^1(\mathbb{R})$ and (2.2.13), we obtain $\theta_0(x) > 0, \quad \forall\, x \in \mathbb{R}$. Considering the maximum principle, it is enough to show that $\theta(0,t) = g(t) > 0, \forall\, t > 0$. For the contrary, assume that $\theta(0,t) = g(t) \leq 0$ for some $t > 0$. We have $g(0) = \theta_0(0) > 0$. This implies that there exists a point $a > 0$ which satisfies $g(t)$ and $g(\tau)$ is non-negative for $\tau \leq a$. For $t > a$, with our assumption, consider the right side integrals of (2.4.40) which simplifies to:

$$\begin{aligned}\int_0^a \sqrt{(t-\tau)}g'(\tau)d\tau &= g(a)\sqrt{t-a} - g(0)\sqrt{t} + \int_0^a \frac{g(\tau)}{2}\frac{1}{\sqrt{t-\tau}}d\tau \\ &= -g(0)\sqrt{t} + \int_0^a \frac{g(\tau)}{2}\frac{1}{\sqrt{t-\tau}}d\tau\end{aligned}$$

and

$$\begin{aligned}\int_0^t \frac{g'(\tau)}{\sqrt{t-\tau}}d\tau &= \frac{g(a)}{\sqrt{t-a}} - \frac{g(0)}{\sqrt{t}} - \int_0^a \frac{g(\tau)}{2}\frac{1}{(t-\tau)^{\frac{3}{2}}}d\tau \\ &= -\frac{g(0)}{\sqrt{t}} - \int_0^a \frac{g(\tau)}{2}\frac{1}{(t-\tau)^{\frac{3}{2}}}d\tau.\end{aligned}$$

Splitting the integral over $0 < a < t$ and substituting the above simplified expressions in (2.4.40), we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \xi\big[\theta_0(\xi) + (1+t)\theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - g(0)\Big[3\sqrt{t} + \frac{2}{\sqrt{t}}\Big] \\ &\quad = \int_0^a \sqrt{(t-\tau)}g'(\tau)d\tau + \int_a^t \sqrt{(t-\tau)}g'(\tau)d\tau + 2(t+1)\int_0^a \frac{g'(\tau)}{\sqrt{t-\tau}}d\tau \\ &\qquad + 2(t+1)\int_a^t \frac{g'(\tau)}{\sqrt{t-\tau}}d\tau.\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \xi\big[\theta_0(\xi) + (1+t)\theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - g(0)\Big[3\sqrt{t} + \frac{2}{\sqrt{t}}\Big] \\ &\quad = -g(0)\sqrt{t} + \int_0^a \frac{g(\tau)}{2\sqrt{t-\tau}}d\tau + \int_a^t \sqrt{(t-\tau)}g'(\tau)d\tau \\ &\qquad + 2(t+1)\Big[-\frac{g(0)}{\sqrt{t}} - \int_0^a \frac{g(\tau)}{2(t-\tau)^{\frac{3}{2}}}d\tau\Big] + 2(t+1)\int_a^t \frac{g'(\tau)}{\sqrt{t-\tau}}d\tau.\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \xi\big[\theta_0(\xi) + (1+t)\theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - g(0)\Big[3\sqrt{t} + \frac{2}{\sqrt{t}}\Big] \\ &\quad = -g(0)\sqrt{t} - \frac{g(0)(2+2t)}{\sqrt{t}} + \int_0^a \frac{g(\tau)}{2\sqrt{t-\tau}}d\tau + \int_a^t \sqrt{(t-\tau)}g'(\tau)d\tau \\ &\qquad - (t+1)\int_0^a \frac{g(\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau + 2(t+1)\int_a^t \frac{g'(\tau)}{\sqrt{t-\tau}}d\tau.\end{aligned}$$

Note that $-g(0)\left[\sqrt{t}+\frac{2(t+1)}{\sqrt{t}}\right]=-g(0)\left[3\sqrt{t}+\frac{2}{\sqrt{t}}\right]$ which simplifies to

$$\frac{1}{2}\int_0^\infty \xi\big[\theta_0(\xi)+(1+t)\theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi$$
$$=\int_0^a \frac{g(\tau)}{2\sqrt{t-\tau}}d\tau+\int_a^t \sqrt{(t-\tau)}g'(\tau)d\tau$$
$$+2(t+1)\int_a^t \frac{g'(\tau)}{\sqrt{t-\tau}}d\tau-(t+1)\int_0^a \frac{g(\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau.$$

$$\frac{1}{2}\int_0^\infty \xi\big[\theta_0(\xi)+(1+t)\theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi$$
$$=\int_0^a \frac{g(\tau)}{2\sqrt{t-\tau}}d\tau-\int_0^a \frac{(t+1)g(\tau)}{(t-\tau)\sqrt{t-\tau}}d\tau+\int_a^t \sqrt{(t-\tau)}g'(\tau)d\tau$$
$$+\int_a^t \frac{2(t+1)g'(\tau)}{\sqrt{t-\tau}}d\tau.$$

$$\frac{1}{2}\int_0^\infty \xi\big[\theta_0(\xi)+(1+t)\theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi$$
$$=\int_0^a\left(\frac{1}{2}-\frac{(t+1)}{t-\tau}\right)\frac{g(\tau)}{\sqrt{t-\tau}}d\tau+\int_a^t\left(\sqrt{(t-\tau)}+\frac{2(t+1)}{\sqrt{t-\tau}}\right)g'(\tau)d\tau.$$

$$\frac{1}{2}\int_0^\infty \xi\big[\theta_0(\xi)+(1+t)\theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi+\int_0^a \frac{2+t+\tau}{2(t-\tau)^{\frac{3}{2}}}\,g(\tau)d\tau=\int_a^t \frac{3t-\tau+2}{\sqrt{t-\tau}}g'(\tau)d\tau. \tag{2.4.41}$$

Using $0\le\tau\le a<t$, we can get $0<\dfrac{2+t}{t^{\frac{3}{2}}}\le\dfrac{2+t+\tau}{(t-\tau)^{\frac{3}{2}}}$ which in turn provides

$$\lim_{t\to a^+}\int_0^a \frac{2+t+\tau}{(t-\tau)^{\frac{3}{2}}}g(\tau)d\tau\ge\frac{2+a}{a^{\frac{3}{2}}}\int_0^a g(\tau)d\tau>0.$$

In fact,

$$0\le\tau\le a<t$$
$$0\ge-\tau\ge-a>-t$$
$$t\ge t-\tau\ge t-a>0$$
$$0<\frac{1}{t}\le\frac{1}{t-\tau}\le\frac{1}{t-a}$$

$$0 < \frac{1}{t^{\frac{3}{2}}} \leq \frac{1}{(t-\tau)^{\frac{3}{2}}}$$

$$0 < \frac{2+t+\tau}{t^{\frac{3}{2}}} \leq \frac{2+t+\tau}{(t-\tau)^{\frac{3}{2}}}$$

$$0 < \frac{2+t}{t^{\frac{3}{2}}} \leq \frac{2+t+\tau}{t^{\frac{3}{2}}} \leq \frac{2+t+\tau}{(t-\tau)^{\frac{3}{2}}}.$$

Multiplying by $g(\tau)$ and integrating over 0 to a, we obtain

$$0 < \frac{2+t}{t^{\frac{3}{2}}} \int_0^a g(\tau)d\tau \leq \int_0^a \frac{2+t+\tau}{(t-\tau)^{\frac{3}{2}}} g(\tau)d\tau$$

$$0 < \lim_{t\to a} \frac{2+t}{t^{\frac{3}{2}}} \int_0^a g(\tau)d\tau \leq \lim_{t\to a^+} \int_0^a \frac{2+t+\tau}{(t-\tau)^{\frac{3}{2}}} g(\tau)d\tau$$

$$\lim_{t\to a^+} \int_0^a \frac{2+t+\tau}{(t-\tau)^{\frac{3}{2}}} g(\tau)d\tau \geq \frac{2+a}{a^{\frac{3}{2}}} \int_0^a g(\tau)d\tau > 0.$$

It is clear that the first term of (2.4.41) admits a non-negative lower bound. i.e.,

$$\lim_{t\to a^+} \frac{1}{2} \int_0^\infty \xi \big[\theta_0(\xi) + (1+t)\theta_0(-\xi)\big] \frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}} d\xi \geq 0.$$

Hence, we get

$$\frac{1}{2} \int_0^\infty \xi \big[\theta_0(\xi)+(1+t)\theta_0(-\xi)\big] \frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}} d\xi + \int_0^a \frac{2+t+\tau}{2(t-\tau)^{\frac{3}{2}}} g(\tau)d\tau \geq \frac{2+a}{a^{\frac{3}{2}}} \int_0^a g(\tau)d\tau > 0.$$

i.e., it is proved that left side of (2.4.41) admits a positive lower bound as $t \to a$, whereas the right side vanishes as $t \to a$, which is a contradiction. □

2.5 Asymptotic analysis

In this section, we discover the large time behavior of the boundary function $g(t)$. Utilizing the same, asymptotic behavior of the solutions $R(x,t)$ and $L(x,t)$ are determined. It is interesting to see that the large time behavior of $g(t)$ will remain same as that of $g(t)$ concerned to the trivial initial data case of (2.2.12).
Since $u_0 \in L^1(\mathbb{R})$, we have the initial data (2.2.15) holds the property;

$$\theta_0(x) \to k \quad \text{as} \quad x \to \infty, \tag{2.5.42}$$

for some real constant k. It is easy to observe that $\theta_0(x) \to 1$ as $x \to -\infty$.

Lemma 2.5.1. *Let $g(t)$ be the boundary condition as in (2.3.39). Then with (2.5.42), we have*

$$\lim_{t\to\infty} g(t) = \frac{1}{3}. \tag{2.5.43}$$

Proof. From (2.3.39) we have the explicit expression for the boundary data given by

$$g(t) = \frac{1}{(t+1)^{\frac{3}{4}}}\left[g(0) - \frac{B(0)}{\pi}\right] + \frac{B(t)}{\pi(t+1)} + \frac{1}{4\pi(t+1)^{\frac{3}{4}}}\int_0^t \frac{B(r)}{(r+1)^{\frac{5}{4}}}\,dr$$

where

$$B(t) = \int_0^t \int_0^\infty \frac{\eta\left[\theta_0(2\sqrt{\tau}\eta) + (1+\tau)\theta_0(-2\sqrt{\tau}\eta)\right]e^{-\eta^2}}{\sqrt{\tau}\,\sqrt{t-\tau}}\,d\eta\,d\tau.$$

In the interest of simplification, let us substitute $\tau = \gamma t$ in the expression of $B(t)$ given above. But then we have

$$\begin{aligned}
B(t) &= \int_0^\infty \int_0^1 \frac{\eta\left[\theta_0(2\eta\sqrt{\gamma t}) + (1+\gamma t)\theta_0(-2\eta\sqrt{\gamma t})\right]e^{-\eta^2}}{\sqrt{\gamma t}\sqrt{t-\gamma t}}\; t\; d\gamma\; d\eta \\
&= \int_0^\infty \int_0^1 \frac{\eta\left[\theta_0(2\eta\sqrt{\gamma t}) + \theta_0(-2\eta\sqrt{\gamma t})\right]e^{-\eta^2}}{\sqrt{\gamma}\sqrt{1-\gamma}}\,d\gamma\,d\eta \\
&\qquad + \int_0^\infty \int_0^1 \frac{\eta\;\gamma\;t\;\theta_0(-2\eta\sqrt{\gamma t})e^{-\eta^2}}{\sqrt{\gamma}\sqrt{1-\gamma}}\;d\gamma\;d\eta \\
&=: I_1 + I_2, \quad \text{(say)},
\end{aligned}$$

where

$$I_1 = \int_0^\infty \int_0^1 \frac{\eta\left[\theta_0(2\eta\sqrt{\gamma t}) + \theta_0(-2\eta\sqrt{\gamma t})\right]e^{-\eta^2}}{\sqrt{\gamma}\sqrt{1-\gamma}}\,d\gamma\,d\eta$$

and

$$I_2 = \int_0^\infty \int_0^1 \frac{\eta\;\gamma\;t\;\theta_0(-2\eta\sqrt{\gamma t})e^{-\eta^2}}{\sqrt{\gamma}\sqrt{1-\gamma}}\;d\gamma\;d\eta.$$

Since u_0 is continuous and essentially bounded, there exists a real $M > 0$ such that $|\theta_0(2\eta\sqrt{\gamma t}) + \theta_0(-2\eta\sqrt{\gamma t})| \leq M$, $\forall\ 2\eta\sqrt{\gamma t} \in \mathbb{R}$. Further, one can see that $\dfrac{M\eta e^{-\eta^2}}{\sqrt{\gamma}\sqrt{1-\gamma}}$ is summable over $[0,\infty] \times [0,1]$. Thus, by dominated convergence theorem, the condition (2.5.42) yields

$$\begin{aligned}
\lim_{t\to\infty} I_1 &= \lim_{t\to\infty} \int_0^\infty \int_0^1 \frac{\eta\big[\theta_0(2\eta\sqrt{\gamma t}) + \theta_0(-2\eta\sqrt{\gamma t})\big] e^{-\eta^2}}{\sqrt{\gamma}\sqrt{1-\gamma}}\, d\gamma\, d\eta \\
&= \int_0^\infty \eta e^{-\eta^2} \int_0^1 \frac{1}{\sqrt{\gamma}\sqrt{1-\gamma}} \lim_{t\to\infty} \big[\theta_0(2\eta\sqrt{\gamma t}) + \theta_0(-2\eta\sqrt{\gamma t})\big]\, d\gamma\, d\eta \\
&= \int_0^\infty \eta e^{-\eta^2} \int_0^1 \frac{k+1}{\sqrt{\gamma}\sqrt{1-\gamma}}\, d\gamma\, d\eta \\
&= \int_0^\infty \eta e^{-\eta^2} (k+1)\ \pi\ d\eta = \frac{(k+1)\pi}{2}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\lim_{t\to\infty} \frac{I_2}{t} &= \int_0^\infty \eta\ e^{-\eta^2} \int_0^1 \frac{\gamma}{\sqrt{\gamma}\sqrt{1-\gamma}} \lim_{t\to\infty} \theta_0(-2\eta\sqrt{\gamma t})\ d\gamma\ d\eta \\
&= \int_0^\infty \eta\ e^{-\eta^2} \int_0^1 \frac{\sqrt{\gamma}}{\sqrt{1-\gamma}} d\gamma\ d\eta \\
&= \frac{\pi}{2} \int_0^\infty \eta\ e^{-\eta^2} d\eta = \frac{\pi}{4}.
\end{aligned}$$

To make easier, we reshuffle the term $\dfrac{B(t)}{(1+t)}$ in terms of I_1 and I_2 as follows;

$$\begin{aligned}
\frac{B(t)}{(1+t)} &= \frac{I_1 + I_2}{(1+t)} \\
&= \frac{I_1}{(1+t)} + \frac{I_2}{(1+t)} = \frac{I_1}{(1+t)} + \frac{I_2}{t}\frac{t}{(1+t)} \\
&= \frac{I_1}{(1+t)} + \frac{I_2}{t}\frac{1}{(1+\frac{1}{t})}.
\end{aligned}$$

Then by using the above values as t approaches ∞ of I_1 and I_2 we obtain

$$\lim_{t\to\infty} \frac{B(t)}{(1+t)} = \lim_{t\to\infty} \frac{I_1}{(1+t)} + \lim_{t\to\infty} \frac{I_2}{t}\frac{1}{(1+\frac{1}{t})} = 0 + \frac{\pi}{4} = \frac{\pi}{4}.$$

i.e,

$$\frac{1}{\pi} \lim_{t\to\infty} \frac{B(t)}{(1+t)} = \frac{1}{4}.$$

From L'Hospital's rule, one can easily examine that the third term in the right side of (2.3.39) approaches to $\dfrac{1}{12}$ as t advances to ∞. i.e.,

$$\lim_{t\to\infty} \frac{1}{4\pi(t+1)^{\frac{3}{4}}} \int_0^t \frac{B(r)}{(r+1)^{\frac{5}{4}}} dr = \lim_{t\to\infty} \frac{1}{3\pi}\frac{B(t)}{(1+t)} = \frac{1}{12}.$$

Hence, from (2.3.39), substituting the estimates we arrive at the desired. □

Theorem 2.5.2. *$xR(x,t)$ and $xR_x(x,t)$ are uniformly convergent on compact sets and the limits are*

$$\lim_{t\to\infty} x\,R(x,t) = \frac{x}{3} \qquad \textit{and} \qquad \lim_{t\to\infty} x\,R_x(x,t) = 0.$$

Proof. By (2.2.20) we have

$$R(x,t) = \frac{1}{1+t}\Bigg[\frac{1}{2\sqrt{\pi t}}\int_0^\infty \theta_0(\xi)\big[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}}\big]d\xi + \int_0^t \big(g(\tau)(1+\tau)\big)' \operatorname{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right)d\tau + g(0)\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)\Bigg].$$

By changing the variable accordingly for the integrals in first term and applying integral by parts to the second and third term we obtain

$$\begin{aligned} R(x,t) &= \frac{1}{1+t}\left[\frac{1}{\sqrt{\pi}}\int_{\frac{-x}{2\sqrt{t}}}^\infty \theta_0(2\sqrt{t}\eta + x)e^{-\eta^2}d\eta - \frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^\infty \theta_0(2\sqrt{t}\eta - x)e^{-\eta^2}d\eta\right] \\ &+ \frac{x}{2\sqrt{\pi}(1+t)}\int_0^t \frac{g(\tau)(1+\tau)e^{\frac{-x^2}{4(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}d\tau \\ &=: I_1 + I_2 + I_3 \quad \text{(say)}. \end{aligned} \tag{2.5.44}$$

Clearly the first term $I_1 + I_2$ vanishes as $t \longrightarrow \infty$ and hence ignore it. Then we consider the term I_3 given by

$$I_3 = \frac{x}{2\sqrt{\pi}(1+t)}\int_0^t \frac{g(\tau)(1+\tau)e^{\frac{-x^2}{4(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}d\tau.$$

Changing the variable by $y = t - \tau$, we simplify the above integral to get

$$\begin{aligned} I_3 &= \frac{x}{2\sqrt{\pi}}\int_0^t g(t-y)\left[\frac{1+t-y}{1+t}\right]\frac{e^{\frac{-x^2}{4y}}}{y^{\frac{3}{2}}}dy \\ &= \frac{x}{2\sqrt{\pi}}\int_0^t g(t-y)\frac{e^{\frac{-x^2}{4y}}}{y^{\frac{3}{2}}}dy - \frac{x}{2\sqrt{\pi}(1+t)}\int_0^t g(t-y)\frac{e^{\frac{-x^2}{4y}}}{\sqrt{y}}dy \\ &= I_3' + I_3'' \quad \text{(say)}. \end{aligned}$$

From (2.5.43) we have the boundary function $g(t)$ converges to $\frac{1}{3}$ for large time t. Changing the variable to $\eta = \frac{x}{2\sqrt{y}}$ in I_3' of the above equation leads to

$$\begin{aligned} I_3' &= \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^\infty g\left(t - \frac{x^2}{4\eta^2}\right)e^{-\eta^2}d\eta \\ &= \frac{2}{\sqrt{\pi}}\int_{-\infty}^\infty \chi_{[\frac{x}{2\sqrt{t}},\infty)}(\eta)\; g\left(t - \frac{x^2}{4\eta^2}\right)e^{-\eta^2}d\eta. \end{aligned}$$

Observe that

$$\left| \frac{2}{\sqrt{\pi}} \chi_{[\frac{x}{2\sqrt{t}},\infty)}(\eta)\ g\left(t - \frac{x^2}{4\eta^2}\right) e^{-\eta^2} \right| \leq c_0 \cdot e^{-\eta^2}$$

for some constant c_0. Also,

$$\lim_{t\to\infty} \frac{2}{\sqrt{\pi}} \chi_{[\frac{x}{2\sqrt{t}},\infty)}(\eta)\ g\left(t - \frac{x^2}{4\eta^2}\right) e^{-\eta^2} = \begin{cases} \frac{2}{3\sqrt{\pi}} e^{-\eta^2}, & 0 \leq \eta < \infty \\ 0, & -\infty < \eta < 0. \end{cases}$$

Hence by Dominated convergence theorem, we get

$$\lim_{t\to\infty} I_3' = \frac{2}{3\sqrt{\pi}} \int_0^\infty e^{-\eta^2} d\eta = \frac{2}{3\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \frac{1}{3}.$$

Clearly the numerator of I_3'' tends to ∞ for large t. Hence by using L'Hospital's rule

$$\lim_{t\to\infty} I_3'' = \frac{-x}{2\sqrt{\pi}} g(0) \lim_{t\to\infty} \frac{e^{\frac{-x^2}{4t}}}{\sqrt{t}} = 0$$

From (2.5.44) we conclude that

$$\lim_{t\to\infty} R(x,t) = \frac{1}{3}.$$

Differentiating (2.5.44) with respect to the space variable x we get

$$\begin{aligned} R_x(x,t) &= \frac{1}{(1+t)} \left[\frac{1}{4\sqrt{\pi} t^{\frac{3}{2}}} \int_0^\infty \theta_0(\xi) \big[(\xi - x) e^{\frac{-(\xi-x)^2}{4t}} + (\xi + x) e^{\frac{-(\xi+x)^2}{4t}} \big] d\xi \right] \\ &\quad + \frac{1}{2\sqrt{\pi}(1+t)} \int_0^t \frac{g(t-\tau)(1+t-\tau) e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}} \left[1 - \frac{x^2}{2\tau} \right] d\tau \\ &=: J_1 + J_2 + J_3 \quad \text{(say)}. \end{aligned}$$

Consider the first term of the above equation

$$J_1 = \frac{1}{(1+t)} \frac{1}{4\sqrt{\pi} t^{\frac{3}{2}}} \int_0^\infty \theta_0(\xi)(\xi - x) e^{\frac{-(\xi-x)^2}{4t}} d\xi$$

and then changing the variable to $\eta = \frac{\xi - x}{2\sqrt{t}}$, J_1 can be expressed as

$$J_1 = \frac{1}{(1+t)} \frac{1}{\sqrt{\pi t}} \int_{\frac{-x}{2\sqrt{t}}}^\infty \theta_0(2\sqrt{t}\eta + x) \eta e^{-\eta^2} d\xi.$$

From the above expression of J_1, clearly we can say that $J_1 \longrightarrow 0$ as $t \longrightarrow \infty$. In similar way we get $J_2 \longrightarrow 0$ as $t \longrightarrow \infty$. Now consider

$$\begin{aligned}
J_3 &= \frac{1}{2\sqrt{\pi}} \int_0^t \frac{g(t-\tau)}{\tau^{\frac{3}{2}}} \left[\frac{1+t-\tau}{1+t}\right] e^{\frac{-x^2}{4\tau}} \left[1 - \frac{x^2}{2\tau}\right] d\tau \\
&= \frac{1}{2\sqrt{\pi}} \int_0^t \frac{g(t-\tau)}{\tau^{\frac{3}{2}}} \left[\frac{1+t-\tau}{1+t}\right] e^{\frac{-x^2}{4\tau}} d\tau - \frac{x^2}{4\sqrt{\pi}} \int_0^t g(t-\tau) \left[\frac{1+t-\tau}{1+t}\right] \frac{e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{5}{2}}} d\tau \\
&= J_3' + J_3'' \quad \text{(say)}.
\end{aligned}$$

The first term of the above equation can be expressed in terms of I_3 and then using the asymptotic behavior we get

$$\begin{aligned}
J_3' &= \frac{1}{2\sqrt{\pi}} \int_0^t \frac{g(t-\tau)}{\tau^{\frac{3}{2}}} \left[\frac{1+t-\tau}{1+t}\right] e^{\frac{-x^2}{4\tau}} d\tau \\
&= \frac{1}{2\sqrt{\pi}(1+t)} \int_0^t \frac{g(\tau)(1+\tau) e^{\frac{-x^2}{4(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} d\tau \\
&= \frac{1}{x} \left[\frac{x}{2\sqrt{\pi}(1+t)} \int_0^t \frac{g(\tau)(1+\tau) e^{\frac{-x^2}{4(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} d\tau\right] \\
&= \frac{I_3}{x}
\end{aligned}$$

$$\lim_{t\to\infty} J_3' = \lim_{t\to\infty} \frac{I_3}{x} = \frac{1}{3x}.$$

Now, the second term J_3'' can be split as follows:

$$\begin{aligned}
J_3'' &= \frac{-x^2}{4\sqrt{\pi}} \int_0^t g(t-\tau) \left[\frac{1+t-\tau}{1+t}\right] \frac{e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{5}{2}}} d\tau \\
&= \frac{-x^2}{4\sqrt{\pi}} \int_0^t \frac{g(t-\tau) e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{5}{2}}} d\tau + \frac{x^2}{4\sqrt{\pi}(1+t)} \int_0^t g(t-\tau) \frac{e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}} d\tau \\
&= M_1 + M_2 \quad \text{(say)}.
\end{aligned}$$

Consider the first term of J_3'' as in the above equation:

$$\begin{aligned}
M_1 &= \frac{-x^2}{4\sqrt{\pi}} \int_0^t \frac{g(t-\tau) e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{5}{2}}} d\tau \\
&= \frac{-x}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{t}}}^{\infty} g\left(t - \frac{x^2}{4\eta^2}\right) \frac{4\eta^2}{x^2} e^{-\eta^2} d\eta \quad \text{(By changing variables)} \\
&= \frac{-4}{x\sqrt{\pi}} \int_{\frac{x}{2\sqrt{t}}}^{\infty} g\left(t - \frac{x^2}{4\eta^2}\right) \eta^2 e^{-\eta^2} d\eta.
\end{aligned}$$

By Dominated convergence theorem,

$$\lim_{t\to\infty} M_1 = \frac{-4}{x\sqrt{\pi}}\int_0^\infty \frac{1}{3}\eta^2 e^{\eta^2}d\eta = \frac{-4}{x\sqrt{\pi}}\frac{1}{3}\frac{\sqrt{\pi}}{4} = -\frac{1}{3x}.$$

Now, the second term of J_3'' can be expressed in terms of I_3':

$$\begin{aligned} M_2 &= \frac{x^2}{4\sqrt{\pi}(1+t)}\int_0^t g(t-\tau)\frac{e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau \\ &= \frac{x}{2(1+t)}\left[\frac{x}{2\sqrt{\pi}}\int_0^t g(t-\tau)\frac{e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau\right] \\ &= \frac{x}{2(1+t)}I_3'. \end{aligned}$$

We know that $I_3' \longrightarrow \frac{1}{3}$ as $t \longrightarrow \infty$.Hence we get

$$\lim_{t\to\infty} M_2 = \frac{x}{2}\lim_{t\to\infty}\frac{I_3'}{(1+t)} = 0$$

Substituting large time asymptotic values of M_1 and M_2 in J_3'' we get

$$\lim_{t\to\infty} J_3'' = -\frac{1}{3x} + 0 = -\frac{1}{3x}.$$

Hence

$$\lim_{t\to\infty} J_3 = \frac{1}{3x} + \left(-\frac{1}{3x}\right) = 0$$

which in turn leads to get

$$\lim_{t\to\infty} R_x(x,t) = 0.$$

i.e., $R(x,t)$ and $R_x(x,t)$ are convergent. Now, with the help of above procedure we show that $xR(x,t)$ and $xR_x(x,t)$ are uniformly convergent on compact sets. First, we prove that $R(x,t)$ converges to $1/3$ uniformly on compact sets. Assume $0 \le x \le A$ for some $A > 0$. Let $g(t)$ be bounded by M. Integrating the second term in (2.2.20) by parts, we have

$$R(x,t) = \frac{1}{1+t}\left[\frac{1}{\sqrt{\pi}}\int_{\frac{-x}{2\sqrt{t}}}^\infty \theta_0(2\sqrt{t}\eta + x)e^{-\eta^2}d\eta - \frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^\infty \theta_0(2\sqrt{t}\eta - x)e^{-\eta^2}d\eta\right]$$
$$+\frac{x}{2\sqrt{\pi}(1+t)}\int_0^t \frac{g(t-\tau)(1+t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau. \quad (2.5.45)$$

It is seen that the first term vanishes uniformly as $t \to \infty$ and hence ignore it. Then consider

$$\left|R(x,t) - \frac{1}{3}\right| = \left|\frac{x}{2\sqrt{\pi}(1+t)}\int_0^t \frac{g(t-\tau)(1+t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau - \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{1}{3}e^{-\eta^2}d\eta\right|.$$

Expanding the second term of (2.5.45) we get

$$\frac{x}{2\sqrt{\pi}}\int_0^t g(t-\tau)\left[\frac{1+t-\tau}{1+t}\right]\frac{e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau = \frac{x}{2\sqrt{\pi}}\int_0^t \frac{g(t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau - \frac{x}{2\sqrt{\pi}}\int_0^t \frac{g(t-\tau)e^{\frac{-x^2}{4\tau}}}{(1+t)\sqrt{\tau}}d\tau$$

and then changing the variables by $\eta = \frac{x}{2\sqrt{\tau}}$ in the first term, we observe that

$$\begin{aligned}\frac{x}{2\sqrt{\pi}}\int_0^t \frac{g(t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau &= \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^\infty g\left(t-\frac{x^2}{4\eta^2}\right)e^{-\eta^2}d\eta\\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty g\left(t-\frac{x^2}{4\eta^2}\right)e^{-\eta^2}d\eta.\\ &\quad - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} g\left(t-\frac{x^2}{4\eta^2}\right)e^{-\eta^2}d\eta.\end{aligned}$$

But then the above expression turns out to be

$$\left|R(x,t)-\frac{1}{3}\right| = \left|\frac{2}{\sqrt{\pi}}\int_0^\infty g\left(t-\frac{x^2}{4\eta^2}\right)e^{-\eta^2}d\eta - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} g\left(t-\frac{x^2}{4\eta^2}\right)e^{-\eta^2}d\eta - \frac{x}{2\sqrt{\pi}}\int_0^t \frac{g(t-\tau)e^{\frac{-x^2}{4\tau}}}{(1+t)\sqrt{\tau}}d\tau - \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{1}{3}e^{-\eta^2}d\eta\right|$$

$$\left|R(x,t)-\frac{1}{3}\right| \leq \frac{2}{\sqrt{\pi}}\int_0^\infty \left|g\left(t-\frac{x^2}{4\eta^2}\right)-\frac{1}{3}\right|e^{-\eta^2}d\eta + \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} \left|g\left(t-\frac{x^2}{4\eta^2}\right)\right|e^{-\eta^2}d\eta + \frac{x}{2\sqrt{\pi}}\int_0^t \frac{|g(t-\tau)|\,e^{\frac{-x^2}{4\tau}}}{(1+t)\sqrt{\tau}}d\tau. \quad (2.5.46)$$

Note that the second and third term of the above equation admits a uniform bound and vanishes uniformly as t tends to ∞. i.e., one can obtain

$$\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} \left|g\left(t-\frac{x^2}{4\eta^2}\right)\right|e^{-\eta^2}d\eta \leq \frac{2\ M}{\sqrt{\pi}}\int_0^{\frac{A}{2\sqrt{t}}} e^{-\eta^2}d\eta \leq M\ \operatorname{erf}\left(\frac{A}{2\sqrt{t}}\right).$$

By the fact that $t \leq 1+t$ for all $t > 0$ and $g(t)$ is bounded, we can see that last term in (2.5.46) is uniformly bounded by

$$\frac{A\ M}{2\sqrt{\pi}}\int_0^t \frac{e^{\frac{-x^2}{4\tau}}}{t\ \sqrt{\tau}}d\tau \leq \frac{A\ M}{2\sqrt{\pi}}\int_0^t \frac{1}{t\ \sqrt{\tau}}d\tau = \frac{A\ M}{\sqrt{\pi}}\frac{1}{\sqrt{t}}.$$

Now, we consider the uniform convergence of $xR_x(x,t)$. We have

$$\begin{aligned}
xR_x(x,t) &= \frac{x}{(1+t)}\frac{1}{4\sqrt{\pi}t^{\frac{3}{2}}}\int_0^\infty \theta_0(\xi)(\xi-x)e^{\frac{-(\xi-x)^2}{4t}} \\
&+ \frac{x}{(1+t)}\frac{1}{4\sqrt{\pi}t^{\frac{3}{2}}}\int_0^\infty \theta_0(\xi)(\xi+x)e^{\frac{-(\xi+x)^2}{4t}}d\xi \\
&+ \frac{x}{2\sqrt{\pi}(1+t)}\int_0^t \frac{g(t-\tau)(1+t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}\left[1-\frac{x^2}{2\tau}\right]d\tau \\
&=: xJ_1 + xJ_2 + xJ_3.
\end{aligned}$$

By changing the variable $\eta = \frac{\xi-x}{2\sqrt{t}}$ in xJ_1 and $\eta = \frac{\xi+x}{2\sqrt{t}}$ in xJ_2, we can observe that both the terms vanishes uniformly. The third term xJ_3 can be expanded as

$$\begin{aligned}
xJ_3 &= \frac{x}{2\sqrt{\pi}}\int_0^t g(t-\tau)\left(\frac{1+t-\tau}{1+t}\right)\frac{e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau + \frac{-x^3}{4\sqrt{\pi}}\int_0^t \frac{g(t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{5}{2}}}d\tau \\
&+ \frac{x^3}{4\sqrt{\pi}}\int_0^t \frac{g(t-\tau)}{(1+t)}\frac{e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau \\
&=: M_1 + M_2 + M_3.
\end{aligned} \tag{2.5.47}$$

Then, the above expression can be modified as

$$|xJ_3| \le \left|M_1 - \frac{1}{3}\right| + \left|M_2 + M_3 + \frac{1}{3}\right| \le \left|R(x,t) - \frac{1}{3}\right| + \left|M_2 + \frac{1}{3}\right| + |M_3|. \tag{2.5.48}$$

Now by changing the variable for M_2, we get

$$\begin{aligned}
M_2 + \frac{1}{3} &= \frac{-4}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^\infty g\left(t - \frac{x^2}{4\eta^2}\right)\eta^2 e^{-\eta^2}d\eta + \frac{4}{\sqrt{\pi}}\int_0^\infty \frac{1}{3}\eta^2 e^{-\eta^2}d\eta \\
&= \frac{-4}{\sqrt{\pi}}\int_0^\infty \left[g\left(t-\frac{x^2}{4\eta^2}\right) - \frac{1}{3}\right]\eta^2 e^{-\eta^2}d\eta + \frac{4}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} g\left(t-\frac{x^2}{4\eta^2}\right)\eta^2 e^{-\eta^2}d\eta.
\end{aligned}$$

The last term on the right side of the above equation is uniformly bounded by

$$\frac{4M}{\sqrt{\pi}}\frac{A^2}{4t}\int_0^{\frac{x}{2\sqrt{t}}} e^{-\eta^2}d\eta \le 2M\frac{A^2}{4t}\,\mathrm{erf}\left(\frac{A}{2\sqrt{t}}\right),$$

which vanishes uniformly as $t \to \infty$. By similar calculations, one can obtain that the third term in (2.5.48) vanishes uniformly as $t \to \infty$. Hence, we conclude that $xR_x(x,t)$ approaches to *zero* uniformly as $t \to \infty$ when x is bounded. □

Theorem 2.5.3. *$xL(x,t)$ and $xL_x(x,t)$ are uniformly convergent on compact sets and the limits are*

$$\lim_{t\to\infty} xL(x,t) = \frac{x}{3} \qquad \textit{and} \qquad \lim_{t\to\infty} xL_x(x,t) = 0.$$

Proof. First, we prove that $L(x,t)$ converges to $1/3$ uniformly on compact sets. Let M_1 and M_2 be the bounds for $g(t)$ and $\left|g(t)-\frac{1}{3}\right|$ respectively for all $t>0$. Let $\epsilon>0$ be given and a positive number A such that $-A\le x\le 0$. Then there exist T_1, T_2 and T_3 such that

$$\left|g(t)-\frac{1}{3}\right| < \frac{\epsilon}{3}, \quad \forall\, t>T_1. \tag{2.5.49}$$

$$\operatorname{erf}\left(\frac{A}{2\sqrt{t}}\right) < \frac{\epsilon}{3M_1}, \quad \forall\, t>T_2. \tag{2.5.50}$$

$$\operatorname{erf}\left(\frac{A}{2\sqrt{t}}\right) < \frac{\epsilon}{3M_2} \quad \forall\, t>T_3. \tag{2.5.51}$$

Hence, $\left|g\left(t-\frac{x^2}{4\eta^2}\right)-\frac{1}{3}\right| < \frac{\epsilon}{3}$ whenever $\eta < \frac{-A}{2\sqrt{t-T_1}}$. Assume that $t\ge T_1+T_2+T_3$. Then

$$\left|L(x,t)-\frac{1}{3}\right| = \left|-\frac{x}{2\sqrt{\pi}}\int_0^t g(t-\tau)\frac{e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{3}{2}}}d\tau - \frac{2}{\sqrt{\pi}}\int_{-\infty}^0 \frac{1}{3}\, e^{-\eta^2}d\eta\right|. \tag{2.5.52}$$

Substituting $\eta=\frac{x}{2\sqrt{\tau}}$ in the first term of right side expression, (2.5.52) reduces to

$$\left|\frac{2}{\sqrt{\pi}}\left[\int_{-\infty}^0 g\left(t-\frac{x^2}{4\eta^2}\right)e^{-\eta^2}d\eta - \int_{\frac{-|x|}{2\sqrt{t}}}^0 g\left(t-\frac{x^2}{4\eta^2}\right)e^{-\eta^2}d\eta\right] - \frac{2}{\sqrt{\pi}}\int_{-\infty}^0 \frac{1}{3}\, e^{-\eta^2}d\eta\right|,$$

which is bounded by

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{-A}{2\sqrt{t-T_1}}} \left|g\left(t-\frac{x^2}{4\eta^2}\right)-\frac{1}{3}\right| e^{-\eta^2}d\eta + \frac{2}{\sqrt{\pi}}\int_{\frac{-A}{2\sqrt{t-T_1}}}^0 \left|g\left(t-\frac{x^2}{4\eta^2}\right)-\frac{1}{3}\right| e^{-\eta^2}d\eta$$
$$+\frac{2}{\sqrt{\pi}}\int_{\frac{-A}{2\sqrt{t}}}^0 \left|g\left(t-\frac{x^2}{4\eta^2}\right)\right| e^{-\eta^2}d\eta$$
$$\le \frac{2}{\sqrt{\pi}}\frac{\epsilon}{3}\int_{-\infty}^0 e^{-\eta^2}d\eta + \frac{2}{\sqrt{\pi}}M_2\int_0^{\frac{A}{2\sqrt{t-T_1}}} e^{-\eta^2}d\eta + \frac{2}{\sqrt{\pi}}M_1\int_0^{\frac{A}{2\sqrt{t}}} e^{-\eta^2}d\eta$$
$$\le \frac{\epsilon}{3} + M_2\operatorname{erf}\left(\frac{A}{2\sqrt{t-T_1}}\right) + M_1\operatorname{erf}\left(\frac{A}{2\sqrt{t}}\right) \le \epsilon.$$

Consider,

$$L_x(x,t) = \frac{-1}{4\sqrt{\pi}t^{\frac{3}{2}}}\int_0^\infty \theta_0(-\xi)(\xi-x)e^{\frac{-(\xi-x)^2}{4t}}d\xi$$
$$+\frac{-1}{4\sqrt{\pi}t^{\frac{3}{2}}}\int_0^\infty \theta_0(-\xi)(\xi+x)e^{\frac{-(\xi+x)^2}{4t}}d\xi$$
$$-\frac{1}{2\sqrt{\pi}}\int_0^t \frac{g(t-\tau)}{\tau^{\frac{3}{2}}}e^{\frac{-x^2}{4\tau}}\left[1-\frac{x^2}{2\tau}\right]d\tau =: P_1+P_2+P_3.$$

Observe that $|xP_1| \le AP_1$ vanishes uniformly as $t \to \infty$. Similarly, xP_2 vanishes uniformly as $t \to \infty$. Hence, it is enough to show that xP_3 converges to zero uniformly when x is bounded. Consider

$$\begin{aligned}
P_3 &= -\frac{1}{2\sqrt{\pi}} \int_0^t \frac{g(t-\tau)}{\tau^{\frac{3}{2}}} e^{\frac{-x^2}{4\tau}} \left[1 - \frac{x^2}{2\tau}\right] d\tau \\
&= \frac{2}{x\sqrt{\pi}} \int_{-\infty}^{\frac{-|x|}{2\sqrt{t}}} g\left(t - \frac{x^2}{4\eta^2}\right) e^{-\eta^2} d\eta - \frac{4}{x\sqrt{\pi}} \int_{-\infty}^{\frac{-|x|}{2\sqrt{t}}} g\left(t - \frac{x^2}{4\eta^2}\right) \eta^2 e^{-\eta^2} d\eta \\
&=: P_3' + P_3''.
\end{aligned}$$

Then,

$$|xP_3| \le \left| xP_3' - \frac{1}{3} \right| + \left| xP_3'' + \frac{1}{3} \right|. \tag{2.5.53}$$

We show that the terms in the right of the above expression (2.5.53) admit uniform bounds which vanish uniformly as t tends to infinity. Consider

$$\begin{aligned}
\left| xP_3' - \frac{1}{3} \right| &= \left| \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\frac{-|x|}{2\sqrt{t}}} g\left(t - \frac{x^2}{4\eta^2}\right) e^{-\eta^2} d\eta - \frac{1}{3} \right| \\
&\le \frac{2}{\sqrt{\pi}} \int_{-\infty}^{0} \left| g\left(t - \frac{x^2}{4\eta^2}\right) - \frac{1}{3} \right| e^{-\eta^2} d\eta + \frac{2}{\sqrt{\pi}} \int_{\frac{-|x|}{2\sqrt{t}}}^{0} \left| g\left(t - \frac{x^2}{4\eta^2}\right) \right| e^{-\eta^2} d\eta.
\end{aligned}$$

Observe that the second term of the above expression

$$\begin{aligned}
\frac{2}{\sqrt{\pi}} \int_{\frac{-|x|}{2\sqrt{t}}}^{0} \left| g\left(t - \frac{x^2}{4\eta^2}\right) \right| e^{-\eta^2} d\eta &\le \frac{2}{\sqrt{\pi}} \int_{\frac{-A}{2\sqrt{t}}}^{0} \left| g\left(t - \frac{x^2}{4\eta^2}\right) \right| e^{-\eta^2} d\eta \\
&\le \frac{2M_1}{\sqrt{\pi}} \int_0^{\frac{A}{2\sqrt{t}}} e^{-\eta^2} d\eta = M_1 \operatorname{erf}\left(\frac{A}{2\sqrt{t}}\right),
\end{aligned}$$

vanishes uniformly. The bound for the second term in right side of (2.5.53) is obtained as follows:

$$\begin{aligned}
\left| xP_3'' + \frac{1}{3} \right| &= \left| \frac{4}{\sqrt{\pi}} \int_{-\infty}^{\frac{-|x|}{2\sqrt{t}}} g\left(t - \frac{x^2}{4\eta^2}\right) \eta^2 e^{-\eta^2} d\eta - \frac{4}{\sqrt{\pi}} \int_{-\infty}^{0} \frac{1}{3} \eta^2 e^{-\eta^2} d\eta \right| \\
&= \frac{4}{\sqrt{\pi}} \left| \int_{-\infty}^{0} g\left(t - \frac{x^2}{4\eta^2}\right) \eta^2 e^{-\eta^2} d\eta - \int_{\frac{-|x|}{2\sqrt{t}}}^{0} g\left(t - \frac{x^2}{4\eta^2}\right) \eta^2 e^{-\eta^2} d\eta - \int_{-\infty}^{0} \frac{1}{3} \eta^2 e^{-\eta^2} d\eta \right| \\
&\le \frac{4}{\sqrt{\pi}} \left[\int_{-\infty}^{0} \left| g\left(t - \frac{x^2}{4\eta^2}\right) - \frac{1}{3} \right| \eta^2 e^{-\eta^2} d\eta + \int_{\frac{-|x|}{2\sqrt{t}}}^{0} \left| g\left(t - \frac{x^2}{4\eta^2}\right) \right| \eta^2 e^{-\eta^2} d\eta \right].
\end{aligned}$$

It is observed that the second term in the above expression vanishes uniformly and is bounded by

$$\frac{4M_1}{\sqrt{\pi}} \int_{\frac{-A}{2\sqrt{t}}}^{0} \eta^2 e^{-\eta^2} d\eta \leq \frac{4M_1}{\sqrt{\pi}} \frac{A^2}{4t} \int_{0}^{\frac{A}{2\sqrt{t}}} e^{-\eta^2} d\eta = \frac{M_1 A^2}{2t} \operatorname{erf}\left(\frac{A}{2\sqrt{t}}\right).$$

This completes the proof of the theorem. □

2.6 Global weak solutions

Definition 2.6.1. *A function $u(x,t)$ defined in $\mathbb{R} \times (0,\infty)$ is said to be a (global) weak solution if $u \in L^2(\mathbb{R} \times (0,\infty)) \cap W^{1,\infty}(\mathbb{R} \times (0,\infty))$ and u satisfies (2.2.11) in the sense of distributions. i.e.,*

$$\int_0^\infty \int_{\mathbb{R}} (u\phi_t + \frac{1}{2}u^2\phi_x - u_x\phi_x)\, dx\, dt + \int_0^\infty \frac{2}{1+t}\phi(0,t)\, dt - \int_{\mathbb{R}} u_0(x)\, \phi(x,0)\, dx = 0, \tag{2.6.54}$$

for all test functions $\phi \in C_0^\infty(\mathbb{R} \times [0,\infty))$.

We now show that the inverse Cole–Hopf transform of $\theta(x,t)$ is the unique weak solution of (2.2.14)-(2.2.15).

Theorem 2.6.2. *With the initial data u_0 and the solution $\theta(x,t)$ of (2.2.14)-(2.2.15), there exists a unique weak solution of (2.2.11)-(2.2.12) given by*

$$u(x,t) = -2\frac{\theta_x(x,t)}{\theta(x,t)}. \tag{2.6.55}$$

Moreover, the solution $u(x,t)$ is in $C^\infty(\mathbb{R} \setminus \{0\} \times (0,\infty))$.

Proof. **(Existence)** Let us prove the existence of a weak solution of (2.2.11)-(2.2.12). It is known that

$$\theta_{xx} = \theta_t + \frac{\theta}{(1+t)}, \quad \text{for} \quad x \geq 0.$$

Hence, we get

$$u_x(0^+,t) = \frac{-2}{\theta(0^+,t)}\left[\theta_t(0^+,t) + \frac{\theta(0^+,t)}{(1+t)} - \frac{\theta_x^2(0^+,t)}{\theta(0^+,t)}\right].$$

and

$$u_x(0^-,t) = \frac{2}{\theta(0^-,t)}\left[\theta_t(0^-,t) - \frac{\theta_x^2(0^-,t)}{\theta(0^-,t)}\right].$$

Thus using the above equations one can obtain

$$u_x(0^+,t) - u_x(0^-,t) + \frac{2}{(1+t)} = \frac{2}{g(t)}\Big[\theta_t(0^-,t) - \theta_t(0^+,t)\Big]. \qquad (2.6.56)$$

Let U and V be the domains of $R(x,t)$ and $L(x,t)$ respectively. Integrating $u\,\phi_t$ by parts gives

$$\iint_U u\,\phi_t\,dx\,dt = -\iint_U u_t\,\phi\,dx\,dt - \int_{\tau=0}^{\infty} u(\tau,0)\,\phi(\tau,0)\,d\tau,$$

and

$$\iint_V u\,\phi_t\,dx\,dt = -\iint_V u_t\,\phi\,dx\,dt - \int_{\tau=-\infty}^{0} u(\tau,0)\,\phi(\tau,0)\,d\tau.$$

Similarly, integrating $u_{xx}\phi$ by parts, we get

$$\iint_U u_{xx}\,\phi\,dx\,dt = -\iint_U u_x\,\phi_x\,dx\,dt - \int_{t=0}^{\infty} u_x(0^+,t)\,\phi(0,t)\,dt,$$

and

$$\iint_V u_{xx}\,\phi\,dx\,dt = -\iint_V u_x\,\phi_x\,dx\,dt + \int_{t=0}^{\infty} u_x(0^-,t)\,\phi(0,t)\,dt.$$

Further, integrating $\frac{u^2}{2}\phi_x$ by parts on U and V provides

$$\int_0^\infty \int_{\mathbb{R}} \left(\frac{u^2}{2}\right)\phi_x\,dx\,dt = -\int_0^\infty \int_{\mathbb{R}} \left(\frac{u^2}{2}\right)_x \phi\,dx\,dt.$$

Hence, for $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$, we obtain

$$\int_0^\infty \int_{\mathbb{R}} \left[u\phi_t + \frac{u^2}{2}\phi_x - u_x\phi_x\right] dx\,dt + \int_0^\infty \frac{2}{(1+t)}\phi(0,t)\,dt$$
$$+ \int_{\mathbb{R}} u_0(x)\phi(x,0)dx = -2\int_{t=0}^{\infty} \frac{\theta_t(0^+,t)-\theta_t(0^-,t)}{g(t)}\phi(0,t)dt. \qquad (2.6.57)$$

Dominated convergence theorem and then by integration by parts reduce right hand side expression of above equation to zero.

(Uniqueness) Let u and v be two solutions of (2.6.54). Take $e := u - v$. Then, we obtain

$$\int_0^\infty \int_{\mathbb{R}} \big(e\phi_t + \frac{1}{2}(u+v)e\phi_x - e_x\phi_x\big)dx\ dt = 0, \qquad (2.6.58)$$

for all test functions $\phi \in C_0^\infty(\mathbb{R}\times[0,\infty))$.

For fixed $T > 0$, we define $\phi = e(x,t)H(T-t)H(x)$ in the domain $\{0 \leq x < \infty, t > 0\}$ and $\phi = e(x,t)H(T-t)H_1(x)$, where $H_1(x) = 1 - H(x)$, in

$\{\infty < x < 0, t > 0\}$. Note that the defined function ϕ in both domains is not a test function. However, we can use this ϕ in (2.6.58) using usual approximation techniques as $C_0^\infty(\mathbb{R} \times [0,\infty))$ is dense in $H_0^1(\mathbb{R} \times [0,\infty))$.

For $0 \leq x < \infty$, the weak derivatives of ϕ with respect to t is $e_t(x,t) - e(x,t)\delta(t-T)$ and the weak derivative of ϕ with respect to x is $e_x(x,t)+e(x,t)\delta(x)$. Similarly for $-\infty < x < 0$, the weak derivatives of ϕ with respect to t is $e_t(x,t) - e(x,t)\delta(t-T)$ and the weak derivative of ϕ with respect to x is $e_x(x,t)-e(x,t)\delta(x)$. For $\phi = e(x,t)H(T-t)H_1(x)$, integral equation (2.6.58) turns out to be

$$\begin{aligned} \int_0^T \int_{-\infty}^0 -e\, e_t \, dx\, dt + \int_{-\infty}^0 e^2(x,T)\, dx \\ - \frac{1}{2}\Big[\int_0^T \int_{-\infty}^0 (u+v) e\, e_x \, dx\, dt - \int_0^T (e\,(u+v)e)(0,t)\, dt \Big] \\ + \int_0^T \int_{-\infty}^0 e_x^2 \, dx\, dt - \int_0^T (e_x e)(0,t)\, dt = 0. \quad (2.6.59) \end{aligned}$$

Similarly for $\phi = e(x,t)H(T-t)H(x)$, integral equation (2.6.58) yields

$$\begin{aligned} \int_0^T \int_0^\infty -e\, e_t \, dx\, dt + \int_0^\infty e^2(x,T)\, dx \\ - \frac{1}{2}\Big[\int_0^T \int_0^\infty (u+v) e\, e_x \, dx\, dt + \int_0^T (e\,(u+v)e)(0,t)\, dt \Big] \\ + \Big[\int_0^T \int_0^\infty e_x^2 \, dx\, dt + \int_0^T (e_x e)(0,t)\, dt \Big] = 0. \quad (2.6.60) \end{aligned}$$

Also using the fact that $e(x,0) = 0$, we deduce

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}} e\, e_t \, dx\, dt + \int_{\mathbb{R}} e^2(x,T) dx &= \frac{-1}{2} \int_{\mathbb{R}} \int_0^T \partial_t(e^2)\, dt\, dx + ||\, e(\cdot\,,T)\, ||^2 \\ &= \frac{1}{2}\, ||\, e(\cdot\,,T)\, ||_2^2 . \end{aligned}$$

Hence, equations (2.6.59)-(2.6.60) with above equation lead to

$$\begin{aligned} &\frac{1}{2}\, ||\, e(\cdot\,,T)\, ||_2^2 + \int_0^T \int_{\mathbb{R}} e_x^2 \, dx\, dt + \int_0^T \big[(e_x e)(0^+,t) - (e_x e)(0^-,t) \big] dt \\ &= \frac{1}{2}\Big[\int_0^T \int_{\mathbb{R}} (u+v) e\, e_x \, dx\, dt + \int_0^T \big[((u+v)e^2)(0^+,t) - ((u+v)e^2)(0^-,t) \big] dt \Big], \end{aligned}$$

which implies

$$
\begin{aligned}
||\, e(\cdot\,,T)\,||_2^2 + 2\int_0^T ||\, e_x(\cdot\,,t)\,||_2^2 \, dt &\leq \frac{1}{2}\int_0^T \int_{\mathbb{R}} ||(u+v)(\cdot\,,t)||_\infty |e(\cdot\,,t)||e_x(\cdot\,,t)| \, dx \, dt \\
&\leq \frac{1}{2}\int_0^T ||(u+v)(\cdot\,,t)||_\infty ||e||_2 ||e_x||_2 \, dt \\
&\leq \frac{1}{4}\int_0^T ||(u+v)(\cdot\,,t)||_\infty^2 ||e||_2^2 dt + \frac{1}{4}\int_0^T ||e_x||_2^2 dt \\
&\leq \frac{M_0}{4}\int_0^T ||\, e(\cdot\,,t)\,||_2^2 \, dt + 2\int_0^T ||e_x(\cdot\,,t)||_2^2 \, dt.
\end{aligned}
$$

Hence,

$$
||\, e(\cdot\,,T)\,||_2^2 \leq \frac{M_0}{4}\int_0^T ||\, e(\cdot\,,t)\,||_2^2 \, dt.
$$

Making use of Gronwall's inequality, we conclude that $e(x,T) = 0$ a.e. for all $T > 0$.

(Smoothness) Since $\theta(x,t)$ is positive solution of the heat equation for $x < 0$ and $(1+t)\theta(x,t)$ is also a positive solution of heat equation for $x > 0$, the solution $u(x,t)$ of the Cauchy problem (2.2.11)-(2.2.12) given in (2.6.55) is well-defined and is smooth on the domain $\mathbb{R} \setminus \{0\} \times (0,\infty)$. □

Theorem 2.6.3. *The unique weak solution $u(x,t)$ of (2.1.1)-(2.1.2) converges to zero uniformly on compact sets.*

Proof. Using the Theorem 2.2.11, the inverse Hopf-Cole transformation is well defined and given by (2.6.55). Hence, for $x > 0$, we have

$$
u(x,t) = -2\frac{x\,R_x(x,t)}{x\,R(x,t)} \to 0 \quad \text{as} \quad t \to \infty.
$$

Similarly, for $x < 0$, we obtain

$$
u(x,t) = -2\frac{x\,L_x(x,t)}{x\,L(x,t)} \to 0 \quad \text{as} \quad t \to \infty.
$$

In view of the uniform convergence of R, R_x, L and L_x, the unique weak solution $u(x,t)$ converges to zero uniformly on compact sets. □

Chapter 3

Riemann problem for a de-coupled system with source term

3.1 Introduction

The viscous Burgers' equation with source term :

$$u_t + uu_x = \nu u_{xx} + f(x,t), \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.1}$$

appears in while modeling several physical phenomena (Xu et al., 2007).
Hopf (1950) studied the vanishing viscosity behavior of solutions to the viscous Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.2}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.1.3}$$

with the assumption that the initial data u_0 satisfies $\int_0^x u_0(y)dy = o(x^2)$ for large $|x|$. And he proved that the solution of (3.1.2)-(3.1.3) converges to the weak solution of the concerned inviscid Burgers' equation as the viscosity $\nu \to 0$.

Joseph (1993) considered a system of conservation laws

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ v_t + (uv)_x = 0, & x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{3.1.4}$$

with the initial conditions

$$(u,v)^t(x,0) = \begin{cases} (u_L, v_L)^t & if\ x < 0, \\ (u_R, v_R)^t & if\ x > 0 \end{cases} \tag{3.1.5}$$

which is known as Riemann problem. He analyzed the solution of above problem by using vanishing viscosity method. For which he took an approximate solution $(u^\epsilon(x,t), v^\epsilon(x,t))$ of (3.1.4)-(3.1.5) which is defined by the Riemann problem

$$\begin{cases} u_t^\nu + \left(\frac{{u^\nu}^2}{2}\right)_x = \frac{1}{2}\nu u_{xx}^\nu, & x \in \mathbb{R}, \quad t > 0, \\ v_t^\nu + (uv)_x^\nu = \frac{1}{2}\nu v_{xx}^\nu, & x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{3.1.6}$$

with the initial conditions

$$(u^\nu, v^\nu)^t(x,0) = \begin{cases} (u_L^\nu, v_L^\nu)^t & if\ x < 0, \\ (u_R^\nu, v_R^\nu)^t & if\ x > 0. \end{cases} \tag{3.1.7}$$

He proved that the solution so obtained for the above Riemann problem will give the solution of (3.1.4)-(3.1.5) in the sense of distribution as $\nu \to 0$. The explicit solution for (3.1.4)-(3.1.5) given by Joseph (1993) is,

(i) $u_L > u_R$

$$(u^0(x,t), v^0(x,t)) = \begin{cases} (u_L, v_L) & \text{if } \ x < st, \\ (\frac{1}{2}(u_L + u_R), \frac{1}{2}(u_L - u_R)(v_L + v_R)t\delta_{x=st}) & \text{if } \ x = st, \\ (u_R, v_R) & \text{if } \ x > st, \end{cases} \tag{3.1.8}$$

where $s = \frac{1}{2}(u_L + u_R)$ and $\delta_{x=st}$ is the usual δ- measure concentrated along the line $x = st$.

(ii) $u_L < u_R$

$$(u^0(x,t), v^0(x,t)) = \begin{cases} (u_L, v_L) & \text{if } \ x < u_L t, \\ (x/t, 0) & \text{if } \ u_L t < x < u_R t, \\ (u_R, v_R) & \text{if } \ x > u_R t, \end{cases} \tag{3.1.9}$$

(iii) $u_L = u_R = \overline{u}$

$$(u^0(x,t), v^0(x,t)) = \begin{cases} (\overline{u}, v_L) & \text{if } \ x < \overline{u} t, \\ (\overline{u}, v_R) & \text{if } \ x > \overline{u} t, \end{cases} \tag{3.1.10}$$

Ding and Ding (2003) showed that, for fixed (x,t), the solution of

$$u_t + uu_x = \nu u_{xx} + 4x, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.11}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.1.12}$$

converges to the weak solution of relevant inviscid forced Burgers' equation as $\nu \to 0$, assuming the initial data satisfies

$$u_0(x) = o(x), \quad |x| \to \infty.$$

In this chapter, we consider a Riemann problem for de-coupled system with general source term and obtain the explicit weak solution with the help of characteristic equations. The cases of shock waves and rarefaction waves are discussed.

3.2 Generalized inhomogeneous Riemann Problem

Consider the Riemann problem for a decoupled system

$$u_t + uu_x = h(t), \quad x \in \mathbb{R}, \quad t > 0, \tag{3.2.13}$$

$$\rho_t + (u\rho)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.2.14}$$

where $h(t) \in L^1_{loc}[0, \infty)$, with the initial condition

$$(u, \rho)^t(x, 0) = \begin{cases} (u_L, \rho_L)^t & \text{if } \ x < 0, \\ (u_R, \rho_R)^t & \text{if } \ x > 0. \end{cases} \tag{3.2.15}$$

Choosing $v(x,t)$ to be $v(x,t) = \int_0^x \rho(z,t)dz$, the system (3.2.13)-(3.2.14) with (3.2.15) reduces to

$$\begin{cases} u_t + uu_x = h(t), \\ v_t + uv_x = 0, \\ u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \\ v(x,0) = \int_0^x \rho(x,0)dx = \begin{cases} \rho_L x, & x < 0, \\ \rho_R x, & x > 0, \end{cases} \end{cases} \tag{3.2.16}$$

where

$$v_x = \rho, \qquad v_t = -u\rho.$$

The characteristic equations of first component in the above Riemann problem is given by

$$\frac{dx}{dt} = u(x(t),t) \qquad \text{and} \qquad \frac{d\,u(x(t),t)}{dt} = h(t). \tag{3.2.17}$$

Integrating the second equation in (3.2.17) over 0 to t:

$$\int_0^t \frac{d\,u(x(t),t)}{dt}\,dt = \int_0^t h(s)\,ds$$

simplifying

$$\begin{aligned} u(x(t),t) - u(x(0),0) &= \int_0^t h(s)\,ds \\ \text{which implies} \qquad u(x(t),t) &= u_0(x_0) + \int_0^t h(s)\,ds \end{aligned}$$

where $x(0) = x_0$. Substituting the above in the first equation of characteristic equations given in (3.2.17) we obtain

$$\frac{dx}{dt} = u_0(x_0) + \int_0^t h(s)\,ds.$$

On integration over 0 to t:

$$\int_0^t \frac{dx}{dt} = \int_0^t u_0(x_0)\,dt + \int_0^t \int_0^\tau h(s)\,ds\,d\tau.$$

which simplifies to

$$x(t) = x_0 + u_0(x_0)\,t + \int_0^t \int_0^\tau h(s)\,ds\,d\tau. \tag{3.2.18}$$

Case 1 : $u_L > u_R$

In this case, we will have shock originated from $(0,0)$ and satisfies the Rankine-Hugoniot condition given by

$$\begin{cases} \frac{dx}{dt} = \frac{1}{2}(u_L + u_R) + \int_0^t h(s)\,ds. \\ x(0) = 0. \end{cases}$$

Then the equation of shock wave is

$$x(t) - x(0) = \frac{1}{2}(u_L + u_R)\,t + \int_0^t \int_0^\tau h(s)\,ds\,d\tau.$$

i.e., with $x(0) = 0$,

$$x(t) = \frac{1}{2}(u_L + u_R)\,t + \int_0^t \int_0^\tau h(s)\,ds\,d\tau := g(t).$$

Then the solution for the Riemann problem along curves is

$$u(x,t) = \begin{cases} u_L + \int_0^t h(s)\,ds, & \text{if} \quad x < g(t), \\ \frac{1}{2}(u_L + u_R) + \int_0^t h(s)\,ds, & \text{if} \quad x = g(t), \\ u_R + \int_0^t h(s)\,ds, & \text{if} \quad x > g(t). \end{cases} \tag{3.2.19}$$

We solve the Riemann problem for the second component (3.2.14) in decoupled system. If $x < g(t)$, then from (3.2.18) we have

$$x(t) = \xi_1 + u_L\,t + \int_0^t \int_0^\tau h(s)\,ds\,d\tau$$

and

$$u(x,t) = u_L + \int_0^t h(s)\,ds \quad \text{where} \quad x(0) = \xi_1.$$

Thus

$$v(x,t) = \rho_L\,\xi_1 = \rho_L\Big(x(t) - u_L\,t - \int_0^t \int_0^\tau h(s)\,ds\,d\tau\Big).$$

If $x > g(t)$, then from (3.2.18) we have

$$x(t) = \xi_2 + u_R\,t + \int_0^t \int_0^\tau h(s)\,ds\,d\tau$$

and

$$u(x,t) = u_R + \int_0^t h(s)\,ds \quad \text{where} \quad x(0) = \xi_2.$$

Thus

$$v(x,t) = \rho_R\,\xi_2 = \rho_R\bigg(x(t) - u_R\,t - \int_0^t\int_0^\tau h(s)\,ds\,d\tau\bigg).$$

Hence to find $\rho(x,t)$, we calculate the following distribution derivative. For simplification, let

$$\begin{aligned} e_L(t) &:= u_L t + \int_0^t\int_0^\tau h(s)\,ds\,d\tau, \\ e_R(t) &:= u_R t + \int_0^t\int_0^\tau h(s)\,ds\,d\tau. \end{aligned}$$

Consider

$$\begin{aligned}
\rho(x,t) &= T_{\frac{\partial v}{\partial x}}(\phi) \\
&= -T_v\left(\frac{\partial \phi}{\partial x}\right) \\
&= -\bigg[\int_0^\infty\int_{\mathbb{R}} v(x,t)\,\frac{\partial\phi}{\partial x}\,dx\,dt\bigg] \\
&= -\bigg[\int_0^\infty\int_{x<g(t)} v(x,t)\,\frac{\partial\phi}{\partial x}\,dx\,dt + \int_0^\infty\int_{x>g(t)} v(x,t)\,\frac{\partial\phi}{\partial x}\,dx\,dt\bigg] \\
&= -\bigg[\int_0^\infty\int_{x<g(t)} \rho_L(x-e_L(t))\,\frac{\partial\phi}{\partial x}\,dx\,dt + \int_0^\infty\int_{x>g(t)} \rho_R(x-e_R(t))\,\frac{\partial\phi}{\partial x}\,dx\,dt\bigg] \\
&= -\bigg[\int_0^\infty \bigg\{\rho_L(x-e_L(t))\,\phi(x,t)\bigg|_{-\infty}^{g(t)} - \int_{-\infty}^{g(t)}\phi(x,t)\rho_L\,dx\bigg\}\,dt \\
&\qquad + \int_0^\infty \bigg\{\rho_R(x-e_R(t))\,\phi(x,t)\bigg|_{g(t)}^{\infty} - \int_{g(t)}^{\infty}\phi(x,t)\rho_R\,dx\bigg\}\,dt\bigg] \\
&= \int_0^\infty\int_{-\infty}^{g(t)}\phi(x,t)\rho_L\,dx\,dt - \int_0^\infty \rho_L(g(t)-e_L(t))\,\phi(g(t),t)\,dt \\
&\qquad + \int_0^\infty\int_{g(t)}^{\infty}\phi(x,t)\rho_R\,dx\,dt + \int_0^\infty \rho_R(g(t)-e_R(t))\,\phi(g(t),t)\,dt \\
&= \int_0^\infty\int_{-\infty}^{g(t)}\phi(x,t)\rho_L\,dx\,dt + \int_0^\infty\int_{g(t)}^{\infty}\phi(x,t)\rho_R\,dx\,dt \\
&\qquad + \int_0^\infty \big[\rho_R\,(g(t)-e_R(t)) - \rho_L\,(g(t)-e_L(t))\big]\,\phi(g(t),t)\,dt.
\end{aligned}$$

For every $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty)$,

$$\rho(x,t) = \begin{cases} \rho_L, & \text{if} \quad x < g(t), \\ \rho_R\,[g(t)-e_R(t)] - \rho_L\,[g(t)-e_L(t)], & \text{if} \quad x = g(t), \\ \rho_R, & \text{if} \quad x > g(t). \end{cases} \tag{3.2.20}$$

We now show that u satisfies the first component equation (3.2.13) of decoupled system in the weak sense.

For every $\phi(x,t) \in C_c^\infty(\mathbb{R} \times [0,\infty))$, consider

$$\int_0^\infty \int_{-\infty}^\infty \left(u\phi_t + \frac{u^2}{2}\phi_x \right) dxdt = \int_0^\infty \int_{-\infty}^{g(t)} \left(u\phi_t + \frac{u^2}{2}\phi_x \right) dxdt + \int_0^\infty \int_{g(t)}^\infty \left(u\phi_t + \frac{u^2}{2}\phi_x \right) dxdt$$

$$= -\int_{-\infty}^0 u_L \phi(x,0)dx - \int_0^\infty \int_{-\infty}^{g(t)} h(t)\phi(x,t)dxdt$$

$$- \int_0^\infty \left(u_L + \int_0^t h(s)ds \right) \phi(g(t),t)g^{'}(t)dt + \int_0^\infty \frac{1}{2} \left(u_L + \int_0^t h(s)ds \right)^2 \phi(g(t),t)dt,$$

$$- \int_0^\infty u_R \phi(x,0)dx - \int_0^\infty \int_{g(t)}^\infty h(t)\phi(x,t)dxdt$$

$$+ \int_0^\infty \left(u_R + \int_0^t h(s)ds \right) \phi(g(t),t)g^{'}(t)dt - \int_0^\infty \frac{1}{2} \left(u_R + \int_0^t h(s)ds \right)^2 \phi(g(t),t)dt.$$

$$= -\int_{-\infty}^0 u_L \phi(x,0)dx - \int_0^\infty u_R \phi(x,0)dx - \int_0^\infty \int_{-\infty}^\infty h(t)\phi(x,t)dxdt.$$

Thus u in (3.2.19) is a weak solution of (3.2.13). Similarly one can show that ρ in (3.2.20) satisfies the second component (3.2.14) of decoupled sytem in the weak sense.

<u>Case 2</u> : $u_L < u_R$

The charecteristic equations originated from $(x_0, 0)$ for the Riemann problem for inviscid Burgers' equations are

$$x(t) = \begin{cases} x_0 + u_L\, t + \int_0^t \int_0^\tau h(s)\, ds\, d\tau & \text{if} \quad x_0 < 0, \\ x_0 + u_R\, t + \int_0^t \int_0^\tau h(s)\, ds\, d\tau & \text{if} \quad x_0 > 0, \end{cases}$$

and the solution along these curves respectively are given by

$$u(x,t) = \begin{cases} u_L + \int_0^t h(s)\, ds, & \text{if} \quad x < 0, \\ u_R + \int_0^t h(s)\, ds, & \text{if} \quad x > 0. \end{cases}$$

Let

$$f_L(t) := \int_0^t \left(u_L + \int_0^\tau h(s)\, ds \right) d\tau$$
$$f_R(t) := \int_0^t \left(u_R + \int_0^\tau h(s)\, ds \right) d\tau.$$

Then we observe that the region $\{f_L(t) < x < f_R(t)\}$ is not covered by the characteristic curves. We fill this region with a fan of characteristics. Therefore, the solution in this case is given by

$$u(x,t) = \begin{cases} u_L + \int_0^t h(s)\,ds, & \text{if} \quad x < f_L(t), \\ \frac{x}{t} + \frac{1}{t}\int_0^t sh(s)\,ds, & \text{if} \quad f_L(t) < x < f_R(t), \\ u_R + \int_0^t h(s)\,ds, & \text{if} \quad x > f_R(t). \end{cases}$$

and

$$\rho(x,t) = \begin{cases} \rho_L, & \text{if} \quad x < f_L(t), \\ 0, & \text{if} \quad f_L(t) < x < f_R(t), \\ \rho_R, & \text{if} \quad x > f_R(t). \end{cases}$$

Now we show that u is a weak solution , i.e., u satisfies the integral equation

$$\int_0^\infty \int_{-\infty}^\infty (u\phi_t + (\frac{u^2}{2})\phi_x)dxdt = -\int_{-\infty}^0 u_L\phi(x,0)dx - \int_0^\infty u_R\phi(x,0)dx \\ - \int_0^\infty \int_{-\infty}^\infty h(t)\phi(x,t)dxdt \quad (3.2.21)$$

for every $\phi(x,t) \in C_c^\infty(\mathbb{R} \times [0,\infty])$.

We prove that $u(x,t)$ given above is a weak solution of (3.2.13). For that we show that $u(x,t)$ satisfies (3.2.21).

Consider

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt &= \int_0^\infty \int_{-\infty}^{f_L(t)} (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt \\ &+ \int_0^\infty \int_{f_L(t)}^{f_R(t)} (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt \\ &+ \int_0^\infty \int_{f_R(t)}^{\infty} (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Then

$$
\begin{aligned}
I_1 = &\int_0^\infty \int_{-\infty}^{f_L(t)} \left[(u_L + \int_0^t h(s)\,ds)\right]\phi_t dxdt \\
&\qquad + \int_0^\infty \int_{-\infty}^{f_L(t)} \left(u_L + \int_0^t h(s)\,ds\right)^2 \phi_x dxdt \\
= &- \int_{-\infty}^0 u_L\phi(x,0)dx - \int_0^\infty \int_{-\infty}^{f_L(t)} h(t)\phi(x,t)dxdt \\
&\quad - \int_0^\infty \left[u_L + \int_0^t h(s)\,ds\right]\phi(f_L(t)(t),t)f_L'(t)dt \\
&\qquad + \int_0^\infty \left[\frac{1}{2}\big(u_L + \int_0^t h(s)\,ds\big)^2\right]\phi(f_L(t),t)dt.
\end{aligned}
$$

Since I_2 has singularity at $t = 0$, we consider $I_2 = \lim_{\epsilon\to 0} I_{2,\epsilon}$.

$$
\begin{aligned}
I_{2,\epsilon} = &\int_\epsilon^\infty \int_{f_L(t)}^{f_R(t)} (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt. \\
= &\int_\epsilon^\infty \int_{f_L(t)}^{f_R(t)} \left(\frac{1}{t}\left[x + \int_0^t sh(s)ds\right]\phi_t dxdt\right. \\
&\quad + \frac{1}{2}\int_\epsilon^\infty \int_{f_L(t)}^{f_R(t)} \left(\frac{1}{t}\left[x + \int_0^t sh(s)ds\right)^2 \phi_x dxdt \\
= &-\frac{1}{2}\int_\epsilon^\infty \left(u_R + \int_0^t h(s)ds\right)^2 \phi(f_R(t),t)dt \\
&\quad + \frac{1}{2}\int_\epsilon^\infty \left(u_L + \int_0^t h(s)ds\right)\Bigg)^2 \phi(f_L(t),t)dt \\
&\qquad - \int_\epsilon^\infty \int_{f_L(t)}^{f_R(t)} h(t)\phi(x,t)dxdt.
\end{aligned}
$$

$$
\begin{aligned}
I_2 = &\lim_{\epsilon\to 0} I_{2,\epsilon} \\
= &-\frac{1}{2}\int_0^\infty \left(u_R + \int_0^t h(s)ds\right)^2 \phi(f_R(t),t)dt \\
&\quad + \frac{1}{2}\int_0^\infty \left(u_L + \int_0^t h(s)ds\right)\Bigg)^2 \phi(f_L(t),t)dt \\
&\qquad - \int_0^\infty \int_{f_L(t)}^{f_R(t)} h(t)\phi(x,t)dxdt.
\end{aligned}
$$

Also,

$$I_3 = \int_0^\infty \int_{f_R(t)}^\infty \left[u_R + \int_0^t h(s)\,ds\right]\phi_t dxdt + \int_0^\infty \int_{f_R(t)}^\infty \left(\frac{1}{2}\left[u_R + \int_0^t h(s)\,ds\right]^2\right)\phi_x dxdt$$
$$= -\int_{-\infty}^0 u_R\phi(x,0)dx - \int_0^\infty \int_{f_R(t)}^\infty h(t)\phi(x,t)dxdt - \frac{1}{2}\int_0^\infty \left[u_R + \int_0^t h(s)\,ds\right]^2 \phi(f_R(t),t)dt.$$

Thus,

$$\int_0^\infty \int_{-\infty}^\infty (u\phi_t + (\frac{u^2}{2})\phi_x)dxdt = I_1 + I_2 + I_3$$
$$= -\int_{-\infty}^0 u_L\phi(x,0)dx - \int_0^\infty u_R\phi(x,0)dx - \int_0^\infty \int_{-\infty}^\infty h(t)\phi(x,t)dxdt.$$

Similarly, one may prove that, the solution $\rho(x,t)$ given above is a weak solution of (3.2.14), i.e.,

$$\int_0^\infty \int_{-\infty}^{f_L(t)} \left[\rho\phi_t + (u\rho)\phi_x\right]dxdt + \int_0^\infty \int_{f_L(t)}^{f_R(t)} \left[\rho\phi_t + (u\rho)\phi_x\right]dxdt + \int_0^\infty \int_{f_R(t)}^\infty \left[\rho\phi_t + (u\rho)\phi_x\right]dxdt = 0. \tag{3.2.22}$$

Chapter 4

Explicit Formula and Large Time Behavior for the Heat Equation in a Strip with Discontinuous Source Term

4.1 Introduction

This chapter is mainly focuses on the existence, continuity and asymptotic analysis of solution to the initial-boundary value problem posed for a heat equation with Heaviside function in the source term given by

$$\begin{aligned} v_t &= v_{xx} - \frac{H(x)}{2}v, \quad -l \leq x \leq l, \quad t > 0, && (4.1.1) \\ v(x,0) &= \cos\left(\frac{m\pi}{l}x\right), \quad -l \leq x \leq l, && (4.1.2) \\ v_x(-l,t) &= 0, \quad t > 0, && (4.1.3) \\ v_x(l,t) &= 0, \quad t > 0, && (4.1.4) \end{aligned}$$

where $H(x)$ is the Heaviside function and m is any natural number.

The general structure of convection diffusion equation with the Cauchy data is given by

$$\begin{aligned} v_t + \Phi(v,x,t)_x + \Psi(v,x,t) &= \Theta(v,t)_{xx}, \quad x \in \mathbb{R}, \quad 0 < t < T, \\ v(x,0) &= v_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

where $\Theta(v,t)$ is the diffusion function which is non-linear, depends explicitly on time, $\Phi(v,x,t)$ is a nonlinear flux function which depend on v, x, and t. $\Psi(v,x,t)$ is the given source term which can also depends on x and t explicitly. The equation appears in several applications as mentioned in Lu et al. (2015) and the references therein. In recent times, the study of Burgers' equation with Dirac delta measure or subsequently studying the heat equation with Heaviside function in the source term grabbed much attention. The relation between the linear heat equation with source term and nonlinear nonhomogeneous Burgers' equation via Hopf-Cole transformation makes the process progressive in the field.

Chung et al. (2014) considered the heat equation with a positive heat source:

$$v_t - v_{xx} = \delta(x)$$

with the initial condition $v_0(x)$ is integrable on $\mathbb{R}$ and discussed that the solution $v(x,t)$ diverges everywhere as $t \to \infty$. Also, the solution trajectory cannot be compact in any L^p space even on compact sets. Additionally, while solving Burgers' equation with Dirac delta measure they encountered the heat equation with Heaviside step function in source term:

$$v_t - v_{xx} = -\frac{H(x)v}{2}, \quad x \in \mathbb{R}, \quad t > 0.$$

It is observed that the continuous solution exists for the initial boundary value problem with discontinuous source term. The article also discuss the existence of boundary data, asymptotic behavior of solutions and common boundary condition converging to nonzero quantity with necessary conditions.

Chidella and Yadav (2010) investigated the heat equation where the source term:

$$v_t - v_{xx} = -\frac{kx^2}{4}v, \quad x \in \mathbb{R}, \quad t > 0. \tag{4.1.5}$$

where initial condition $v_0(x)$ is in class of rapidly decaying functions. Rescaling the above equation can lead us to get

$$v_t - v_{xx} = -x^2 v, \quad x \in \mathbb{R}, \quad t > 0. \tag{4.1.6}$$

With Ding et al. (2001) approach, where initial data is square integrable on $\mathbb{R}$, the solution of above equation (4.1.6) can be represented in Fourier-Hermite expansion. Otherwise, to deal with bounded and compactly supported initial data, the above equation (4.1.6) is transformed into Cauchy problem for heat equation. If the transformed initial data is square integrable with weight function $e^{z^2/2}$, the Kloosterziel (1990) approach is applied and the solution can be represented in an infinite series of self similar solutions of the heat equation. Also, large time behavior of the solution which converge to a nonzero quantity is discovered.

Rao and Yadav (2010b) studied the initial value problem for heat equation with source term depending on x and t explicitly:

$$v_t - v_{xx} = -\frac{kx^2}{4(2\beta t+1)^2} v, \quad x \in \mathbb{R}, \quad t > 0. \tag{4.1.7}$$

where the initial data $v(x, 0)$ is square integrable on $\mathbb{R}$ with weight function $e^{\beta x^2/2}$. They constructed a family of self-similar solutions $\{v_n(x,t)\}_{n=0}^{\infty}$ of (4.1.7) where the family $\{v_n(x,0)\}_{n=0}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R}, e^{\beta x^2/2})$. The solution of (4.1.7) is expressed in an infinite series form with these self-similar solutions with variable coefficients and the approach quickly reveals the large time behavior of the IVP.

Gianni and Hulshof (1992) considered the semi-linear parabolic equation with a discontinuous source term:

$$v_t - v_{xx} = H(v),$$

where H is the Heaviside graph, on a bounded interval with Dirichlet boundary condition and discussed the existence, uniqueness and smoothness of solutions. The article also indicates that solutions of the semi-linear heat equation with a Heaviside source term are usually well behaved, with the exceptions that the initial profiles are either identically zero, or touch zero from below.

Cannon and DuChateau (1998) examined one-dimensional heat equation involving a state-dependent source term with first order spatial derivatives at the boundaries:

$$\begin{aligned} v_t - v_{xx} &= F(v(x,t)), \quad 0 < x < 1, \quad 0 < t < T \\ v(x,0) &= 0, \quad 0 < x < 1 \\ v_x(0,t) &= g(t) \\ v_x(1,t) &= 0, \quad 0 < t < T. \end{aligned}$$

Further, they assumed that $g(t)$ is continuous, $g(0) = 0$ and F is continuous with piecewise differentiability on the real line. Assuming the priori bound for the solution of above system, various properties and integral identities are obtained which are useful to understand the system with change in the source term. Also, the inverse problems are formulated and the identities are used to estimate the unknown source term in an appropriate class of source functions.

Engu et al. (2021) studied initial-boundary value problem for heat equation with time-dependent source involving Heaviside function:

$$\begin{aligned} v_t - v_{xx} &= -\frac{H(x)}{(1+t)}v\,, \quad x \in \mathbb{R}, \quad t > 0, \\ v(x,0) &= v_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

where the initial data $v_0(x) \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R})$. The authors then investigated the corresponding two initial boundary value problems by introducing a common boundary on two upper quarter planes separately. The existence of the boundary function is obtained with the aid of Abel's integral equation. Also, the solutions of associated initial boundary value problems converge uniformly to a nonzero real constant uniformly on compact sets as t tends to infinity.

The following chapter is dedicated to discuss whether the solution exists for the Heat equation with Heaviside function in the source term, the continuity of solution due to the derivative of boundary condition imposed at $x = 0$ and the asymptotic behavior of the solutions in respective domains for large time t. The presence of Heaviside function immediately splits the problem space of (4.1.1)-(4.1.4) into two domains, namely, right side domain $\{0 \leq x \leq l; t > 0\}$ and left side domain $\{-l \leq x \leq 0; t > 0\}$. Hence, one needs to study the problem on two upper quarter plane slits separately with common condition at $x = 0$ along the positive t-axis. The Neumann condition is applied in various fields like thermodynamics, solving Laplace equation and poisson's equation, solid mechanics and fluid mechanics. In addition, the Neumann boundary constraints leads us to get the solution involving Green's function which is associated with Sturm-Liouville problem including eigenfunctions and eigenvalues. Also, it is notable that the boundary conditions emerge in the corresponding Sturm-Liouville system is also of second type boundary condition. The modified Green's function provides the solution in an integral representation where the kernel is composed with the Green's function.

With the help of Volterra's integral equation of first kind with difference kernel, we aspire to establish the existence and uniqueness of the common condition, derivative of boundary function, enforced at $x = 0$ to the linearized partial differential equations. For the same we rearrange the Volterra's integral into Abel's integral equation of first kind where the kernel can be expressed in terms of Jacobi theta function. Further, we seek the asymptotic behavior to the solution of linearized partial differential equation for large time t. Making use of the Laplace transformation techniques on convolution integral and final value theorems we observe the rate of convergence as t tends to ∞. The following observations are used in this chapter.

1. Orthogonality of the trigonometric system $\left\{ \cos(\frac{n\pi}{l}x) : n \in \mathbb{N} \cup \{0\} \right\}$ is given by

$$\begin{aligned} \int_0^l \cos\left(\frac{n\pi}{l}x\right) \cos\left(\frac{m\pi}{l}x\right) dx &= 0, \quad m \neq n \\ \int_0^l \cos^2\left(\frac{n\pi}{l}x\right) dx &= \frac{l}{2}, \quad n \in \mathbb{N}. \end{aligned}$$

2. The Sturm-Liouville equation on a finite interval $a \leq x \leq b$ together with

two separated end point conditions is of the form:

$$\begin{aligned}
\frac{d}{dx}\Big[p(x)\frac{d}{dx}\phi\Big] &= \big[q(x) - \lambda\, s(x)\big]\phi, \\
a_1\,\phi_x + b_1\,\phi &= 0 \quad \text{at} \quad x = a, \\
a_2\,\phi_x + b_2\,\phi &= 0 \quad \text{at} \quad x = b,
\end{aligned}$$

where a_1, a_2, b_1 and b_2 are given real numbers.

3. There are infinitely many eigenvalues to the above system and all are real and distinct. It can be ordered as $\lambda_1 < \lambda_2 < \lambda_3 < ......$ with $\lambda_n \to \infty$ as $n \to \infty$. Each eigenvalue is of multiplicity 1.

4. Eigenfunctions of above Sturm-Liouville system are orthogonal with weight $s(x)$ on the interval $a \leq x \leq b$. i.e.,

$$\int_a^b s(x)\phi_n(x)\phi_m(x)\,dx = 0, \quad \text{with} \quad m \neq n,$$

where $\phi_n(x)$ represents the n-th eigenfunction.

5. If $q(x) \geq 0$, $a_1 b_1 \leq 0$ and $a_2 b_2 \geq 0$, then there are no negative eigenvalues. Further, if $q \equiv 0$ with $b_1 = b_2 = 0$, then $\lambda_1 = 0$ is the least eigenvalue, to which there corresponds the eigenfunction ϕ_1=constant.

The arrangements of this chapter is as follows. Section (4.2) deals with the solution of heat equation imposed with initial condition and Neumann boundary condition. Also, the continuity of the solution due to the derivative of boundary condition and Volterra's integral equation of first kind is discussed. In section (4.3), asymptotic behavior and convergence rate of the derivative of boundary condition, $g(t)$, right side domain solution $R(x,t)$ and left side domain solution $L(x,t)$ is explained.

4.2 Heat Equation with Heaviside Function in Source Term

In view of the Heaviside function in source term, we can split the initial-boundary value problem (4.1.1)-(4.1.4) into two domains, namely, right side domain $\{0 \leq x \leq l; t > 0\}$ and left side domain $\{-l \leq x \leq 0; t > 0\}$. In this process we establish derivative of boundary condition at $x = 0$, namely, $\phi(t)$. Existence of the same will be shown later.

Let $R(x,t)$ be the solution in the right side domain $\{0 \leq x \leq l; t > 0\}$ satisfying

$$R_t = R_{xx} - \frac{R}{2}, \quad 0 \leq x \leq l, \quad t > 0, \tag{4.2.8}$$

$$R(x,0) = \cos\left(\frac{m\pi}{l}x\right), \quad 0 \leq x \leq l, \tag{4.2.9}$$

$$R_x(0,t) = \phi(t), \quad t > 0, \tag{4.2.10}$$

$$R_x(l,t) = 0, \quad t > 0. \tag{4.2.11}$$

To rescale R, we consider the transformation,

$$w(x,t) = e^{\frac{t}{2}} R(x,t), \tag{4.2.12}$$

which transforms (4.2.8)-(4.2.11) into following linearized initial-boundary value problem;

$$w_t = w_{xx}, \quad 0 \leq x \leq l, \quad t > 0, \tag{4.2.13}$$

$$w(x,0) = \cos\left(\frac{m\pi}{l}x\right), \quad 0 \leq x \leq l, \quad t > 0, \tag{4.2.14}$$

$$w_x(0,t) = e^{\frac{t}{2}} \phi(t), \quad t > 0, \tag{4.2.15}$$

$$w_x(l,t) = 0, \quad t > 0. \tag{4.2.16}$$

The above system is a linear partial differential equation equipped with initial data and boundary conditions of the form Polyanin and Nazaikinskii (2015):

$$s(x)\,\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\,\frac{\partial z}{\partial x}\right] - q(x)\,z + \Phi(x,t), \tag{4.2.17}$$

where s, p, p'_x and q are continuous functions in x with $s > 0$, $p > 0$ and $x_1 \leq x \leq x_2$. The solution of (4.2.17) equipped with the initial condition

$$z(x,0) = \alpha(x), \tag{4.2.18}$$

and the arbitrary linear nonhomogenoeous boundary conditions

$$a_1\, z_x + b_1\, z \;=\; \eta_1(t) \quad \text{at} \quad x = x_1, \tag{4.2.19}$$

$$a_2\, z_x + b_2\, z \;=\; \eta_2(t) \quad \text{at} \quad x = x_2, \tag{4.2.20}$$

can be represented as the sum expressed below

$$\begin{aligned} z(x,t) = &\int_0^t \int_{x_1}^{x_2} \Phi(\xi,t)\, G(x,\xi,t-\tau) d\xi\, d\tau + \int_{x_1}^{x_2} s(\xi)\alpha(\xi) G(x,\xi,t) d\xi \\ &+ p(x_1) \int_0^t \eta_1(\tau) \Lambda_1(x,t-\tau) d\tau + p(x_2) \int_0^t \eta_2(\tau) \Lambda_2(x,t-\tau) d\tau, \end{aligned} \tag{4.2.21}$$

where $G(x,\xi,t)$ is the modified Green's function given by

$$G(x,\xi,t) = \sum_{n=1}^{\infty} \frac{y_n(x)\, y_n(\xi)}{||y_n||^2}\, e^{-\lambda_n t}, \qquad ||y_n||^2 = \int_{x_1}^{x_2} s(x)\, y_n^2(x)\, dx. \tag{4.2.22}$$

The notations λ_n and $y_n(x)$ in (4.2.24) are the eigenvalues and the respective eigenfunctions of the following Sturm–Liouville problem for a second-order linear ordinary differential equation:

$$\big[p(x)\, y_x'\big]_x' + \big[\lambda\, s(x) - q(x)\big] y = 0,$$

$$a_1\, y_x + b_1\, y = 0 \quad \text{at} \quad x = x_1,$$

$$a_2\, y_x + b_2\, y = 0 \quad \text{at} \quad x = x_2.$$

The functions $\Lambda_1(x,t)$ and $\Lambda_2(x,t)$ arising in the integrands of last two terms in (4.2.21) are expressed via the Green's function (4.2.22). In particular, for the second type boundary value problem (i.e., $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$), the values of Λ_1 and Λ_2 are given by

$$\Lambda_1(x,t) = -G(x,x_1,t) \quad \text{and} \quad \Lambda_2(x,t) = G(x,x_2,t). \tag{4.2.23}$$

Hence, From (4.2.17)-(4.2.23), the solution of initial-boundary value problem (4.2.13)-(4.2.16) is given by

$$w(x,t) = \int_0^l \cos\left(\frac{m\pi}{l}\xi\right) G(x,\xi,t)\, d\xi - \int_0^t e^{\frac{\tau}{2}} \phi(\tau) G(x,0,t-\tau)\, d\tau,$$

where $G(x,\xi,t)$ is the modified Green's function given by

$$G(x,\xi,t) = \sum_{n=0}^{\infty} \frac{y_n(x)\, y_n(\xi)}{||y_n||^2}\, e^{-\lambda_n t}, \quad ||y_n||^2 = \int_0^l y_n^2(x)\, dx, \tag{4.2.24}$$

with λ_n and $y_n(x)$ are the eigenvalues and the corresponding eigenfunctions of the following Sturm-Liouville problem for a second-order linear ordinary differential equation:

$$y''(x) + \lambda\, y(x) = 0, \tag{4.2.25}$$

$$\text{with} \quad y'(0) = 0 \quad \text{and} \quad y'(l) = 0. \tag{4.2.26}$$

Now, we calculate the eigenfunctions and their corresponding eigenvalues by considering the auxiliary equation. i.e., the associated auxiliary equation for (4.2.25) is given by

$$m^2 + \lambda = 0.$$

Solving the above quadratic equation we obtain

$$m = \pm\sqrt{-\lambda}.$$

Based on the sign of λ, we consider following cases:

1. If $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta > 0$. But then $m = \pm i\beta$, where i is the imaginary number, and the solution is given by

$$y(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x).$$

Differentiating the above equation,

$$\begin{aligned} y'(x) &= c_1(-\sin(\beta x))\beta + c_2(\cos(\beta x))\beta \\ &= \beta(-c_1 \sin(\beta x) + c_2 \cos(\beta x)). \end{aligned}$$

Making use of the condition $y'(0) = 0$ we get $c_2 = 0$. Hence, the solution $y(x)$ reduces to

$$y(x) = c_1 \cos(\beta x) \quad \text{and hence} \quad y'(x) = -c_1 \sin(\beta x).$$

Also, from (4.2.26) we have $y'(-l) = 0$ which in turn leads us to get $\sin(-\beta l) = 0$. Hence, $\beta l = n\pi$, $n \in \mathbb{N}$ or $\sqrt{\lambda} = \frac{n\pi}{l}$, $n \in \mathbb{N}$. But then

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \quad n \in \mathbb{N} \quad \text{with} \quad y_n(x) = \cos(\beta x) = \cos(\sqrt{\lambda_n}\, x).$$

i.e.,

$$y_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n \in \mathbb{N}.$$

2. If $\lambda = 0$, then we have $y(x) = c_3 x + c_4$. Again, using (4.2.26) we obtain $y(x) =$ a constant, which is a special case of case (1).

3. Suppose $\lambda < 0$. Let $\lambda = -\beta^2$ where $\beta > 0$. But then $m = \pm\beta$ and the solution is given by

$$y(x) = c_5 e^{\beta x} + c_6 e^{-\beta x}.$$

Differentiating the above equation,

$$\begin{aligned} y'(x) &= c_5\, e^{\beta x}\, \beta - c_6\, e^{-\beta x}\, \beta \\ &= \beta(c_5\, e^{\beta x} - c_6\, e^{-\beta x}). \end{aligned}$$

Using (4.2.26) we obtain the system

$$\begin{aligned} c_5 - c_6 &= 0, \\ c_5\, e^{-\beta l} - c_6\, e^{\beta l} &= 0, \end{aligned}$$

in $(c_5, c_6)^T$ for which the determinant is a nonzero number, which concludes the above system has trivial solution.

Clubbing case (1) and (2), the explicit expressions for the eigenvalues and corresponding eigenfunctions are given by

$$\hat{\lambda}_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{and} \quad \hat{y}_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n \in \mathbb{N} \cup \{0\}.$$

Substituting the values in (4.2.24) we note that for $n = 0$ we have $||y_0||^2 = l$ and for $n \in \mathbb{N}$,

$$||y_n||^2 = \int_0^l \cos^2\left(\frac{n\pi x}{l}\right) dx = \int_0^l \frac{1 + \cos\left(\frac{2n\pi x}{l}\right)}{2}\, dx = \frac{l}{2}.$$

Subsequently, from (4.2.24), we can express the modified Green's function by

$$\begin{aligned} G(x, \xi, t) &= \frac{\cos(0)\cos(0)}{l} e^{0\,t} + \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi\xi}{l}\right)\cos\left(\frac{n\pi x}{l}\right)}{\frac{l}{2}} e^{\lambda_n t} \\ &= \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{l}\right)\cos\left(\frac{n\pi x}{l}\right) e^{\lambda_n t}. \end{aligned} \tag{4.2.27}$$

Hence, simplifying (4.2.27) we get the modified Green's function given by

$$G(x,\xi,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}\xi\right) e^{-(\frac{n\pi}{l})^2 t}. \tag{4.2.28}$$

Tracing back the solution of (4.2.8)-(4.2.11) via (4.2.13), we get

$$R(x,t) = \int_0^l e^{-\frac{t}{2}} \cos\left(\frac{m\pi}{l}\xi\right) G(x,\xi,t)\, d\xi - \int_0^t e^{\frac{-(t-\tau)}{2}} \phi(\tau)\, G(x,0,t-\tau)\, d\tau. \tag{4.2.29}$$

Similarly, let $L(x,t)$ be the solution of (4.1.1)-(4.1.4) in the left side domain $\{-l \leq x \leq 0; t > 0\}$ satisfying

$$\begin{aligned} L_t &= L_{xx}, \quad -l \leq x \leq 0, \quad t > 0, &(4.2.30)\\ L(x,0) &= \cos\left(\frac{m\pi}{l}x\right), \quad -l \leq x \leq 0, &(4.2.31)\\ L_x(-l,t) &= 0, \quad t > 0, &(4.2.32)\\ L_x(0,t) &= \phi(t), \quad t > 0. &(4.2.33)\end{aligned}$$

Take $x = -y$. Then $L(x,t) = \hat{L}(y,t), \ 0 \leq y \leq l$. But then (4.2.12)-(4.2.15) turns into

$$\begin{aligned} \hat{L}_t &= \hat{L}_{yy}, \quad 0 \leq y \leq l, \quad t > 0, &(4.2.34)\\ \hat{L}(y,0) &= \cos\left(-\frac{m\pi}{l}y\right), \quad 0 \leq y \leq l, &(4.2.35)\\ \hat{L}_y(l,t) &= 0, \quad t > 0, &(4.2.36)\\ \hat{L}_y(0,t) &= -\phi(t), \quad t > 0. &(4.2.37)\end{aligned}$$

From (4.2.17)-(4.2.23), the solution of (4.2.34)-(4.2.37) is given by

$$\hat{L}(y,t) = \int_0^l \cos\left(\frac{m\pi}{l}\xi\right) \hat{G}(y,\xi,t)\, d\xi + \int_0^t \phi(\tau)\, \hat{G}(y,0,t-\tau))\, d\tau,$$

which in turns leads to get the solution of (4.2.30)-(4.2.33) given by

$$L(x,t) = \int_0^l \cos\left(\frac{m\pi}{l}\xi\right) \hat{G}(-x,\xi,t)\, d\xi + \int_0^t \phi(\tau)\, \hat{G}(-x,0,t-\tau)\, d\tau, \tag{4.2.38}$$

where

$$\hat{G}(-x,\xi,t) = \sum_{n=0}^{\infty} \frac{\hat{y}_n(-x)\,\hat{y}_n(\xi)}{||\hat{y}_n||^2} e^{-\hat{\lambda}_n t}, \quad ||\hat{y}_n||^2 = \int_{-l}^0 \hat{y}_n^2(x)\, dx, \tag{4.2.39}$$

is the modified Green's function with $\hat{\lambda}_n$ and $\hat{y}_n(x)$ are the eigenvalues and the associated eigenfunctions of the following Sturm–Liouville problem (4.2.30)-(4.2.33) for a second-order linear ordinary differential equation:

$$\hat{y}''(x) + \hat{\lambda}\,\hat{y}(x) = 0, \tag{4.2.40}$$

$$\text{with} \quad \hat{y}'(-l) = 0 \quad \text{and} \quad \hat{y}'(0) = 0. \tag{4.2.41}$$

In fact, we calculate the eigenfunctions and their associated eigenvalues by considering the auxiliary equation. i.e., the associated auxiliary equation for the above given by

$$m^2 + \hat{\lambda} = 0.$$

Solving the above equation as before to obtain the left side domain's modified Green's function, we get the explicit expressions for the eigenvalues and corresponding eigenfunctions:

$$\hat{\lambda}_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{and} \quad \hat{y}_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n \in \mathbb{N} \cup \{0\}.$$

But then with the norm for $n = 0$ we have $||\hat{y}_0||^2 = l$ and for $n \in \mathbb{N}$

$$||\hat{y}_n||^2 = \int_{-l}^{0} \cos^2\left(\frac{n\pi x}{l}\right) dx = \frac{l}{2}.$$

Hence, the Green's function (4.2.39) reduces to

$$\hat{G}(-x, \xi, t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos(\frac{n\pi}{l}x)\, \cos(\frac{n\pi}{l}\xi)\, e^{-(\frac{n\pi}{l})^2 t}.$$

It can be seen that the modified Green's function satisfies the relation

$$\hat{G}(-x, \xi, t) = G(-x, \xi, t) = G(x, \xi, t).$$

But then (4.2.38) reduces to

$$L(x,t) = \int_0^l \cos\left(\frac{m\pi}{l}\xi\right) G(-x, \xi, t)\, d\xi + \int_0^t \phi(\tau)\, G(-x, 0, t-\tau)\, d\tau. \tag{4.2.42}$$

Note that in the argument for space variable in Green's function G of $L(x,t)$, we have maintained *minus* sign to avoid the confusion with notation of Green's function in right side solution $R(x,t)$. Also, $\frac{\partial}{\partial x}G(0,\xi,t)=0$ and $\frac{\partial}{\partial x}G(l,\xi,t)=0$. In fact,

$$\frac{\partial}{\partial x}G(x,\xi,t)=-\frac{2}{l}\sum_{n=1}^{\infty}\frac{n\pi}{l}\sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}\xi\right)e^{-(\frac{n\pi}{l})^2t}.$$

To establish the continuity of the solution $v(x,t)$ via $R(x,t)$ and $L(x,t)$, we impose the condition:

$$\lim_{x\to 0}L(x,t)=\lim_{x\to 0}R(x,t)\quad \text{for all}\quad t>0. \tag{4.2.43}$$

But then from (4.2.29) and (4.2.42) we have

$$\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)G(0,\xi,t)\,d\xi+\lim_{x\to 0}\int_0^t \phi(\tau)\,G(-x,0,t-\tau)\,d\tau$$
$$=e^{-\frac{t}{2}}\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)G(0,\xi,t)\,d\xi-\lim_{x\to 0}\int_0^t e^{-\frac{(t-\tau)}{2}}\phi(\tau)\,G(x,0,t-\tau)\,d\tau.$$

Assuming that we can pass the limit inside the integral, we get

$$\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)G(0,\xi,t)\,d\xi-e^{-\frac{t}{2}}\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)G(0,\xi,t)\,d\xi$$
$$=-\int_0^t e^{-\frac{(t-\tau)}{2}}\phi(\tau)\,G(0,0,t-\tau)\,d\tau-\int_0^t \phi(\tau)\,G(0,0,t-\tau)\,d\tau,$$

which implies

$$(e^{-\frac{t}{2}}-1)\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)G(0,\xi,t)\,d\xi=\int_0^t\left[e^{-\frac{(t-\tau)}{2}}+1\right]\phi(\tau)\,G(0,0,t-\tau)\,d\tau. \tag{4.2.44}$$

The orthogonality of the trigonometric system $\left\{\cos(\frac{n\pi}{l}x): n\in\mathbb{N}\cup\{0\}\right\}$ simplifies the left side of the equation (4.2.44) given by

$$\begin{aligned}\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)G(0,\xi,t)\,d\xi&=\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\left[\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}\cos(\frac{n\pi}{l}\xi)\,e^{-(\frac{n\pi}{l})^2t}\right]d\xi\\&=\frac{1}{l}\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)d\xi\\&\quad+\frac{2}{l}\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\sum_{n=1}^{\infty}\cos(\frac{n\pi}{l}\xi)\,e^{-(\frac{n\pi}{l})^2t}d\xi\\&=\sum_{n=1}^{\infty}\frac{2}{l}\left(\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\cos\left(\frac{n\pi}{l}\xi\right)d\xi\right)e^{-(\frac{n\pi}{l})^2t}\\&=\frac{2}{l}\left(\frac{l}{2}\right)e^{-(\frac{m\pi}{l})^2t}=e^{-(\frac{m\pi}{l})^2t}.\end{aligned}$$

Hence, (4.2.44) leads us to get

$$\left(e^{-\frac{t}{2}}-1\right)e^{-(\frac{m\pi}{l})^2 t}=\int_0^t\left[\left(e^{-\frac{(t-\tau)}{2}}+1\right)\left(\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}\right)\right]\phi(\tau)\,d\tau,$$

which is in the form of Volterra's integral equation of the first kind with difference kernel given by

$$f(t)=\int_0^t K(t-\tau)\,\phi(\tau)\,d\tau, \tag{4.2.45}$$

where f and difference kernel K are given by

$$f(t)=\left(e^{-\frac{t}{2}}-1\right)e^{-(\frac{m\pi}{l})^2 t} \tag{4.2.46}$$

and

$$K(t,\tau)=\left[\left(e^{-\frac{(t-\tau)}{2}}+1\right)\left(\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}\right)\right]. \tag{4.2.47}$$

Theorem 4.2.1 (Gorenflo and Vessella (1991)). *Let the kernel $K\in C^{p+1}(T)$ with $T=\{(t,\tau)\in\mathbb{R}^2 : 0<\tau<t<\infty\}$ and assume $K(t,t)=1$ for every $t\in(0,\infty)$. Let γ be a function defined on $(0,\infty)$ with*

$$(\Omega\gamma)(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{\gamma(\tau)}{(t-\tau)^{\alpha}}dt\in C^p(0,\infty),$$

for $0<\alpha<1$ and $\Gamma(.)$ representing the gamma function. Then the equation

$$\frac{1}{\Gamma(\alpha)}\int_0^t\frac{K(t,\tau)\gamma(\tau)}{(t-\tau)^{1-\alpha}}dt=\omega(t)$$

has a unique solution in $C^p(0,\infty)$ and

$$\|\gamma(t)\|_{C^p(0,\infty)}\le\Delta(\sigma_2)\,\|(\Omega\gamma)(t)\|_{C^p(0,\infty)},$$

where $\sigma_2=\|K\|_{C^{p+1}(0,\infty)}$ and $\Delta(\sigma_2)$ is a constant depending on σ_2.

Theorem 4.2.2. *There exists a unique continuously differentiable function $\phi(t)$ satisfying Volterra's integral equation of the first kind (4.2.45)-(4.2.47).*

Proof: To establish the existence of the derivative of boundary function, $\phi(t)$, we rearrange (4.2.45)-(4.2.47) in terms of Abel's integral equation of first kind as follows:

$$\int_0^t\frac{\hat{K}(t,\tau)}{\sqrt{t-\tau}}\phi(\tau)d\tau=F(t),$$

where

$$\hat{K}(t,\tau) = \sqrt{\frac{\pi}{8}}\sqrt{t-\tau}\Big(e^{-\frac{(t-\tau)}{2}}+1\Big)\left(\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}\right)$$

and

$$F(t) = \sqrt{\frac{\pi}{8}}f(t) = \sqrt{\frac{\pi}{8}}\big(e^{-\frac{t}{2}}-1\big)e^{-(\frac{m\pi}{l})^2 t}.$$

Let $T = \{(t,\tau) \in \mathbb{R}^2 : 0 < \tau < t < \infty\}$. We show that $\hat{K}(t,\tau) \to 1$ as $\tau \to t$. The kernel $\hat{K}(t,\tau)$ of Abel's integral equation with rearrangement yields us to get

$$\begin{aligned}
\hat{K}(t,\tau) &= \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{t-\tau}\Big(e^{-\frac{(t-\tau)}{2}}+1\Big)\left(\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}\right)\\
&= \frac{\sqrt{\pi}}{2\sqrt{2}}\sqrt{t-\tau}\Big(e^{-\frac{(t-\tau)}{2}}+1\Big)\left(\frac{1}{l}+\frac{2}{l}\left[\sum_{n=0}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}-1\right]\right)\\
&= \frac{\sqrt{\pi}}{2\sqrt{2}}\sqrt{t-\tau}\Big(e^{-\frac{(t-\tau)}{2}}+1\Big)\left(\frac{2}{l}\sum_{n=0}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}-\frac{1}{l}\right)\\
&= \Big(e^{-\frac{(t-\tau)}{2}}+1\Big)\left[\sqrt{\frac{\pi}{2}}\frac{\sqrt{t-\tau}}{l}\sum_{n=0}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}-\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{\sqrt{t-\tau}}{l}\right]\\
&= \frac{(e^{-\frac{(t-\tau)}{2}}+1)}{2}\left[\frac{\sqrt{2\pi(t-\tau)}}{l}\sum_{n=0}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}-\frac{1}{l}\sqrt{\frac{\pi}{2}(t-\tau)}\right]\\
&= \frac{(e^{-\frac{(t-\tau)}{2}}+1)}{2}\left[\sqrt{\frac{2}{\pi}\left(\frac{\pi}{l}\right)^2(t-\tau)}\sum_{n=0}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}-\frac{1}{l}\sqrt{\frac{\pi}{2}(t-\tau)}\right].
\end{aligned}$$

Using the result in Wang (2009), $\sqrt{2\theta/\pi}\sum_{n=0}^{\infty}e^{-n^2\theta} \sim 1$ as $\theta \to 0$, we obtain

$$\hat{K}(t,\tau) \longrightarrow 1, \text{ as } \tau \to t.$$

We now express the kernel $\hat{K}(t,\tau)$ as

$$\begin{aligned}
\hat{K}(t,\tau) &= \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{t-\tau}\Big(e^{-\frac{(t-\tau)}{2}}+1\Big)\left(\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2(t-\tau)}\right)\\
&= \frac{1}{l}\sqrt{\frac{\pi}{8}}\sqrt{t-\tau}\Big(e^{-\frac{(t-\tau)}{2}}+1\Big)\,\vartheta_3\Big(0, e^{-(\frac{\pi}{l})^2(t-\tau)}\Big),
\end{aligned}$$

where $\vartheta_3(z,q)$ is the *Jacobi 3-Theta function* [Borwein and Borwein (1987); Qiu and Vuorinen (2005)] defined by

$$\vartheta_3(z,q) = 1 + 2\sum_{n=1}^{\infty}q^{n^2}\cos(2nz), \text{ for } z,\ q \in \mathbb{C} \text{ and } |q| < 1.$$

Since the *Jacobi 3-Theta function* is analytic for all z, $q \in \mathbb{C}$ and $|q| < 1$, the kernel $\hat{K}(t,\tau)$ is twice continuously differentiable on T.

We now show that

$$(\Omega F)(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{F(\tau)}{\sqrt{t-\tau}} d\tau \in C^1(0,\infty). \tag{4.2.48}$$

Consider

$$\begin{aligned}
F'(t) &= -\sqrt{\frac{\pi}{8}} \left(\frac{m\pi}{l}\right)^2 (e^{-\frac{t}{2}} - 1) e^{-(\frac{m\pi}{l})^2 t} - \frac{1}{2}\sqrt{\frac{\pi}{8}}\, e^{-(\frac{m\pi}{l})^2 t} e^{-\frac{t}{2}} \\
&= -\sqrt{\frac{\pi}{8}} e^{-(\frac{m\pi}{l})^2 t} \left[(e^{-\frac{t}{2}} - 1) \left(\frac{m\pi}{l}\right)^2 + \frac{1}{2}\, e^{-\frac{t}{2}} \right] \\
&= -\sqrt{\frac{\pi}{8}} e^{-(\frac{m\pi}{l})^2 t} \left[e^{-\frac{t}{2}} \left(\left(\frac{m\pi}{l}\right)^2 + \frac{1}{2} \right) - \left(\frac{m\pi}{l}\right)^2 \right] \\
&= -\sqrt{\frac{\pi}{8}} \left(\left(\frac{m\pi}{l}\right)^2 + \frac{1}{2} \right) e^{-(\frac{m\pi}{l})^2 t}\, e^{-\frac{t}{2}} + \sqrt{\frac{\pi}{8}} \left(\frac{m\pi}{l}\right)^2 e^{-(\frac{m\pi}{l})^2 t}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\sqrt{t}F'(t)\right| &\leq \left| \sqrt{\frac{\pi}{8}} \left(\left(\frac{m\pi}{l}\right)^2 + \frac{1}{2} \right) \sqrt{t}\, e^{-((\frac{m\pi}{l})^2 + \frac{1}{2})t} \right| + \left| \sqrt{\frac{\pi}{8}} \left(\frac{m\pi}{l}\right)^2 \sqrt{t}\, e^{-(\frac{m\pi}{l})^2 t} \right| \\
&\leq C, \text{ a constant for all } t > 0.
\end{aligned}$$

Clearly, we have $F(0) = 0$. Integrating by parts for (4.2.48) leads us to get

$$(\Omega F)(t) := \frac{2}{\sqrt{\pi}} \frac{d}{dt} \int_0^t F'(\tau)\sqrt{t-\tau}\, d\tau = \frac{1}{\sqrt{\pi}} \int_0^t \frac{F'(\tau)}{\sqrt{t-\tau}} d\tau.$$

Thus, we have

$$|(\Omega F)(t)| = \frac{1}{\sqrt{\pi}} \left| \int_0^t \frac{F'(\tau)}{\sqrt{t-\tau}} d\tau \right| \leq \frac{C}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\tau}\sqrt{t-\tau}} d\tau = C\sqrt{\pi}, \quad \forall\, t > 0.$$

i.e., $(\Omega F)(t)$ is bounded for all $t > 0$. Also, $(\Omega F)(t)$ has no singularity which concludes the continuity for all $t > 0$. From $F'(t)$ we calculate $F'(0)$ given by

$$\begin{aligned}
F'(0) &= -\sqrt{\frac{\pi}{8}} \left(\left(\frac{m\pi}{l}\right)^2 + \frac{1}{2} \right) + \sqrt{\frac{\pi}{8}} \left(\frac{m\pi}{l}\right)^2 \\
&= -\sqrt{\frac{\pi}{8}} \left(\frac{m\pi}{l}\right)^2 - \frac{1}{2}\sqrt{\frac{\pi}{8}} + \sqrt{\frac{\pi}{8}} \left(\frac{m\pi}{l}\right)^2 = -\frac{1}{2}\sqrt{\frac{\pi}{8}}.
\end{aligned}$$

Also, differentiating $F'(t)$, we get

$$F''(t) = \sqrt{\frac{\pi}{8}} \left(\left(\frac{m\pi}{l}\right)^2 + \frac{1}{2} \right)^2 e^{-(\frac{m\pi}{l})^2 t}\, e^{-\frac{t}{2}} + \frac{-\sqrt{\pi}}{\sqrt{8}} \left(\frac{m\pi}{l}\right)^4 e^{-(\frac{m\pi}{l})^2 t}.$$

Hence,

$$\left|\sqrt{t}F''(t)\right| \le \left|\sqrt{\frac{\pi}{8}}\left(\left(\frac{m\,\pi}{l}\right)^2+\frac{1}{2}\right)^2\sqrt{t}\,e^{-((\frac{m\,\pi}{l})^2+\frac{1}{2})t}\right| + \left|\sqrt{\frac{\pi}{8}}\left(\frac{m\,\pi}{l}\right)^4\sqrt{t}\,e^{-(\frac{m\,\pi}{l})^2 t}\right|$$
$$\le B,\ \text{ a constant for all } t>0.$$

Consider

$$(\Omega F)(t) = \frac{2}{\sqrt{\pi}}\frac{d}{dt}\int_0^t F'(\tau)\sqrt{t-\tau}\,d\tau$$

and integrating by parts followed by substitution of $F'(0)$, we obtain

$$\begin{aligned}
(\Omega F)(t) &= \frac{2}{\sqrt{\pi}}\frac{d}{dt}\Big[-\frac{2}{3}F'(t)(t-t)^{\frac{3}{2}}+\frac{2}{3}F'(0)(t-0)^{\frac{3}{2}}+\frac{2}{3}\int_0^t F''(\tau)(t-\tau)^{\frac{3}{2}}d\tau\Big]\\
&= \frac{4}{3\sqrt{\pi}}\frac{d}{dt}\Big[F'(0)\,t^{\frac{3}{2}}+\int_0^t F''(\tau)(t-\tau)^{\frac{3}{2}}d\tau\Big]\\
&= \frac{4}{3\sqrt{\pi}}\frac{d}{dt}\Big[-\frac{\sqrt{\pi}}{2\sqrt{8}}\,t^{\frac{3}{2}}+\int_0^t F''(\tau)(t-\tau)^{\frac{3}{2}}d\tau\Big]\\
&= \frac{4}{3\sqrt{\pi}}\Big[-\frac{\sqrt{\pi}}{2\sqrt{8}}\frac{3}{2}\,\sqrt{t}+\frac{3}{2}\int_0^t F''(\tau)\,\sqrt{t-\tau}\,d\tau\Big]\\
&= \frac{2}{\sqrt{\pi}}\Big[-\frac{\sqrt{\pi}}{2\sqrt{8}}\,\sqrt{t}+\int_0^t F''(\tau)\,\sqrt{t-\tau}\,d\tau\Big].
\end{aligned}$$

Differentiating with respect to t, $(\Omega F)(t)$ can be expressed as

$$\begin{aligned}
\frac{d}{dt}(\Omega F)(t) &= \frac{2}{\sqrt{\pi}}\frac{d}{dt}\Big[-\frac{\sqrt{\pi}}{2\sqrt{8}}\,\sqrt{t}+\int_0^t F''(\tau)\,\sqrt{t-\tau}\,d\tau\Big]\\
&= \frac{2}{\sqrt{\pi}}\Big[-\frac{\sqrt{\pi}}{2\sqrt{8}}\,\frac{1}{2\sqrt{t}}+\int_0^t F''(\tau)\,\frac{1}{2\sqrt{t-\tau}}\,d\tau\Big]\\
&= \frac{1}{\sqrt{\pi}}\Big[-\frac{\sqrt{\pi}}{2\sqrt{8}}\,\frac{1}{\sqrt{t}}+\int_0^t \frac{F''(\tau)}{\sqrt{t-\tau}}\,d\tau\Big]\\
&= \frac{-1}{2\sqrt{8}}\,\frac{1}{\sqrt{t}}+\frac{1}{\sqrt{\pi}}\int_0^t \frac{F''(\tau)}{\sqrt{t-\tau}}\,d\tau.
\end{aligned}$$

Note that

$$\frac{1}{\sqrt{\pi}}\left|\int_0^t \frac{F''(\tau)}{\sqrt{t-\tau}}\,d\tau\right| \le \frac{B}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}\sqrt{t-\tau}}d\tau \le B\sqrt{\pi}, \quad \forall\, t>0.$$

Thus, $(\Omega F)(t)$ is continuously differentiable for all $t>0$. Hence, by standard results [Gorenflo and Vessella (1991), Theorem 5.1.4], there exists a unique continuously differentiable solution $\phi(t)$ for the integral equation (4.2.45)-(4.2.47).

With the existence of derivative of boundary function, $\phi(t)$, from theorem (4.2.2), the classical solution of (4.1.1)-(4.1.4) is well defined and given by

$$v(x,t) = \begin{cases} L(x,t), & -l \leq x < 0, \ t > 0, \\ R(x,t), & 0 < x \leq l, \ t > 0, \end{cases} \tag{4.2.49}$$

where $L(x,t)$ and $R(x,t)$ are given in (4.2.42) and (4.2.29) respectively. Further note that $v(x,t)$ is continuous when $x = 0$.

Remark. Using the results given in Gorenflo and Vessella (1991), we state the bounds for the function $\phi(t)$ in $H\ddot{o}lder'$s norm as follows:

$$\|\phi(t)\|_{C^1(0,\infty)} \leq \Delta(\sigma_2) \ \|(\Omega F)(t)\|_{C^1(0,\infty)},$$

where $\sigma_2 = \|\hat{K}\|_{C^2(0,\infty)}$ and $\Delta(\sigma_2)$ is a constant depending on σ_2.

4.3 Asymptotic behavior of the solutions

Theorem 4.3.1 (Extended Final Value Theorem. Chen et al. (2007))**.** *Assume that every pole of $Y(s)$ is either in the open left half plane or at the origin. Then* $\lim_{t\to\infty} y(t)$ *exists and is given by*

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s),$$

where $Y(s) = \mathcal{L}\{y(t)\}$.

Theorem 4.3.2 (Generalized Final Value Theorem. Chen et al. (2007))**.** *Let $y(t)$ be Laplace transformable, let $\lambda > -1$, and assume that* $\lim_{t\to\infty} \frac{y(t)}{t^\lambda}$ *and* $\lim_{s\to 0} s^{\lambda+1}Y(s)$ *exist. Then*

$$\lim_{t\to\infty} \frac{y(t)}{t^\lambda} = \frac{1}{\Gamma(\lambda+1)} \lim_{s\to 0} s^{\lambda+1}Y(s),$$

where $Y(s) = \mathcal{L}\{y(t)\}$.

Theorem 4.3.3. *The derivative of boundary function, $\phi(t)$, converges to zero with rate of convergence*

$$\phi(t) = o(t^{-1}) \ \text{as} \ t \to \infty. \tag{4.3.50}$$

Proof. Let $\breve{G}(s)$, $J(s)$ and $\bar{K}(s)$ denotes the Laplace transformation of $\phi(t)$, $f(t)$ and difference kernel $K(t)$ respectively. Applying the Laplace transformation to the convolution of K and g in (4.2.45), one gets

$$\mathcal{L}\{\phi(t)\} = \frac{\mathcal{L}\{f(t)\}}{\mathcal{L}\{K(t)\}}. \quad \text{i.e.,} \quad \breve{G}(s) = \frac{J(s)}{\bar{K}(s)}. \tag{4.3.51}$$

Now we calculate the Laplace transformation of kernel $K(t)$:

$$\begin{aligned}
\bar{K}(s) &= \mathcal{L}\bigg\{\Big(e^{-\frac{t}{2}}+1\Big)\Big(\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2 t}\Big)\bigg\} \\
&= \mathcal{L}\bigg\{\frac{e^{-\frac{t}{2}}}{l}+\frac{2}{l}e^{-\frac{t}{2}}\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2 t}+\frac{1}{l}+\frac{2}{l}e^{-\frac{t}{2}}\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2 t}\bigg\} \\
&= \frac{1}{l}\mathcal{L}\{e^{-\frac{t}{2}}\}+\frac{2}{l}\mathcal{L}\bigg\{e^{-\frac{t}{2}}\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2 t}\bigg\}+\frac{1}{l}\mathcal{L}\{1\}+\frac{2}{l}\mathcal{L}\bigg\{\sum_{n=1}^{\infty}e^{-(\frac{n\pi}{l})^2 t}\bigg\} \\
&= \frac{1}{l}\frac{1}{\Big(s+\frac{1}{2}\Big)}+\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\Big(s+\frac{1}{2}+(\frac{n\pi}{l})^2\Big)}+\frac{1}{l}\frac{1}{s}+\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\Big(s+(\frac{n\pi}{l})^2\Big)} \\
&= \frac{1}{l}\Bigg[\frac{1}{(s+\frac{1}{2})}+\frac{1}{s}+2\Big(\sum_{n=1}^{\infty}\frac{1}{s+\frac{1}{2}+(\frac{n\pi}{l})^2}+\sum_{n=1}^{\infty}\frac{1}{s+(\frac{n\pi}{l})^2}\Big)\Bigg].
\end{aligned} \tag{4.3.52}$$

From the Handbook of linear partial differential equations Polyanin and Nazaikinskii (2015) we have

$$\sum_{k=1}^{\infty}\frac{1}{k^2+a^2} = \frac{\pi}{2\,a}\coth\left(\pi\,a\right)-\frac{1}{2\,a^2}.$$

To simplify (4.3.52), we calculate the limit of the series via the above formula:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{s+(\frac{n\pi}{l})^2} &= \frac{l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2+(\frac{l\sqrt{s}}{\pi})^2} \\
&= \frac{l^2}{\pi^2}\left[\frac{\pi}{2(\frac{l\sqrt{s}}{\pi})}\coth\left(\pi\frac{l\sqrt{s}}{\pi}\right)-\frac{1}{2(\frac{l\sqrt{s}}{\pi})^2}\right] \\
&= \frac{l^2}{\pi^2}\left[\frac{\pi^2}{2l\sqrt{s}}\coth\left(l\sqrt{s}\right)-\frac{\pi^2}{2\,l^2\,s}\right] \\
&= \frac{l}{2\sqrt{s}}\coth\left(l\sqrt{s}\right)-\frac{1}{2\,s} \\
&= \frac{1}{2}\left[\frac{l}{\sqrt{s}}\coth\left(l\sqrt{s}\right)-\frac{1}{s}\right].
\end{aligned}$$

Similarly, we get

$$\sum_{n=1}^{\infty} \frac{1}{s+\frac{1}{2}+(\frac{n\pi}{l})^2} = \frac{1}{2}\left[\frac{l}{\sqrt{s+\frac{1}{2}}} \coth\left(l\sqrt{s+\frac{1}{2}}\right) - \frac{1}{(s+\frac{1}{2})}\right].$$

Substituting the above in (4.3.52), we obtain

$$\begin{aligned}\bar{K}(s) &= \frac{1}{l}\left[\frac{1}{(s+\frac{1}{2})} + \frac{1}{s} + \frac{l}{\sqrt{s+\frac{1}{2}}} \coth\left(l\sqrt{s+\frac{1}{2}}\right)\right. \\ &\qquad \left. - \frac{1}{(s+\frac{1}{2})} + \frac{l}{\sqrt{s}} \coth\left(l\sqrt{s}\right) - \frac{1}{s}\right] \\ &= \frac{1}{l}\left[\frac{l}{\sqrt{s+\frac{1}{2}}} \coth\left(l\sqrt{s+\frac{1}{2}}\right) + \frac{l}{\sqrt{s}} \coth\left(l\sqrt{s}\right)\right].\end{aligned}$$

i.e.,

$$\mathcal{L}\{K(t)\} = \bar{K}(s) = \frac{1}{\sqrt{s+\frac{1}{2}}} \coth\left(l\sqrt{s+\frac{1}{2}}\right) + \frac{1}{\sqrt{s}} \coth\left(l\sqrt{s}\right). \tag{4.3.53}$$

Now,

$$\begin{aligned}\mathcal{L}\{f(t)\} = J(s) &= \mathcal{L}\left\{\left(e^{-\frac{t}{2}} - 1\right)e^{-(\frac{m\pi}{l})^2 t}\right\} \\ &= \mathcal{L}\left\{e^{-\frac{t}{2}} e^{-(\frac{m\pi}{l})^2 t}\right\} - \mathcal{L}\left\{e^{-(\frac{m\pi}{l})^2 t}\right\} \\ &= \frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} - \frac{1}{s+(\frac{m\pi}{l})^2}.\end{aligned}$$

i.e.,

$$J(s) = \frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} - \frac{1}{s+(\frac{m\pi}{l})^2}. \tag{4.3.54}$$

Note that

$$J(0) = \frac{1}{\frac{1}{2}+(\frac{m\pi}{l})^2} - \frac{1}{(\frac{m\pi}{l})^2} \neq 0, \quad \text{a non-zero real number.}$$

Hence, substituting (4.3.53) and (4.3.54) in (4.3.51) will lead us to get the Laplace transformation of $\phi(t)$, given by

$$
\begin{aligned}
\mathcal{L}\{\phi(t)\} &= \frac{\left[\frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} - \frac{1}{s+(\frac{m\pi}{l})^2}\right]}{\frac{1}{\sqrt{s+\frac{1}{2}}}\coth\left(l\sqrt{s+\frac{1}{2}}\right) + \frac{1}{\sqrt{s}}\coth\left(l\sqrt{s}\right)} \\
&= \frac{\left[\frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} - \frac{1}{s+(\frac{m\pi}{l})^2}\right]\sqrt{s}\sqrt{s+\frac{1}{2}}}{\sqrt{s}\coth\left(l\sqrt{s+\frac{1}{2}}\right) + \sqrt{s+\frac{1}{2}}\coth\left(l\sqrt{s}\right)} \\
&= \frac{\sqrt{s}\sqrt{s+\frac{1}{2}}\left[\frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} - \frac{1}{s+(\frac{m\pi}{l})^2}\right]}{\sqrt{s}\left(\frac{e^{2l\sqrt{s+\frac{1}{2}}}+1}{e^{2l\sqrt{s+\frac{1}{2}}}-1}\right) + \sqrt{s+\frac{1}{2}}\left(\frac{e^{2l\sqrt{s}}+1}{e^{2l\sqrt{s}}-1}\right)} \\
&= \frac{\sqrt{s}\sqrt{s+\frac{1}{2}}\left[\frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} - \frac{1}{s+(\frac{m\pi}{l})^2}\right]\left(e^{2l\sqrt{s}}-1\right)\left(e^{2l\sqrt{s+\frac{1}{2}}}-1\right)}{\sqrt{s}\left(e^{2l\sqrt{s+\frac{1}{2}}}+1\right)\left(e^{2l\sqrt{s}}-1\right) + \sqrt{s+\frac{1}{2}}\left(e^{2l\sqrt{s}}+1\right)\left(e^{2l\sqrt{s+\frac{1}{2}}}-1\right)} \\
\breve{G}(s) &=: \frac{\sqrt{s}\sqrt{s+\frac{1}{2}}\,A(s)}{B(s)} = \frac{Z(s)}{B(s)} \quad \text{(say)},
\end{aligned}
\tag{4.3.55}
$$

where

$$
A(s) = J(s)\ \left(e^{2l\sqrt{s}}-1\right)\left(e^{2l\sqrt{s+\frac{1}{2}}}-1\right), \tag{4.3.56}
$$

$$
Z(s) = \sqrt{s}\sqrt{s+\frac{1}{2}}\,A(s), \tag{4.3.57}
$$

and

$$
B(s) = \sqrt{s}\left(e^{2l\sqrt{s+\frac{1}{2}}}+1\right)\left(e^{2l\sqrt{s}}-1\right) + \sqrt{s+\frac{1}{2}}\left(e^{2l\sqrt{s}}+1\right)\left(e^{2l\sqrt{s+\frac{1}{2}}}-1\right). \tag{4.3.58}
$$

Note that the numerator, $Z(s)$, in the right side of the equation (4.3.55) approaches to *zero* as $s \to 0$ and the denominator $B(s)$ approaches to $\sqrt{2}(e^{\sqrt{2}l}-1)$, as $s \to 0$. Hence, we have $\breve{G}(s) \to 0$ as $s \to 0$. Also, $B(s)$ has no zeros which in turn $\breve{G}(s)$ has no poles. Thus, using Extended Final Value Theorem [Chen et al. (2007)], we conclude that $\phi(t)$ converges to *zero* as t approaches infinity. To evaluate the order of convergence we differentiate $\breve{G}(s)$ in its argument. Differentiating $A(s)$

given in (4.3.56) with respect to s we obtain

$$A'(s) = \frac{2\,l\,J(s)\,e^{2l\sqrt{s+\frac{1}{2}}}\big(e^{2l\sqrt{s}}-1\big)}{2\sqrt{s+\frac{1}{2}}} + \frac{2\,l\,J(s)\,e^{2l\sqrt{s}}\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)}{2\sqrt{s}}$$

$$+\big(e^{2l\sqrt{s}}-1\big)\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)\Bigg(\frac{1}{\big(s+\frac{1}{2}+(\frac{m\pi}{l})^2\big)^2} - \frac{1}{\big(s+(\frac{m\pi}{l})^2\big)^2}\Bigg)$$

$$= J(s)\Bigg[\frac{le^{2l\sqrt{s+\frac{1}{2}}}\big(e^{2l\sqrt{s}}-1\big)}{\sqrt{s+\frac{1}{2}}} + \frac{le^{2l\sqrt{s}}\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)}{\sqrt{s}}$$

$$+\big(e^{2l\sqrt{s}}-1\big)\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)\Bigg(\frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} + \frac{1}{s+(\frac{m\pi}{l})^2}\Bigg)\Bigg]$$

$$= J(s)\Bigg[I_1 + \frac{I_2}{\sqrt{s}} + I_3\Bigg], \quad \text{(say)},$$

where

$$I_1(s) = \frac{le^{2l\sqrt{s+\frac{1}{2}}}\big(e^{2l\sqrt{s}}-1\big)}{\sqrt{s+\frac{1}{2}}} \to 0 \quad \text{as} \quad s \to 0,$$

$$I_2(s) = le^{2l\sqrt{s}}\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big) \to l\,(e^{\sqrt{2}l}-1) \neq 0 \quad \text{as} \quad s \to 0,$$

and

$$I_3(s) = \big(e^{2l\sqrt{s}}-1\big)\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)\Bigg(\frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} + \frac{1}{s+(\frac{m\pi}{l})^2}\Bigg) \to 0 \text{ as } s \to 0.$$

Differentiating the numerator $Z(s)$ of $\breve{G}(s)$ and then substituting $A'(s)$ given above, we obtain

$$Z'(s) = \frac{\sqrt{s+\frac{1}{2}}A(s)}{2\sqrt{s}} + \frac{\sqrt{s}A(s)}{2\sqrt{s+\frac{1}{2}}} + \sqrt{s}\sqrt{s+\frac{1}{2}}\,A'(s)$$

$$= \frac{\sqrt{s+\frac{1}{2}}A(s)}{2\sqrt{s}} + \frac{\sqrt{s}A(s)}{2\sqrt{s+\frac{1}{2}}} + \sqrt{s}\sqrt{s+\frac{1}{2}}\,J(s)\Bigg[I_1 + \frac{I_2}{\sqrt{s}} + I_3\Bigg]$$

$$= \frac{\sqrt{s+\frac{1}{2}}A(s)}{2\sqrt{s}} + \frac{\sqrt{s}A(s)}{2\sqrt{s+\frac{1}{2}}} + \sqrt{s+\frac{1}{2}}\,J(s)\Big[\sqrt{s}I_1 + I_2 + \sqrt{s}I_3\Big].$$

Multiplying by s,

$$s\,Z'(s) = \frac{\sqrt{s}\sqrt{s+\frac{1}{2}}A(s)}{2} + \frac{s^{\frac{3}{2}}A(s)}{2\sqrt{s+\frac{1}{2}}} + s\sqrt{s+\frac{1}{2}}\,J(s)\Big[\sqrt{s}I_1 + I_2 + \sqrt{s}I_3\Big]$$

and letting $s \to 0$, we get

$$s\, Z'(s) \to 0 \quad \text{as} \quad s \to 0.$$

Since $B(s)$ approaches to $\sqrt{2}(e^{\sqrt{2}l} - 1) \neq 0$, we obtain

$$s\, B(s)\, Z'(s) \to 0 \quad \text{as} \quad s \to 0. \tag{4.3.59}$$

Now, differentiating $B(s)$ with respect to s, we obtain

$$\begin{aligned} B'(s) = {} & l\big(e^{2l\sqrt{s+\frac{1}{2}}} + 1\big)e^{2l\sqrt{s}} + \frac{l\sqrt{s}e^{2l\sqrt{s+\frac{1}{2}}}\big(e^{2l\sqrt{s}} - 1\big)}{\sqrt{s+\frac{1}{2}}} \\ & + l\big(e^{2l\sqrt{s}} + 1\big)e^{2l\sqrt{s+\frac{1}{2}}} + \frac{\big(e^{2l\sqrt{s+\frac{1}{2}}} - 1\big)\big(e^{2l\sqrt{s}} + 1\big)}{2\sqrt{s+\frac{1}{2}}} \\ & + \frac{1}{\sqrt{s}}\left[\frac{\big(e^{2l\sqrt{s+\frac{1}{2}}} + 1\big)\big(e^{2l\sqrt{s}} - 1\big)}{2} + l\sqrt{s+\frac{1}{2}}e^{2l\sqrt{s}}\big(e^{2l\sqrt{s+\frac{1}{2}}} - 1\big)\right]. \end{aligned}$$

Multiplying by s,

$$\begin{aligned} s\, B'(s) = {} & l\, s\big(e^{2l\sqrt{s+\frac{1}{2}}} + 1\big)e^{2l\sqrt{s}} + \frac{l\, s^{\frac{3}{2}}\, e^{2l\sqrt{s+\frac{1}{2}}}\big(e^{2l\sqrt{s}} - 1\big)}{\sqrt{s+\frac{1}{2}}} \\ & + l\, s\,\big(e^{2l\sqrt{s}} + 1\big)e^{2l\sqrt{s+\frac{1}{2}}} + \frac{s\,\big(e^{2l\sqrt{s+\frac{1}{2}}} - 1\big)\big(e^{2l\sqrt{s}} + 1\big)}{2\sqrt{s+\frac{1}{2}}} \\ & + \sqrt{s}\left[\frac{\big(e^{2l\sqrt{s+\frac{1}{2}}} + 1\big)\big(e^{2l\sqrt{s}} - 1\big)}{2} + l\sqrt{s+\frac{1}{2}}e^{2l\sqrt{s}}\big(e^{2l\sqrt{s+\frac{1}{2}}} - 1\big)\right] \end{aligned}$$

and letting $s \to 0$, we get

$$s\, B'(s) \to 0 \quad \text{as} \quad s \to 0. \tag{4.3.60}$$

Since the numerator, $Z(s)$, also approaches to *zero* as $s \to 0$, we obtain

$$s\, Z(s)\, B'(s) \to 0 \quad \text{as} \quad s \to 0. \tag{4.3.61}$$

Differentiating (4.3.55) and then multiplying by s, we have

$$s\, \breve{G}'(s) = \frac{s\, B(s)\, Z'(s) - s\, Z(s)\, B'(s)}{(B(s))^2}.$$

From (4.3.59), (4.3.61) and $B(s)$ tending to $\sqrt{2}(e^{\sqrt{2}l}-1)\neq 0$ as $s\to 0$, we get

$$s\,\breve{G}'(s)\to 0 \quad \text{as} \quad s\to 0.$$

From Extended Final Value Theorem Chen et al. (2007), we have

$$\lim_{t\to\infty} t\,\phi(t) = \lim_{s\to 0} s\,\mathcal{L}\{t\,\phi(t)\} = -\lim_{s\to 0} s\,\breve{G}'(s)$$

Hence, we obtain

$$\phi(t) = o(t^{-1}) \text{ as } t\to\infty.$$

□

Theorem 4.3.4. *Assume that Laplace transformation of $R_2(x,t)$ and $Q^x(t)$ exists. Then the solution $R(x,t)$ of the initial-boundary value problem (4.2.8)-(4.2.11) converges to zero with rate of convergence*

$$R(x,t) = o(t^{-1}) \text{ as } t\to\infty.$$

Proof: From (4.2.29) we have

$$\begin{aligned}
R(x,t) &= \int_0^l e^{-\frac{t}{2}}\cos\left(\frac{m\pi}{l}\xi\right) G(x,\xi,t)\,d\xi - \int_0^t e^{\frac{-(t-\tau)}{2}}\,G(x,0,t-\tau)\,\phi(\tau)\,d\tau \\
&= R_1(x,t) - R_2(x,t), \quad \text{(say).}
\end{aligned}$$

Clearly the first term $R_1(x,t)$ of the above equation converges to *zero* as $t\to\infty$. We see that

$$\begin{aligned}
R_1(x,t) &= \int_0^l e^{-\frac{t}{2}}\cos\left(\frac{m\pi}{l}\xi\right) G(x,\xi,t)\,d\xi \\
&= e^{-\frac{t}{2}}\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\left[\frac{1}{l}+\frac{2}{l}\sum_{n=1}^{\infty}\cos(\frac{n\pi}{l}x)\,\cos(\frac{n\pi}{l}\xi)\,e^{-(\frac{n\,\pi}{l})^2 t}\right]d\xi \\
&= \frac{e^{-\frac{t}{2}}}{l}\int_0^l \cos\,(\frac{m\pi}{l}\xi)d\xi \\
&\qquad + \frac{2e^{-\frac{t}{2}}}{l}\sum_{n=1}^{\infty}\left(\int_0^l \cos\,(\frac{m\pi}{l}\xi)\,\cos(\frac{n\pi}{l}x)\,\cos(\frac{n\pi}{l}\xi)d\xi\right)e^{-(\frac{n\,\pi}{l})^2 t} \\
&= \frac{2\,e^{-\frac{t}{2}}}{l}\sum_{n=1}^{\infty}\cos(\frac{n\pi}{l}x)\left(\int_0^l \cos\,(\frac{m\pi}{l}\xi)\,\cos(\frac{n\pi}{l}\xi)d\xi\right)e^{-(\frac{n\,\pi}{l})^2 t} \\
&= \frac{2\,e^{-\frac{t}{2}}}{l}\cos(\frac{m\,\pi}{l}x)\,\frac{l}{2}e^{-(\frac{m\,\pi}{l})^2 t} = e^{-\frac{t}{2}}e^{-(\frac{m\,\pi}{l})^2 t}\,\cos(\frac{m\,\pi}{l}x) \\
&= e^{-((\frac{m\,\pi}{l})^2+\frac{1}{2})t}\,\cos(\frac{m\,\pi}{l}x).
\end{aligned}$$

In fact, without evaluating the integral representation of $R_1(x,t)$, it can be seen that $R_1(x,t)$ converges to *zero* as $t \to \infty$ by applying the Dominated convergence theorem. Now, in the second term $R_2(x,t)$, we consider the space variable x as a parameter. But then, in this view, we can see $R_2(x,t)$ as convolution of boundary function ϕ with difference kernel $Q^x(t-\tau) = e^{\frac{-(t-\tau)}{2}}\, G(x,0,t-\tau)$;

$$R_2(x,t) = \int_0^t e^{\frac{-(t-\tau)}{2}}\, G(x,0,t-\tau)\,\phi(\tau)\,d\tau = \int_0^t Q^x(t-\tau)\,\phi(\tau)\,d\tau.$$

Applying the Laplace transformation for the above convolution integral we get,

$$\mathcal{L}\{R_2(x,t)\} = \mathcal{L}\{Q^x(t)\}\,\mathcal{L}\{\phi(t)\}. \tag{4.3.62}$$

From (4.3.55) we have

$$\mathcal{L}\{\phi(t)\} = \frac{\sqrt{s}\,\sqrt{s+\frac{1}{2}}\,A(s)}{B(s)},$$

where $A(s)$ and $B(s)$ is given in (4.3.56) and (4.3.58) respectively. To calculate $\mathcal{L}\{R_2(x,t)\}$, we determine the Laplace transformation of the Kernel $Q^x(t)$:

$$\begin{aligned}
\mathcal{L}\{Q^x(t)\} &= \mathcal{L}\Big\{e^{-\frac{t}{2}}\Big[\frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos(\frac{n\pi}{l}x)\,e^{-(\frac{n\pi}{l})^2 t}\Big]\Big\} \\
&= \frac{1}{l}\Big[\mathcal{L}\{e^{-\frac{t}{2}}\} + 2\sum_{n=1}^{\infty}\cos(\frac{n\pi}{l}x)\mathcal{L}\Big\{e^{-\frac{t}{2}}\,e^{-(\frac{n\pi}{l})^2 t}\Big\}\Big] \\
&= \frac{1}{l}\Big[\frac{1}{s+\frac{1}{2}} + 2\sum_{n=1}^{\infty}\frac{\cos(\frac{n\pi}{l}x)}{s+\frac{1}{2}+(\frac{n\pi}{l})^2}\Big].
\end{aligned} \tag{4.3.63}$$

From the Handbook of linear partial differential equations Polyanin and Nazaikinskii (2015) we have

$$\sum_{k=1}^{\infty}\frac{\cos(k\,y)}{k^2+a^2} = \frac{\pi}{2\,a}\frac{\cosh\big(a(\pi-y)\big)}{\sinh\,(\pi\,a)} - \frac{1}{2\,a^2}, \quad 0 \le y \le 2\pi.$$

Let $y = \frac{\pi x}{l}$. But then note that $0 \le y \le \pi \le 2\pi$ as $0 \le x \le l$. Now consider

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\cos(\frac{n\pi}{l}x)}{s+\frac{1}{2}+(\frac{n\pi}{l})^2} &= \frac{l^2}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(n\,y)}{n^2+\left(\frac{l\sqrt{s+\frac{1}{2}}}{\pi}\right)^2}, \quad 0 \le y \le 2\pi \\
&= \frac{l^2}{\pi^2}\left[\frac{\pi}{2\left(\frac{l\sqrt{s+\frac{1}{2}}}{\pi}\right)} \frac{\cosh\left(\frac{l\sqrt{s+\frac{1}{2}}}{\pi}(\pi-y)\right)}{\sinh\left(\pi\frac{l\sqrt{s+\frac{1}{2}}}{\pi}\right)} - \frac{1}{2\left(\frac{l\sqrt{s+\frac{1}{2}}}{\pi}\right)^2}\right] \\
&= \frac{l^2}{2\,\pi^2}\left[\frac{\pi^2}{l\sqrt{s+\frac{1}{2}}} \frac{\cosh\left(\frac{l\sqrt{s+\frac{1}{2}}}{\pi}(\pi-\frac{\pi x}{l})\right)}{\sinh\left(l\sqrt{s+\frac{1}{2}}\right)} - \frac{\pi^2}{l^2(s+\frac{1}{2})}\right] \\
&= \frac{1}{2}\left[\frac{l}{\sqrt{s+\frac{1}{2}}} \frac{\cosh\left(\sqrt{s+\frac{1}{2}}\,(l-x)\right)}{\sinh\left(l\sqrt{s+\frac{1}{2}}\right)} - \frac{1}{(s+\frac{1}{2})}\right].
\end{aligned}$$

Substituting in(4.3.63),

$$\begin{aligned}
\mathcal{L}\{Q^x(t)\} &= \frac{1}{l}\left[\frac{1}{s+\frac{1}{2}} + \frac{l}{\sqrt{s+\frac{1}{2}}} \frac{\cosh\left(\sqrt{s+\frac{1}{2}}\,(l-x)\right)}{\sinh\left(l\sqrt{s+\frac{1}{2}}\right)} - \frac{1}{s+\frac{1}{2}}\right] \\
&= \frac{1}{\sqrt{s+\frac{1}{2}}} \frac{\cosh\left(\sqrt{s+\frac{1}{2}}\,(l-x)\right)}{\sinh\left(l\sqrt{s+\frac{1}{2}}\right)}.
\end{aligned} \tag{4.3.64}$$

In view of (4.3.55) and (4.3.64), (4.3.62) reduces to

$$\mathcal{L}\{R_2(x,t)\} = \Phi(s) = \frac{\cosh\left(\sqrt{s+\frac{1}{2}}\,(l-x)\right)}{\sinh\left(l\sqrt{s+\frac{1}{2}}\right)} \frac{\sqrt{s}\,A(s)}{B(s)} = \frac{E(s)}{W(s)} \text{ (say)}, \tag{4.3.65}$$

where

$$E(s) = \cosh\left(\sqrt{s+\frac{1}{2}}\,(l-x)\right)\sqrt{s}\,A(s) \tag{4.3.66}$$

and

$$W(s) = \sinh\left(l\sqrt{s+\frac{1}{2}}\right)B(s). \tag{4.3.67}$$

From the expression of $A(s)$ in (4.3.56), $\sqrt{s}\,A(s)$ converges to *zero* as $s \to 0$. In view of (4.3.58), we see that the denominator term of (4.3.65) converges to a non-zero real number as $s \to 0$. i.e.,

$$\sinh\big(l\sqrt{s+\frac{1}{2}}\big)B(s) \to \sqrt{2}\,\sinh\left(\frac{l}{\sqrt{2}}\right)(e^{\sqrt{2}\,l}-1) \quad \text{as} \quad s \to 0.$$

Therefore, from (4.3.65), we have $\mathcal{L}\{R_2(x,t)\} \to 0$ as $s \to 0$. Note that $s = -1/2$ satisfies $W(s)$ which in turn a simple pole of $\Phi(s)$ and lies in the left half-plane. Hence, using Extended Final Value Theorem we conclude that $R_2(x,t) \to 0$ as $t \to \infty$. Finally from the convergences of R_1 and R_2, we conclude that $R(x,t)$ converges to *zero* as t approaches ∞. To evaluate the order of convergence we calculate $\Phi'(s)$. Differentiating $W(s)$, we get

$$W'(s) = \sinh\big(l\sqrt{s+\frac{1}{2}}\big)\,B'(s) + \frac{lB(s)\cosh\big(l\sqrt{s+\frac{1}{2}}\big)}{2\sqrt{s+\frac{1}{2}}}.$$

From (4.3.60) we know that

$$s\,B'(s) \to 0 \quad \text{and} \quad B(s) \to \sqrt{2}(e^{\sqrt{2}l}-1) \quad \text{as} \quad s \to 0.$$

Hence, we have

$$s\,W'(s) \to 0 \quad \text{as} \quad s \to 0.$$

From (4.3.66), clearly $E(s)$ approaches to *zero* as s tends to 0. Hence we have

$$s\,E(s)\,W'(s) \to 0 \quad \text{as} \quad s \to 0. \tag{4.3.68}$$

Now, differentiating $E(s)$ with respect to s, we obtain

$$E'(s) = \cosh\big((l-x)\sqrt{s+\frac{1}{2}}\big)\,\frac{d}{ds}\big(\sqrt{s}\,A(s)\big) + \frac{\sqrt{s}\,A(s)\,(l-x)\sinh\big((l-x)\sqrt{s+\frac{1}{2}}\big)}{2\sqrt{s+\frac{1}{2}}}.$$

Multiplying by s,

$$s\,E'(s) = \cosh\big((l-x)\sqrt{s+\frac{1}{2}}\big)\left[s\,\frac{d}{ds}\big(\sqrt{s}\,A(s)\big)\right] + \frac{s\,\big(\sqrt{s}\,A(s)\big)(l-x)\sinh\big((l-x)\sqrt{s+\frac{1}{2}}\big)}{2\sqrt{s+\frac{1}{2}}}.$$

Differentiating $\sqrt{s}\,A(s)$, we get

$$\frac{d}{ds}\big(\sqrt{s}\,A(s)\big) = \sqrt{s}\;A'(s) + \frac{A(s)}{2\sqrt{s}}.$$

We have $\sqrt{s}\,A'(s) = J(s)\big[\sqrt{s}I_1+I_2+\sqrt{s}I_3\big] \to l\,J(0)\,(e^{\sqrt{2}l}-1) \neq 0$ and $\sqrt{s}\,A(s) \to 0$ as $s \to 0$. Hence, we obtain

$$s\,\frac{d}{ds}\big(\sqrt{s}\,A(s)\big) \to 0 \quad \text{as} \quad s \to 0,$$

which in turn leads us to get

$$s\,E'(s) \to 0 \quad \text{as} \quad s \to 0.$$

Clearly, we have

$$W(s) \to \sqrt{2}\,\sinh\left(\frac{l}{\sqrt{2}}\right)(e^{\sqrt{2}\,l}-1) \neq 0 \quad \text{as} \quad s \to 0.$$

Hence,

$$s\,W(s)\,E'(s) \to 0 \quad \text{as} \quad s \to 0. \tag{4.3.69}$$

Differentiating (4.3.65) and then multiplying by s, we have

$$s\,\Phi'(s) = \frac{s\,W(s)\,E'(s) - s\,E(s)\,W'(s)}{(W(s))^2}.$$

From (4.3.68), (4.3.69) and $W(s)$ approching to a nonzero real number as $s \to 0$, we get

$$s\,\Phi'(s) \to 0 \quad \text{as} \quad s \to 0.$$

Hence, From Extended Final Value Theorem [Chen et al. (2007)], we obtain that

$$R_2(x,t) = o(t^{-1}) \text{ as } t \to \infty.$$

Finally from the convergences of R_1 and R_2 we conclude that $R(x,t)$ converges to *zero* with rate of convergence

$$R(x,t) = o(t^{-1}) \text{ as } t \to \infty.$$

Theorem 4.3.5. *Assume that Laplace transformation of $L_2(x,t)$ and $P^x(t)$ exists. Then the solution $L(x,t)$ of (4.2.30)-(4.2.33) converges to zero and with sufficiently small $\epsilon > 0$*

$$L(x,t) = o(t^{\epsilon-1}) \textit{ as } t \to \infty.$$

Proof: From (4.2.42), the solution of (4.2.30)-(4.2.33) in the left side domain $\{-l \le x \le 0; t > 0\}$ is given by

$$\begin{aligned} L(x,t) &= \int_0^l \cos\left(\frac{m\pi}{l}\xi\right) G(-x,\xi,t)\,d\xi + \int_0^t \phi(t)\,G(-x,0,t-\tau)\,d\tau \\ &= L_1(x,t) + L_2(x,t). \end{aligned}$$

Clearly, the first term $L_1(x,t)$ of the above equation converges to *zero* as $t \to \infty$. Let us now evaluate the integral expression of $L_1(x,t)$:

$$\begin{aligned} L_1(x,t) &= \int_0^l \cos\left(\frac{m\pi}{l}\xi\right) G(-x,\xi,t)\,d\xi \\ &= \int_0^l \cos\left(\frac{m\pi}{l}\xi\right) \left[\frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos(\frac{n\pi}{l}x)\,\cos(\frac{n\pi}{l}\xi)\,e^{-(\frac{n\pi}{l})^2 t}\right] d\xi \\ &= \frac{1}{l}\int_0^l \cos\,(\frac{m\pi}{l}\xi)d\xi \\ &\quad + \frac{2}{l}\sum_{n=1}^{\infty}\left(\int_0^l \cos\,(\frac{m\pi}{l}\xi)\,\cos(\frac{n\pi}{l}x)\,\cos(\frac{n\pi}{l}\xi)d\xi\right)e^{-(\frac{n\pi}{l})^2 t} \\ &= \frac{2}{l}\sum_{n=1}^{\infty}\cos(\frac{n\pi}{l}x)\left(\int_0^l \cos\,(\frac{m\pi}{l}\xi)\,\cos(\frac{n\pi}{l}\xi)d\xi\right)e^{-(\frac{n\pi}{l})^2 t} \\ &= \frac{2}{l}\cos(\frac{m\,\pi}{l}x)\,\frac{l}{2}e^{-(\frac{m\,\pi}{l})^2 t} \\ &= e^{-(\frac{m\,\pi}{l})^2 t}\,\cos(\frac{m\,\pi}{l}x). \end{aligned}$$

In fact, without evaluating the integral representation of $L_1(x,t)$, it can be seen that $L_1(x,t)$ converges to *zero* as $t \to \infty$ by applying the Dominated convergence theorem. To evaluate the second term $L_2(x,t)$, we consider the space variable x as a parameter. But then, in this view, we can see $L_2(x,t)$ as convolution of ϕ with difference kernel $P^x(t-\tau) = G(-x,0,t-\tau)$. i.e.,

$$L_2(x,t) = \int_0^t G(-x,0,t-\tau)\,\phi(\tau)\,d\tau = \int_0^t P^x(t-\tau)\,\phi(\tau)\,d\tau.$$

Applying the Laplace transformation for the above convolution integral we get,

$$\mathcal{L}\{L_2(x,t)\} = \mathcal{L}\{P^x(t)\}\,\mathcal{L}\{\phi(t)\}. \tag{4.3.70}$$

From (4.3.55) the Laplace transformation of $\phi(t)$ is given by

$$\mathcal{L}\{\phi(t)\} = \frac{\sqrt{s}\,\sqrt{s+\frac{1}{2}}\,A(s)}{B(s)}.$$

where $A(s)$ and $B(s)$ is given in (4.3.56) and (4.3.58) respectively. To calculate $\mathcal{L}\{L_2(x,t)\}$, we calculate the Laplace transformation of the Kernel $P^x(t)$:

$$\begin{aligned}\mathcal{L}\{P^x(t)\} &= \mathcal{L}\left\{\frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos(\frac{n\pi}{l}x)\, e^{-(\frac{n\pi}{l})^2 t}\right\} \\ &= \frac{1}{l}\left[\mathcal{L}\{1\} + 2\sum_{n=1}^{\infty}\cos(\frac{n\pi}{l}x)\mathcal{L}\left\{e^{-(\frac{n\pi}{l})^2 t}\right\}\right] \\ &= \frac{1}{l}\left[\frac{1}{s} + 2\sum_{n=1}^{\infty}\frac{\cos(\frac{n\pi}{l}x)}{s+(\frac{n\pi}{l})^2}\right].\end{aligned} \tag{4.3.71}$$

Let $y = \frac{\pi x}{l}$. But then note that as $-l \le x \le 0$ we have $-2\pi \le -\pi \le y \le 0$. Consider

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\cos(\frac{n\pi}{l}x)}{s+(\frac{n\pi}{l})^2} &= \frac{l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\,y)}{n^2+\left(\frac{l\sqrt{s}}{\pi}\right)^2} \\ &= \frac{l^2}{\pi^2}\left[\frac{\pi}{2\left(\frac{l\sqrt{s}}{\pi}\right)}\frac{\cosh\left(\frac{l\sqrt{s}}{\pi}(\pi-y)\right)}{\sinh\left(\pi\frac{l\sqrt{s}}{\pi}\right)} - \frac{1}{2\left(\frac{l\sqrt{s}}{\pi}\right)^2}\right] \\ &= \frac{l^2}{2\,\pi^2}\left[\frac{\pi^2}{l\sqrt{s}}\frac{\cosh\left(\frac{l\sqrt{s}}{\pi}(\pi-\frac{\pi x}{l})\right)}{\sinh\left(l\sqrt{s}\right)} - \frac{\pi^2}{l^2\,s}\right] \\ &= \frac{1}{2}\left[\frac{l}{\sqrt{s}}\frac{\cosh\left(\sqrt{s}\,(l-x)\right)}{\sinh\left(l\sqrt{s}\right)} - \frac{1}{s}\right].\end{aligned}$$

Substituting in (4.3.71) yields

$$\begin{aligned}\mathcal{L}\{P^x(t)\} &= \frac{1}{l}\left[\frac{1}{s} + \frac{l}{\sqrt{s}}\frac{\cosh\left(\sqrt{s}\,(l-x)\right)}{\sinh\left(l\sqrt{s}\right)} - \frac{1}{s}\right] \\ &= \frac{1}{\sqrt{s}}\frac{\cosh\left(\sqrt{s}\,(l-x)\right)}{\sinh\left(l\sqrt{s}\right)}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{L}\{L_2(x,t)\} &= \frac{1}{\sqrt{s}}\frac{\cosh\left(\sqrt{s}\,(l-x)\right)}{\sinh\left(l\sqrt{s}\right)}\frac{\sqrt{s}\sqrt{s+\frac{1}{2}}\,A(s)}{B(s)} \\ &= \frac{\cosh\left(\sqrt{s}\,(l-x)\right)}{\sinh\left(l\sqrt{s}\right)}\frac{\sqrt{s+\frac{1}{2}}\,A(s)}{B(s)},\end{aligned} \tag{4.3.72}$$

where $A(s)$ and $B(s)$ are given by (4.3.56) and (4.3.58). Note that, from the definition of $A(s)$, the numerator converges to *zero* as s tends to *zero*, where the denominator $B(s) \to \sqrt{2}\,(e^{\sqrt{2}\,l} - 1)$ as $s \to 0$. But the presence of "sinh" function makes the denominator term converges to *zero*. Thus, we obtain both the numerator and denominator terms of $\mathcal{L}\{L_2(x,t)\}$ converges to *zero*. Hence, we apply L'Hospital's rule to obtain the limit. For simplification purpose we write

$$\mathcal{L}\{L_2(x,t)\} = \frac{\cosh\left(\sqrt{s}\,(l-x)\right)}{\sinh\left(l\sqrt{s}\right)} \frac{\sqrt{s+\frac{1}{2}}\,A(s)}{B(s)} = \frac{N(s)}{D(s)} \quad \text{(say)},$$

where the numerator is given by

$$\begin{aligned} N(s) &= \cosh\left(\sqrt{s}\,(l-x)\right)\sqrt{s+\frac{1}{2}}\,A(s) \\ &= J(s)\cosh\left(\sqrt{s}\,(l-x)\right)\sqrt{s+\frac{1}{2}}\left(e^{2l\sqrt{s}}-1\right)\left(e^{2l\sqrt{s+\frac{1}{2}}}-1\right), \end{aligned}$$

with

$$J(s) = \left[\frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} - \frac{1}{s+(\frac{m\pi}{l})^2}\right]$$

and the denominator is given by

$$D(s) = \sinh\left(l\sqrt{s}\right)\left[\sqrt{s}\left(e^{2l\sqrt{s+\frac{1}{2}}}+1\right)\left(e^{2l\sqrt{s}}-1\right)+\sqrt{s+\frac{1}{2}}\left(e^{2l\sqrt{s}}+1\right)\left(e^{2l\sqrt{s+\frac{1}{2}}}-1\right)\right].$$

Differentiating the numerator $N(s)$ with respect to s, we get

$$\begin{aligned} N'(s) = J(s)\sqrt{s+\frac{1}{2}}\left(e^{2l\sqrt{s+\frac{1}{2}}}-1\right)&\left[\frac{l\,\cosh(\sqrt{s}\,(l-x))e^{2l\sqrt{s}}}{\sqrt{s}}\right. \\ &+ \left.\frac{(l-x)\sinh(\sqrt{s}\,(l-x))\left(e^{2l\sqrt{s}}-1\right)}{2\sqrt{s}}\right] + \frac{N(s)}{J(s)}\left[\frac{J(s)\,l\,e^{2l\sqrt{s+\frac{1}{2}}}}{\sqrt{s+\frac{1}{2}}\left(e^{2l\sqrt{s+\frac{1}{2}}}-1\right)}\right. \\ &\left.+\frac{J(s)}{2s+1} + \frac{1}{\left(s+(\frac{m\pi}{l})^2\right)^2} - \frac{1}{\left(s+\frac{1}{2}+(\frac{m\pi}{l})^2\right)^2}\right], \end{aligned}$$

where

$$J(s) = \frac{1}{s+\frac{1}{2}+(\frac{m\pi}{l})^2} - \frac{1}{s+(\frac{m\pi}{l})^2}.$$

By taking $\frac{1}{\sqrt{s}}$ as common term in the above expression, we rewrite $N'(s)$ as:

$$\begin{aligned}
N'(s) = \frac{1}{\sqrt{s}}\bigg\{ & \frac{J(s)}{2}\sqrt{s+\frac{1}{2}}\,\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)\Big[2\,l\,\cosh(\sqrt{s}\,(l-x))e^{2l\sqrt{s}} \\
& + (l-x)\sinh(\sqrt{s}\,(l-x))\big(e^{2l\sqrt{s}}-1\big)\Big] + \frac{\sqrt{s}N(s)}{J(s)}\Bigg[\frac{J(s)\,l\,e^{2l\sqrt{s+\frac{1}{2}}}}{\sqrt{s+\frac{1}{2}}\,\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)} \\
& + \frac{J(s)}{2\,s+1} + \frac{1}{\big(s+(\frac{m\,\pi}{l})^2\big)^2} - \frac{1}{\big(s+\frac{1}{2}+(\frac{m\,\pi}{l})^2\big)^2}\Bigg]\bigg\} \\
= \frac{1}{\sqrt{s}}\,N_1(s) \quad & \text{(say)}.
\end{aligned}$$

Also, note that

$$\text{as} \quad s \to 0,\ N_1(s) \to \frac{l\,J(0)\,\big(e^{\sqrt{2}l}-1\big)}{\sqrt{2}} \neq 0,\ \text{a real number.} \tag{4.3.73}$$

Now, we differentiate the denominator $D(s)$ with respect to s, we have

$$\begin{aligned}
D'(s) = & \frac{lB(s)\cosh(l\sqrt{s})}{2\,\sqrt{s}} + \sinh(l\,\sqrt{s})\Bigg[\big(e^{2l\sqrt{s+\frac{1}{2}}}+1\big)l\,e^{2l\sqrt{s}} \\
& + \frac{l\sqrt{s}\,e^{2l\sqrt{s+\frac{1}{2}}}\big(e^{2l\sqrt{s}}-1\big)}{\sqrt{s+\frac{1}{2}}} + l\,e^{2l\sqrt{s+\frac{1}{2}}}\big(e^{2l\sqrt{s}}+1\big) \\
& + \frac{\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)\big(e^{2l\sqrt{s}}+1\big)}{2\sqrt{s+\frac{1}{2}}} + \frac{1}{\sqrt{s}}\bigg\{\frac{\big(e^{2l\sqrt{s+\frac{1}{2}}}+1\big)\big(e^{2l\sqrt{s}}-1\big)}{2} \\
& + l\sqrt{s+\frac{1}{2}}\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)e^{2l\sqrt{s}}\bigg\}\Bigg].
\end{aligned}$$

By taking $\frac{1}{\sqrt{s}}$ as common term in $D'(s)$ in the above expression, we rewrite $D'(s)$ as:

$$\begin{aligned}
D'(s) = \frac{1}{\sqrt{s}}\bigg\{ & \frac{lB(s)\cosh(l\sqrt{s})}{2} + \sinh(l\,\sqrt{s})\Bigg[l\sqrt{s}\big(e^{2l\sqrt{s+\frac{1}{2}}}+1\big)\,e^{2l\sqrt{s}} \\
& + \frac{l\,s\,e^{2l\sqrt{s+\frac{1}{2}}}\big(e^{2l\sqrt{s}}-1\big)}{\sqrt{s+\frac{1}{2}}} + l\,\sqrt{s}e^{2l\sqrt{s+\frac{1}{2}}}\big(e^{2l\sqrt{s}}+1\big) \\
& + \frac{\sqrt{s}\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)\big(e^{2l\sqrt{s}}+1\big)}{2\sqrt{s+\frac{1}{2}}} + \frac{\big(e^{2l\sqrt{s+\frac{1}{2}}}+1\big)\big(e^{2l\sqrt{s}}-1\big)}{2} \\
& + l\sqrt{s+\frac{1}{2}}\big(e^{2l\sqrt{s+\frac{1}{2}}}-1\big)e^{2l\sqrt{s}}\Bigg]\bigg\} \\
= \frac{1}{\sqrt{s}}\,D'(s) \quad & \text{(say)}.
\end{aligned} \tag{4.3.74}$$

Note that

$$\text{as} \quad s \to 0, \; D_1(s) \to \frac{l\left(e^{\sqrt{2}l} - 1\right)}{\sqrt{2}} \neq 0, \; \text{a real number.} \tag{4.3.75}$$

By L'Hospital's rule,

$$\begin{aligned}
\lim_{s\to 0} \mathcal{L}\{L_2(x,t)\} &= \lim_{s\to 0} \frac{N'(s)}{D'(s)} = \lim_{s\to 0} \frac{\frac{1}{\sqrt{s}} N_1(s)}{\frac{1}{\sqrt{s}} D_1(s)} \\
&= \lim_{s\to 0} \frac{N_1(s)}{D_1(s)} \\
&= \frac{\left[\frac{l\,J(0)\left(e^{\sqrt{2}l}-1\right)}{\sqrt{2}}\right]}{\left[\frac{l\left(e^{\sqrt{2}l}-1\right)}{\sqrt{2}}\right]} = J(0) \\
&= \frac{1}{\frac{1}{2} + (\frac{m\pi}{l})^2} - \frac{1}{(\frac{m\pi}{l})^2} \neq 0, \; \text{a real number.}
\end{aligned}$$

Hence, we have

$$\text{as } s \to 0, \; s^{\epsilon}\mathcal{L}\{L_2(x,t)\} \to 0, \quad \text{for any} \quad \epsilon > 0. \tag{4.3.76}$$

Note that in (4.3.72), $\mathcal{L}\{L_2(x,t)\}$ has a simple pole at the origin. From Generalized Final value theorem [Chen et al. (2007)],

$$\lim_{t\to\infty} \frac{L_2(x,t)}{t^{\epsilon-1}} = 0. \quad \text{i.e.,} \quad L_2(x,t) = o(t^{\epsilon-1}) \quad \text{as} \quad t \to \infty. \tag{4.3.77}$$

Finally from the convergences of L_1 and L_2 we conclude that $L(x,t)$ converges to 0 as t approaches ∞.

Chapter 5

Conclusions and future work

- We consider the non homogeneous Burgers' equation with time dependent point source with initial condition u_0 and showed that the continuous solution exists and converges to zero uniformly on compact sets. In this process, we encounter the associated initial boundary value problem involving Heaviside function in source term. It is interesting to see that the solution of the corresponding heat equation converges to a non zero quantity and using this solution we establish the unique weak solution to the Cauchy problem for Burgers' equation with Dirac delta function via inverse Hopf-Cole transformation. The continuity of the solution to Heat equation with Heaviside function is established via common boundary condition imposed and the existence of the same is done with the help of Abel's integral equation. Explicit formula to express the boundary function is found via classical Abel's integral equation. Also, the alternative expression for the same is also given using Laplace transformation.

- It will be more challenging to consider the Cauchy problem for Burgers' equation with time dependent point source in general. i.e.,

$$u_t + uu_x - u_{xx} \quad = \quad f(t)\,\delta(x), \quad x \in \mathbb{R}, \quad t > 0, \tag{5.0.1}$$

subject to the initial condition

$$u(x,0) \quad = \quad u_0(x), \quad x \in \mathbb{R}. \tag{5.0.2}$$

The conditions should be imposed on the general function $f(t)$ depending only on time variable t with Dirac delta function in the forcing term complicates the problem even more. Also, one can study the problem with different suitable initial conditions $u_0(x)$ to understand the behavior of the solutions at large time and continuity of the solution. Corresponding initial boundary value problems for the above in-homogeneous Burgers' equation is given by

$$\begin{aligned} R_t - R_{xx} &= -\frac{f(t)R}{2}, \quad 0 \leq x, \quad 0 < t, \\ R(x,0) &= \theta_0(x), \quad 0 \leq x, \\ R(0,t) &= g_1(t), \quad 0 < t. \end{aligned}$$

and

$$\begin{aligned} L_t - L_{xx} &= 0, \quad x \leq 0, \quad 0 < t, \\ L(x,0) &= \theta_0(x), \quad x \leq 0, \\ L(0,t) &= g_2(t), \quad 0 < t. \end{aligned}$$

The solutions of the above problems is given by

$$\begin{aligned} R(x,t) = e^{-\frac{h(t)}{2}} \Bigg[\frac{e^{\frac{h(0)}{2}}}{2\sqrt{\pi t}} \int_0^\infty \theta_0(\xi) \big[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}} \big] d\xi + \\ \int_0^t \big(g_1(\tau) e^{\frac{h(\tau)}{2}} \big)' \ \operatorname{erfc}\left(\frac{x}{2\sqrt{t-\tau}} \right) d\tau + g_1(0) e^{\frac{h(0)}{2}} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}} \right) \Bigg] \end{aligned}$$

and

$$\begin{aligned} L(x,t) = \frac{-1}{2\sqrt{\pi t}} \int_0^\infty \theta_0(-\xi) \big[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}} \big] d\xi \\ + \int_0^t g_2'(\tau) \operatorname{erfc}\left(\frac{-x}{2\sqrt{t-\tau}} \right) d\tau + g_2(0) \operatorname{erfc}\left(\frac{-x}{2\sqrt{t}} \right) \end{aligned}$$

respectively. If the same boundary data, i.e., $g_1(t) = g_2(t)$, is considered in the associated initial boundary problems, the work on existence, uniqueness and asymptotic behavior of the boundary function will be more fruitful to the field of non homogeneous Burgers' equation. It is also interesting to consider distinct boundary conditions at the initial boundary problems (as shown above) occurred in the first two quadrants due to unit step function and see the relation between the same to establish the continuity of the solution to corresponding heat equation with source term.

- Chapter 4 is devoted to examine whether the solution exists for the Heat equation with a discontinuous function like unit step function in the source term, the continuity of solution due to the condition imposed at $x = 0$ and the large time behavior of the solutions in the corresponding domains. The presence of unit step function splits the problem space in a strip of length l each in first two quadrants. Hence, one needs to study the problem on two upper quarter plane slits separately with common condition at $x = 0$ along the positive t-axis.

  With the help of Volterra's integral equation of first kind with difference kernel, we aspire to establish the existence and uniqueness of the common condition, derivative of boundary function, enforced at $x = 0$ to the linearized partial differential equations. For the same we rearrange the Volterra's integral into Abel's integral equation of first kind where the kernel can be expressed in terms of Jacobi theta function. Further, we seek the asymptotic behavior to the solution of linearized partial differential equation for large time t. Making use of the Laplace transformation techniques on convolution integral and final value theorems we observe the rate of convergence as t tends to ∞.

- It will be interesting to consider the problem of heat equation with discontinuous Heaviside function in source term equipped with any suitable initial data $v_0(x)$ and secondary boundary data. i.e.,

$$\begin{aligned}
v_t &= v_{xx} - \frac{H(x)}{2} v, \quad -l \le x \le l, \quad t > 0, && (5.0.3)\\
v(x,0) &= v_0(x), \quad -l \le x \le l, && (5.0.4)\\
v_x(-l,t) &= b_1(t), \quad t > 0, && (5.0.5)\\
v_x(l,t) &= b_2(t), \quad t > 0, && (5.0.6)
\end{aligned}$$

  where $H(x)$ is the Heaviside function. It is notable that in chapter 4, the problem is considered with cosine function in the initial data. With the same approach, for some extent we can solve the problem with *even function* in the initial condition. It will be fascinating to see the problem with other class of initial data also. Also, choosing appropriate conditions on $b_1(t)$ and $b_2(t)$ to establish the necessary properties to the solutions is very challenging.

Further, we know that study of Burgers' equation and Heat equation holds recognizable importance in the fields of partial differential equations due to its numerous applications. In recent times, the study of Burgers' equation with Dirac delta measure by Ablowitz and De Lillo (1991, 1993, 1996) started a progressive discussion in the field of non homogeneous Partial Differential Equations by considering Burgers' equation with continuous time dependent function and Dirac delta function in forcing term. They could only impose zero initial data due to the level of difficulty occurred. The struggling however met by Chung et al. (2014) for some extent by considering time independent Dirac delta function in the force term. The relation between linear Heat equation with source term via Cole-Hopf transformation with nonlinear Burgers' equation can be seen in the process. In fact, the relation can be described as follows:

$$
\begin{aligned}
u_t + \left(\frac{u^2}{2}\right)_x &= u_{xx} + \delta(x), \quad -l \leq x \leq l, \quad t > 0 \\
u(x,0) &= u_0(x), \quad -l \leq x \leq l \\
u(-l,t) &= 0, \quad t > 0, \\
u(l,t) &= 0, \quad t > 0.
\end{aligned}
$$

Applying the Cole-Hopf transformation to the above equation $u = \dfrac{-2v_x}{v}$, we obtain the corresponding heat equation with source term involving discontinuous Heaviside function. Also, the problem is imposed with initial condition and second type boundary condition (Neumann condition) given by

$$
\begin{aligned}
v_t &= v_{xx} - \frac{H(x)}{2} v, \quad -l \le x \le l, \quad t > 0, \\
v(x,0) &= v_0(x), \quad -l \le x \le l, \\
v_x(-l,t) &= 0, \quad t > 0, \\
v_x(l,t) &= 0, \quad t > 0,
\end{aligned}
$$

where $v_0(x) = \exp\left\{ -\frac{1}{2} \int u_0(y)dy \right\}$. The presence of Heaviside function immediately splits the problem space into two domains in the upper quadrants and the continuity of the solution in the whole domain is a challenge with newly imposed condition at $x = 0$. Now, the above heat like equation can be seen in two ways. In the first case, we can introduce a boundary condition at $x = 0$ which makes the above problem with mixed boundary condition. Otherwise, we can introduce derivative of the boundary function at $x = 0$ which amounts with the secondary type boundary condition or Neumann boundary condition.

In this view, study of the problem (5.0.3)-(5.0.6) will be significant to understand Burgers' equation with Dirac delta function and time dependent function in forcing term in a strip.

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