METRIC, SCHAUDER AND OPERATOR-VALUED FRAMES

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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Dedicated to the memory of

John von Neumann

(28 December 1903 - 08 February 1957)

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the research thesis entitled **METRIC, SCHAUDER AND OPERATOR-VALUED FRAMES** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me.* The material contained in this research thesis has not been submitted to any University or Institution for the award of any degree.

Place: NITK, Surathkal

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CERTIFICATE

This is to *certify* that the research thesis entitled **METRIC, SCHAUDER AND OPERATOR-VALUED FRAMES** submitted by **Mr. MAHESH KRISHNA K** (Register Number : 165106MA16F02) as the record of the research work carried out by him is *accepted as the research thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Place: NITK, Surathkal

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ABSTRACT

Notion of frames and Bessel sequences for metric spaces have been introduced. This notion is related with the notion of Lipschitz free Banach spaces. It is proved that every separable metric space admits a metric $\mathscr{M}_d$-frame. Through Lipschitz-free Banach spaces it is showed that there is a correspondence between frames for metric spaces and frames for subsets of Banach spaces. Several characterizations of metric frames are obtained. Stability results are also presented. Non linear multipliers are introduced and studied. This notion is connected with the notion of Lipschitz compact operators. Continuity properties of multipliers are discussed.

For a subclass of approximated Schauder frames for Banach spaces, characterization result is derived using standard Schauder basis for standard sequence spaces. Duals of a subclass of approximate Schauder frames are completely described. Similarity of this class is characterized and interpolation result is derived using orthogonality. A dilation result is obtained. A new identity is derived for Banach spaces which admit a homogeneous semi-inner product. Some stability results are obtained for this class.

A generalization of operator-valued frames for Hilbert spaces are introduced which unifies all the known generalizations of frames for Hilbert spaces. This notion has been studied in depth by imposing factorization property of the frame operator. Its duality, similarity and orthogonality are addressed. Connections between this notion and unitary representations of groups and group-like unitary systems are derived. Paley-Wiener theorem for this class are derived.

Keywords: Frame, Riesz basis, Bessel sequence, Lipschitz function, multiplier, operator-valued frame, metric space.

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CHAPTER 1

INTRODUCTION

1.1 GENERAL INTRODUCTION

A vector in a vector space is usually obtained as a linear combination of elements of a basis for the vector space. Thus a vector is fully known if we know the coefficients in the representation of it using basis elements. However, as dimension of the space increases, it is difficult to get the coefficients. Hence we look for nice spaces and certain bases which give coefficients of a given vector easily. For this purpose, Hilbert spaces and orthonormal bases become a very handy tool to obtain representation of a vector.
Orthonormal bases for Hilbert spaces have practical disadvantages. Since each coefficient in the expansion is very important, a small error in one of the coefficient leads to significant variation in the resultant vector and the actual vector. Thus we seek a collection in Hilbert space which gives representation as well as a small change in coefficient need not effect much to the original vector. This is where the theory of frames becomes important.
Historically it was Gabor (1946), who first studied representation of functions using translations and modulations of a single function (Feichtinger et al. (2009, 2007); Feichtinger and Strohmer (1998, 2003); Gröchenig (2001); Sondergaard (2007)). In 1947, Sz. Nagy studied sequences which are close to orthonormal bases using Paley-Wiener type results (de Sz. Nagy (1947)). Modern definition of frames was set by Duffin and Schaeffer (1952) in the study of sequences of type $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}, \lambda_n \in \mathbb{C}, x \in (-r,r), r > 0$. After this work, Young (1980) made an account of frames in his book 'An introduction to nonharmonic Fourier series'.
Paper of Daubechies, Grossmann, and Meyer (Daubechies et al. (1986)) triggered the area of frames for Hilbert spaces. Later, the paper of Benedetto and Fickus (2003) influenced the development of frame theory for finite dimensional Hilbert spaces. Today

theory of frames find its uses in many areas such as wireless communication (Strohmer (2001)), signal processing (Mallat (2009)), image processing (Donoho and Elad (2003)), sampling theory (Benedetto and Ferreira (2001)), filter banks (Fickus et al. (2013)), psycho acoustics (Balazs et al. (2017)), quantum design (Bodmann and Haas (2020)), quantum channels (Han and Juste (2019)), quantum optics (Jamioikowski (2010)), quantum measurement (Eldar and Forney (2002)), numerical approximation (Adcock and Huybrechs (2019)), Sigma-Delta quantization (Benedetto et al. (2006)), coding (Strohmer and Heath (2003)) and graph theory (Bodmann and Paulsen (2005)). For a comprehensive look on the theory of frames, we refer (Christensen (2016), Han and Larson (2000), Han et al. (2007), Casazza and Kutyniok (2013), Waldron (2018), Heil (2011), Okoudjou (2016), Pesenson et al. (2017)).

Since many spaces appearing both in theoretical and practicals are Banach spaces which may not be Hilbert spaces, there is a need for extending the notion of frames to Banach spaces. This was first done by Gröchenig (1991). After the study of several function spaces (Feichtinger (2015)), Gröchenig first studied the notion of an atomic decomposition for Banach spaces and then defined the notion of a Banach frame. Feichtinger and Gröchenig (1988, 1989a,b) in 90's developed a theory of atomic decompositions and frames for a large class of function spaces such as modulation spaces (Feichtinger (2006)) and coorbit spaces (Berge (2022)), via, group representations and projective representations.

Abstract study of atomic decompositions and frames for Banach spaces started from the fundamental paper (Casazza et al. (1999)). Further study and variations of the frames for Banach spaces are done in Carando et al. (2011), Terekhin (2010), Terekhin (2009), Terekhin (2004), Fornasier (2007), Casazza et al. (2005a), Stoeva (2009), Stoeva (2012), Aldroubi et al. (2008), Gröchenig (2004) and so on.

1.2 ORTHONORMAL BASES, RIESZ BASES, FRAMES AND BESSEL SEQUENCES FOR HILBERT SPACES

In the study of integral equations, Hilbert studied the space of square integrable sequences (Blanchard and Bruning (2003)). Later, John von Neumann (1930) formulated the notion of Hilbert spaces.

Definition 1.2.1. *(cf. Limaye (2014)) A vector space $\mathscr{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is said to be a* ***Hilbert space*** *if there exists a map $\langle\cdot,\cdot\rangle : \mathscr{H}\times\mathscr{H}\to\mathbb{K}$ such that the following axioms hold.*

(i) $\langle h,h\rangle \geq 0$, $\forall h\in\mathscr{H}$.

(ii) *If $h \in \mathscr{H}$ is such that $\langle h, h \rangle = 0$, then $h = 0$.*

(iii) $\langle h, h_1 \rangle = \overline{\langle h_1, h \rangle}$, $\forall h, h_1 \in \mathscr{H}$.

(iv) $\langle \alpha h + h_1, h_2 \rangle = \alpha \langle h, h_2 \rangle + \langle h_1, h_2 \rangle$, $\forall h, h_1, h_2 \in \mathscr{H}$, $\forall \alpha \in \mathbb{K}$.

(v) *$\mathscr{H}$ is complete with respect to the norm $\|h\| := \sqrt{\langle h, h \rangle}$.*

We now mention two important examples of Hilbert spaces.

Example 1.2.2. *(cf. Limaye (2014))*

(i) *Let $n \in \mathbb{N}$. The space $\mathbb{K}^n$ equipped with the inner product*

$$\langle (a_k)_{k=1}^n, (b_k)_{k=1}^n \rangle := \sum_{k=1}^{n} a_k \overline{b_k}, \quad \forall (a_k)_{k=1}^n, (b_k)_{k=1}^n \in \mathbb{K}^n$$

is a finite dimensional separable Hilbert space.

(ii) *The space $\ell^2(\mathbb{N}) := \{\{a_n\}_n : a_n \in \mathbb{K}, \forall n \in \mathbb{N}, \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$ equipped with the inner product*

$$\langle \{a_n\}_n, \{b_n\}_n \rangle := \sum_{n=1}^{\infty} a_n \overline{b_n}, \quad \forall \{a_n\}_n, \{b_n\}_n \in \ell^2(\mathbb{N})$$

is an infinite dimensional separable Hilbert space. The space $\ell^2(\mathbb{N})$ is known as the standard separable Hilbert space.

Throughout this thesis, we assume that all of Hilbert spaces are separable. Among all kinds of sets in a Hilbert space, orthonormal sets are the easiest to handle, whose definition reads as follows.

Definition 1.2.3. *(cf. Limaye (2014)) A collection $\{\tau_n\}_n$ in a Hilbert space $\mathscr{H}$ is called an* ***orthonormal set*** *in $\mathscr{H}$ if $\langle \tau_j, \tau_k \rangle = \delta_{j,k}, \forall j, k \in \mathbb{N}$.*

Following theorem, known as Gram-Schmidt orthonormalization (cf. Leon et al. (2013)) shows that a linearly independent sequence of vectors can be converted into an orthonormal set such that at each stage of conversion the spaces spanned by the original set and transformed set are the same.

Theorem 1.2.4. *(cf. Limaye (2014)) (**Gram-Schmidt orthonormalization**) Let $\{\tau_n\}_n$ be a linearly independent subset of $\mathscr{H}$. Define $\omega_1 := \tau_1$, $\rho_1 := \omega_1 / \|\omega_1\|$ and*

$$\omega_n := \tau_n - \sum_{k=1}^{n-1} \langle \tau_n, \rho_k \rangle \rho_k, \quad \rho_n := \frac{\omega_n}{\|\omega_n\|}, \quad \forall n \geq 2.$$

Then $\{\rho_n\}_n$ is orthonormal and

$$\operatorname{span}\{\rho_k\}_{k=1}^{n} = \operatorname{span}\{\tau_k\}_{k=1}^{n}, \quad \forall n \geq 1.$$

One of the most important inequalities associated with an orthonormal sequence is the Bessel's inequality. It is a generalization of Cauchy-Schwarz inequality.

Theorem 1.2.5. *(cf. Limaye (2014)) (**Bessel's inequality**) If $\{\tau_n\}_n$ is an orthonormal set in $\mathscr{H}$, then the series $\sum_{n=1}^{\infty} |\langle h, \tau_n\rangle|^2$ converges for all $h \in \mathscr{H}$ and*

$$\sum_{n=1}^{\infty} |\langle h, \tau_n\rangle|^2 \leq \|h\|^2, \quad \forall h \in \mathscr{H}.$$

Next theorem characterizes convergence of series in a Hilbert space with that of sequence of scalars.

Theorem 1.2.6. *(cf. Limaye (2014)) (**Riesz-Fisher theorem**) Let $\{a_n\}_n$ be a sequence of scalars and $\{\tau_n\}_n$ be an orthonormal set in $\mathscr{H}$. Then*

$$\sum_{n=1}^{\infty} a_n\tau_n \text{ converges in } \mathscr{H} \text{ if and only if } \sum_{n=1}^{\infty} |a_n|^2 \text{ converges in } \mathbb{R}.$$

Next theorem shows that given any orthonormal set and an element in a Hilbert space, the inner product of the element with the members of an orthonormal set can be non zero at most countably many times.

Theorem 1.2.7. *(cf. Limaye (2014)) Let $\{\tau_n\}_n$ be an orthonormal set in $\mathscr{H}$ and $h \in \mathscr{H}$. Then the set $E_h := \{\tau_n : \langle h, \tau_n\rangle \neq 0, n \in \mathbb{N}\}$ is either finite or countable.*

A natural analogue of basis for finite dimensional vector spaces to that of infinite dimensional Hilbert spaces is the notion of Schauder basis and orthonormal basis.

Definition 1.2.8. *(cf. Christensen (2016)) A collection $\{\tau_n\}_n$ in $\mathscr{H}$ is called*

*(i) a **Schauder basis** for $\mathscr{H}$ if for each $h \in \mathscr{H}$, there exists a unique collection $\{a_n(h)\}_n$ of scalars such that $\sum_{n=1}^{\infty} a_n(h)\tau_n$ converges in $\mathscr{H}$ and $h = \sum_{n=1}^{\infty} a_n(h)\tau_n$.*

*(ii) an **orthonormal basis** for $\mathscr{H}$ if it is a Schauder basis for $\mathscr{H}$ and it is orthonormal.*

We now give various examples of orthonormal bases for Hilbert spaces.

Example 1.2.9. *(cf. Christensen (2016))* *(i)* *Define* $e_n := \{\delta_{n,k}\}_k$*, where* $\delta_{\cdot,\cdot}$ *is the Kronecker delta. Then* $\{e_n\}_n$ *is an orthonormal basis for* $\ell^2(\mathbb{N})$*. This is known as the standard orthonormal basis for* $\ell^2(\mathbb{N})$*.*

(ii) *Define*

$$\mathscr{L}^2[0,1] := \left\{ f : [0,1] \to \mathbb{C} \text{ is measurable and } \int_0^1 |f(x)|^2\,dx < \infty \right\}$$

equipped with the inner product

$$\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}\,dx.$$

Let $n \in \mathbb{Z}$*. Define* $e_n : [0,1] \ni x \mapsto e^{2\pi i n x} \in \mathbb{C}$*. Then* $\{e_n\}_{n=-\infty}^{\infty}$ *is an orthonormal basis for* $\mathscr{L}^2[0,1]$*.*

(iii) ***(Gabor basis)*** *Define*

$$\mathscr{L}^2(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \text{ is measurable and } \int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty \right\}$$

equipped with the inner product

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx.$$

Let $\chi_{[0,1]}$ *be the characteristic function on* $[0,1]$*. For* $j,k \in \mathbb{Z}$*, define* $f_{j,k}(x) := e^{2\pi i j x}\chi_{[0,1]}(x-k), \forall x \in \mathbb{R}$*. Then* $\{f_{j,k}\}_{j,k\in\mathbb{Z}}$ *is an orthonormal basis for* $\mathscr{L}^2(\mathbb{R})$*.*

(iv) ***(Haar system)*** *Let* ψ *be the Haar function defined on* $\mathbb{R}$ *by*

$$\psi(x) := \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise}. \end{cases}$$

For $j,k \in \mathbb{Z}$*, let* $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \forall x \in \mathbb{R}$*. Then* $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ *is an orthonormal basis for* $\mathscr{L}^2(\mathbb{R})$*.*

(v) *(cf. Christensen (2016)) For* $a, b > 0$*, define*

$$T_a : \mathscr{L}^2(\mathbb{R}) \ni f \mapsto T_a f \in \mathscr{L}^2(\mathbb{R}), \quad T_a f : \mathbb{R} \mapsto (T_a f)(x) := f(x-a) \in \mathbb{C}$$

and

$$E_b : \mathscr{L}^2(\mathbb{R}) \ni f \mapsto E_b f \in \mathscr{L}^2(\mathbb{R}), \quad E_b f : \mathbb{R} \mapsto (E_b f)(x) := e^{2\pi i b x} f(x) \in \mathbb{C}.$$

Let $g : \mathbb{R} \to \mathbb{C}$ be a continuous function with compact support. Then, for any $a, b > 0$, $\{E_{mb}T_{na}g\}_{n,m\in\mathbb{Z}}$ is not an orthonormal basis for $\mathscr{L}^2(\mathbb{R})$.

Following theorem shows that given an orthonormal set, we can check whether it is an orthonormal basis by checking several equivalent conditions rather appealing to Definition 1.2.8, which is difficult in many cases.

Theorem 1.2.10. *(cf. Christensen (2016)) Let $\{\tau_n\}_n$ be an orthonormal set in $\mathscr{H}$. The following are equivalent.*

(i) $\{\tau_n\}_n$ *is an orthonormal basis for* $\mathscr{H}$.

(ii) ***(Fourier expansion)*** $h = \sum_{n=1}^{\infty} \langle h, \tau_n \rangle \tau_n, \forall h \in \mathscr{H}$.

(iii) ***(Parseval identity for the inner product)*** $\langle h, g \rangle = \sum_{n=1}^{\infty} \langle h, \tau_n \rangle \langle \tau_n, g \rangle, \forall h, g \in \mathscr{H}$.

(iv) ***(Parseval identity for the norm)*** $\|h\|^2 = \sum_{n=1}^{\infty} |\langle h, \tau_n \rangle|^2, h \in \mathscr{H}$.

(v) $\overline{\operatorname{span}}\{\tau_n\}_n = \mathscr{H}$.

(vi) *If* $h \in \mathscr{H}$ *is such that* $\langle h, \tau_n \rangle = 0, \forall n \in \mathbb{N}$, *then* $h = 0$.

Theorem 1.2.11. *(cf. Christensen (2016)) A Hilbert space is separable if and only if it has a countable orthonormal basis.*

Hilbert spaces have the remarkable property that every bounded linear functional from the space to the scalar field is given by the inner product with a unique element in the space.

Theorem 1.2.12. *(cf. Limaye (2014))* ***(Riesz representation theorem)*** *Let $f : \mathscr{H} \ni\to \mathbb{K}$ be a bounded linear functional. Then there exists a unique $\tau_f \in \mathscr{H}$ such that*

$$f(h) = \langle h, \tau_f \rangle, \quad \forall h \in \mathscr{H} \quad \text{and} \quad \|f\| = \|\tau_f\|.$$

Riesz representation theorem opens the door to the following definition.

Definition 1.2.13. *(cf. Limaye (2014)) Let $T : \mathscr{H} \to \mathscr{H}_0$ be a bounded linear operator. The unique bounded linear operator $T^* : \mathscr{H}_0 \to \mathscr{H}$ such that*

$$\langle Th, h_0 \rangle = \langle h, T^* h_0 \rangle, \quad \forall h \in \mathscr{H}, \, \forall h_0 \in \mathscr{H}_0$$

is called as the ***adjoint*** *of* T.

Hilbert spaces are studied along with various kinds of operators. These are defined as follows.

Definition 1.2.14. *(cf. Limaye (2014)) Let* $T : \mathscr{H} \to \mathscr{H}_0$ *be a bounded linear operator. The operator* T *is said to be*

(i) ***invertible*** *if there exists a bounded linear operator* $S : \mathscr{H}_0 \to \mathscr{H}$ *such that* $ST = I_{\mathscr{H}}$ *and* $TS = I_{\mathscr{H}_0}$.

(ii) ***isometry*** *if* $\|Th\| = \|h\|, \forall h \in \mathscr{H}$.

(iii) ***unitary*** *if* $TT^* = I_{\mathscr{H}_0}$, $T^*T = I_{\mathscr{H}}$.

Definition 1.2.15. *(cf. Limaye (2014)) Let* $T : \mathscr{H} \to \mathscr{H}$ *be a bounded linear operator.*

(i) *Operator* T *is said to be a* ***normal*** *operator if* $TT^* = T^*T$.

(ii) *Operator* T *is said to be a* ***projection*** *if* $T^2 = T = T^*$.

(iii) *Operator* T *is said to be a* ***self-adjoint*** *operator if* $T = T^*$.

(iv) *Operator* T *is said to be a* ***positive*** *operator if* $T = T^*$ *and* $\langle Th, h\rangle \geq 0$, $\forall h \in \mathscr{H}$.

Theorem 1.2.16. *(cf. Limaye (2014)) Let* $\mathscr{H}$ *be a separable Hilbert space.*

(i) *If* $\mathscr{H}$ *is finite dimensional, then* $\mathscr{H}$ *is isometrically isomorphic to* $\mathbb{K}^n$*, for some* n.

(ii) *If* $\mathscr{H}$ *is infinite dimensional, then* $\mathscr{H}$ *is isometrically isomorphic to* $\ell^2(\mathbb{N})$.

Orthonormal bases have the nice property that given a single orthonormal basis, we can generate all of them just by acting unitary operators.

Theorem 1.2.17. *(cf. Christensen (2016)) Let* $\{\tau_n\}_n$ *be an orthonormal basis for* $\mathscr{H}$*. Then the set of all orthonormal bases for* $\mathscr{H}$ *are precisely the families* $\{U\tau_n\}_n$*, where* $U \in \mathscr{B}(\mathscr{H})$ *is unitary.*

Hilbert spaces have another nice property that closed subspaces decompose the original space.

Theorem 1.2.18. *(cf. Limaye (2014)) (****Orthogonal complement theorem****) If* $\mathscr{W}$ *is a closed subspace of* $\mathscr{H}$*, then* $\mathscr{H} = \mathscr{W} \oplus \mathscr{W}^\perp$*, where* $\mathscr{W}^\perp$ *is the orthogonal complement of* $\mathscr{W}$ *in* $\mathscr{H}$.

First level of generalization of orthonormal basis is that of Riesz basis. These are defined as follows.

Definition 1.2.19. *(cf. Christensen (2016)) A collection $\{\omega_n\}_n$ in $\mathscr{H}$ is called a **Riesz basis** for $\mathscr{H}$ if there exist an orthonormal basis $\{\tau_n\}_n$ for $\mathscr{H}$ and an invertible $T \in \mathscr{B}(\mathscr{H})$ such that $\omega_n = T\tau_n, \forall n$.*

As written by Simon (2015), the origin of the term "Riesz basis" is unknown. By taking T as the identity operator, we easily see that every orthonormal basis is a Riesz basis.

Example 1.2.20. *(i) Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of scalars such that there exist $a, b > 0$ with $a \leq |\lambda_n| \leq b, \forall n \in \mathbb{N}$. Then $\{\lambda_n e_n\}_{n=1}^\infty$ is a Riesz basis for $\ell^2(\mathbb{N})$, since it is image of the standard orthonormal basis $\{e_n\}_{n=1}^\infty$ under the invertible operator $T : \ell^2(\mathbb{N}) \ni \{x_n\}_{n=1}^\infty \mapsto \{\lambda_n x_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$. We note further that if $|\lambda_n| \neq 1$, for at least one n, then $\{\lambda_n e_n\}_{n=1}^\infty$ is not an orthonormal basis for $\ell^2(\mathbb{N})$.*

*(ii) (Kadec (1964)) (**Kadec 1/4 theorem**) Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of reals such that*

$$\sup_{n\in\mathbb{N}} |\lambda_n - n| < \frac{1}{4}, \quad \forall n \in \mathbb{Z}.$$

Define $f_n : (-\pi, \pi) \ni x \mapsto e^{i\lambda_n x} \in \mathbb{C}$, $\forall n \in \mathbb{Z}$. Then $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathscr{L}^2(-\pi, \pi)$. It was shown that 1/4 is the optimal constant (Levinson (1936), cf. Christensen (2001)).

*(iii) (Casazza (1998)) (**Kalton-Casazza theorem**) A linear combination of two orthonormal bases is a Riesz basis.*

(iv) (cf. Christensen (2016)) For $a, b > 0$, define

$$T_a : \mathscr{L}^2(\mathbb{R}) \ni f \mapsto T_a f \in \mathscr{L}^2(\mathbb{R}), \quad T_a f : \mathbb{R} \mapsto (T_a f)(x) := f(x - a) \in \mathbb{C}$$

and

$$E_b : \mathscr{L}^2(\mathbb{R}) \ni f \mapsto E_b f \in \mathscr{L}^2(\mathbb{R}), \quad E_b f : \mathbb{R} \mapsto (E_b f)(x) := e^{2\pi i b x} f(x) \in \mathbb{C}.$$

Let $g : \mathbb{R} \to \mathbb{C}$ be a continuous function with compact support. Then, for any $a, b > 0$, $\{E_{mb} T_{na} g\}_{n,m\in\mathbb{Z}}$ is not a Riesz basis for $\mathscr{L}^2(\mathbb{R})$.

Remark 1.2.21. *Let $\{\tau_n\}_n$ be an orthonormal basis for $\mathscr{H}$. Since an invertible map preserves the cardinality, it follows that for each $n \in \mathbb{N}$, the set $\{\tau_j\}_{j=1}^n$ can not be a Riesz basis for $\mathscr{H}$.*

Like orthonormal basis, Riesz basis will also give a series representation of every element in a Hilbert space.

Theorem 1.2.22. *(cf. Christensen (2016)) Let $\{\tau_n\}_n$ be a Riesz basis for $\mathscr{H}$.*

(i) There exists a unique collection $\{\omega_n\}_n$ in $\mathscr{H}$ such that

$$h = \sum_{n=1}^{\infty} \langle h, \omega_n \rangle \tau_n, \quad \forall h \in \mathscr{H}. \tag{1.2.1}$$

Moreover, $\{\omega_n\}_n$ is a Riesz basis for $\mathscr{H}$ and the series in Eq. (1.2.1) converges unconditionally for all $h \in \mathscr{H}$.

(ii) There exist $a, b > 0$ such that $a\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, \tau_n \rangle|^2 \leq b\|h\|^2, \forall h \in \mathscr{H}$.

Next result says that there is a characterization of Riesz basis which is free from orthonormal basis. It also gives a tool to check whether a collection is a Riesz basis for a given Hilbert space. To state the result, we need two definitions.

Definition 1.2.23. *(cf. Christensen (2016)) A sequence $\{\tau_n\}_n$ in a Hilbert space $\mathscr{H}$ is said to be **complete** if $\overline{span}_{n\in\mathbb{N}}\{\tau_n\} = \mathscr{H}$.*

Definition 1.2.24. *(cf. Christensen (2016)) A sequence $\{\omega_n\}_n$ in a Hilbert space $\mathscr{H}$ is said to be **biorthogonal** to a sequence $\{\tau_n\}_n$ in $\mathscr{H}$ if $\langle \omega_n, \tau_m \rangle = \delta_{n,m}$ for all n, m.*

Theorem 1.2.25. *(cf. Christensen (2016); Heil (2011); Stoeva (2020)) For a sequence $\{\tau_n\}_n$ in $\mathscr{H}$, the following are equivalent.*

(i) $\{\tau_n\}_n$ is a Riesz basis for $\mathscr{H}$.

(ii) $\overline{\text{span}}\{\tau_n\}_n = \mathscr{H}$ and there exist $a, b > 0$ such that for every finite subset $\mathbb{S}$ of $\mathbb{N}$,

$$a \sum_{n\in\mathbb{S}} |c_n|^2 \leq \left\| \sum_{n\in\mathbb{S}} c_n \tau_n \right\|^2 \leq b \sum_{n\in\mathbb{S}} |c_n|^2, \quad \forall c_n \in \mathbb{K}. \tag{1.2.2}$$

*(iii) $\{\tau_n\}_n$ is complete in $\mathscr{H}$ and the operator given by the **infinite Gram matrix** $[\langle \tau_m, \tau_n \rangle]_{1\leq n,m<\infty}$ defined by*

$$\ell^2(\mathbb{N}) \ni \{c_m\}_m \mapsto \left\{ \sum_{m=1}^{\infty} \langle \tau_n, \tau_m \rangle c_m \right\}_n \in \ell^2(\mathbb{N})$$

is a bounded invertible operator on $\ell^2(\mathbb{N})$.

(iv) $\{\tau_n\}_n$ *is a bounded unconditional Schauder basis for* $\mathscr{H}$.

(v) $\{\tau_n\}_n$ *is a Schauder basis for* $\mathscr{H}$ *such that* $\sum_{n=1}^{\infty} c_n\tau_n$ *converges in* $\mathscr{H}$ *if and only if* $\sum_{n=1}^{\infty} |c_n|^2 < \infty$.

Remark 1.2.26. *Using Inequality 1.2.2, we can show that certain collection of vectors is not a Riesz basis. As an illustration, let* $\{e_n\}_{n=1}^{\infty}$ *be the standard orthonormal basis for* $\ell^2(\mathbb{N})$. *We claim that* $\{e_1\} \cup \{e_n\}_{n=1}^{\infty}$ *is not a Riesz basis for* $\ell^2(\mathbb{N})$. *Suppose the claim fails, then we get an* $a > 0$ *such that first inequality in Inequality 1.2.2 holds. By taking* 1 *and* -1 *we see that*

$$a(|1|^2 + |-1|^2) \leq \|1 \cdot e_1 + (-1) \cdot e_1\|^2 = 0 \Rightarrow a = 0,$$

which is a contradiction. Hence $\{e_1\} \cup \{e_n\}_{n=1}^{\infty}$ *can not be a Riesz basis for* $\ell^2(\mathbb{N})$.

Theorem 1.2.25 leads to the following definition.

Definition 1.2.27. *(cf. Christensen (2016)) A sequence* $\{\omega_n\}_n$ *in a Hilbert space* $\mathscr{H}$ *is said to be a* ***Riesz sequence*** *for* $\mathscr{H}$ *if there exist* $a, b > 0$ *such that for every finite subset* $\mathbb{S}$ *of* $\mathbb{N}$,

$$a \sum_{n \in \mathbb{S}} |c_n|^2 \leq \left\| \sum_{n \in \mathbb{S}} c_n \tau_n \right\|^2 \leq b \sum_{n \in \mathbb{S}} |c_n|^2, \quad \forall c_n \in \mathbb{K}.$$

Theorem 1.2.25 says that every Riesz basis is a Riesz sequence. It is easy to see that a Riesz sequence need not be a Riesz basis. Following theorem gives another characterization of Riesz basis which is also free from orthonormal basis. It also helps to check whether a collection is a Riesz basis.

Theorem 1.2.28. *(cf. Gohberg and Krein (1969)) (**Kothe-Lorch theorem**) A sequence* $\{\tau_n\}_n$ *is a Riesz basis for* $\mathscr{H}$ *if and only if the following three conditions hold.*

(i) $\{\tau_n\}_n$ *is an unconditional Schauder basis for* $\mathscr{H}$.

(ii) $0 < \inf_{n \in \mathbb{N}} \|\tau_n\| \leq \sup_{n \in \mathbb{N}} \|\tau_n\| < \infty$.

Remark 1.2.29. *Since the collection* $\{e_1\} \cup \{e_n\}_{n=1}^{\infty}$ *in Remark 1.2.26 is not a Schauder basis for* $\ell^2(\mathbb{N})$, *using Theorem 1.2.28, we again conclude that it is not a Riesz basis. However, the collection* $\{e_1\} \cup \{e_n\}_{n=1}^{\infty}$ *satisfies conditions (ii) and (iii) in Theorem 1.2.28.*

It is clear that whenever we perturb an orthonormal basis, we may not get an orthonormal basis. However, it is a classical theorem of Paley and Wiener which says whenever we perturb an orthonormal basis, we get a Riesz basis.

Theorem 1.2.30. *(Paley and Wiener (1987), cf. Young, 1980) (**Paley-Wiener theorem**) Let $\{\tau_n\}_n$ be an orthonormal basis for $\mathcal{H}$. If $\{\omega_n\}_n$ in $\mathcal{H}$ is such that there exists $0 < \alpha < 1$ and for every $m = 1, 2, \dots,$*

$$\left\| \sum_{n=1}^{m} c_n(\tau_n - \omega_n) \right\| \leq \alpha \left(\sum_{n=1}^{m} |c_n|^2 \right)^{\frac{1}{2}}, \quad \forall c_n \in \mathbb{K},$$

then $\{\omega_n\}_n$ is a Riesz basis for $\mathcal{H}$.

Next level of generalization of Riesz basis for Hilbert spaces is the notion of frame.

Definition 1.2.31. *(Duffin and Schaeffer (1952)) A collection $\{\tau_n\}_n$ in a Hilbert space $\mathcal{H}$ is said to be a **frame** for $\mathcal{H}$ if there exist $a, b > 0$ such that*

$$\textit{(\textbf{Frame inequalities})} \quad a\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, \tau_n \rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathcal{H}. \tag{1.2.3}$$

*Constants a and b are called as **lower frame bound** and **upper frame bound**, respectively. Supremum (resp. infimum) of the set of all lower (resp. upper) frame bounds is called **optimal lower frame bound** (resp. **optimal upper frame bound**). If the optimal frame bounds are equal, then the frame is called as **tight frame**. A tight frame whose optimal frame bound is one is termed as **Parseval frame**.*

As recorded by Kovacevic and Chebira (2007), and Heil (2013), the reason for using the term “frame" is unknown.

Note that in Definition 1.2.31 we indexed the frame by natural numbers. Since the convergence of series in Definition 1.2.31 is unconditional, any rearrangement of a frame is again a frame. We also note that Definition 1.2.31 can be formulated for arbitrary indexing set $\mathbb{J}$. In this case, by the convergence of the series we mean the convergence of the net obtained by the set inclusion, on the collections of all finite subsets of $\mathbb{J}$.

From (ii) in Theorem 1.2.22 we can conclude that every Riesz basis is a frame. On the other hand, every frame can be written as a finite union of Riesz sequences (which is called as **Feichtinger conjecture** (Casazza et al. (2005b))) and is known as **Marcus-Spielman-Srivastava Theorem** (cf. Casazza and Edidin (2007); Casazza and Tremain (2016)). We now give various examples of frames.

Example 1.2.32. *(i) Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathscr{H}$ and let $m \in \mathbb{N}$. Then the collection $\{e_1, \dots, e_m\} \cup \{e_n\}_{n=1}^\infty$ is a frame for $\mathscr{H}$ with bounds 1 and 2. In fact, for all $h \in \mathscr{H}$,*

$$1 \cdot \|h\|^2 = \sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2 \leq \sum_{k=1}^{m} |\langle h, e_k\rangle|^2 + \sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2 \leq 2\sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2 = 2\|h\|^2.$$

In particular, the collection in Remark 1.2.26 is a frame for $\ell^2(\mathbb{N})$.

*(ii) (cf. Christensen (2016)) (**Harmonic frame**) Let $m, n \in \mathbb{N}$, $n \leq m$ and $\omega_1, \dots, \omega_m$ be the distinct m^{th} roots of unity. Define*

$$\eta_k \coloneqq \frac{1}{\sqrt{m}}(\omega_1^k, \dots, \omega_n^k), \quad 1 \leq k \leq m.$$

Then $\{\eta_k\}_{k=1}^m$ is a Parseval frame for $\mathbb{C}^n$.

*(iii) (cf. Kovacevic and Chebira (2007), Shor (2004)) (**Mercedes-Benz frame** or **Peres-Wooters states**) $\{(0,1), (-\sqrt{3}/2, -1/2), (\sqrt{3}/2, -1/2)\}$ is a tight frame for $\mathbb{R}^2$ with bound $3/2$.*

(iv) (cf. Han et al. (2007)) For $n \geq 3$, the collection $\{(\cos(2\pi j/n), \sin(2\pi j/n)\}_{j=0}^{n-1}$ is a tight frame for $\mathbb{R}^2$. It has to be noted that we can not take $n = 2$ because $\{(1,0), (-1,0)\}$ is not a frame for $\mathbb{R}^2$.

*(v) (cf. Christensen (2016)) (**Gabor frame** or **Weyl-Heisenberg frame**) Let g be the Gaussian defined by $g : \mathbb{R} \ni x \mapsto g(x) \coloneqq e^{-x^2} \in \mathbb{R}$ and let $a, b > 0$. Define $f_{n,m} : \mathbb{R} \ni x \mapsto e^{2\pi imbx} g(x - na) \in \mathbb{R}$, $\forall n, m \in \mathbb{Z}$. Then $\{f_{n,m}\}_{n,m\in\mathbb{Z}}$ is a (Gabor) frame for $\mathscr{L}^2(\mathbb{R})$ if and only if $ab < 1$. Moreover, if $\{f_{n,m}\}_{n,m\in\mathbb{Z}}$ is a frame for $\mathscr{L}^2(\mathbb{R})$, then $ab = 1$ if and only if $\{f_{n,m}\}_{n,m\in\mathbb{Z}}$ is a Riesz basis for $\mathscr{L}^2(\mathbb{R})$.*

(vi) (Duffin and Schaeffer (1952)) Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of scalars such that there are constants $d, L, \delta > 0$ satisfying

$$\left|\lambda_n - \frac{n}{d}\right| \leq L, \quad \forall n \in \mathbb{Z} \quad \text{and} \quad |\lambda_n - \lambda_m| \geq \delta, \quad \forall n, m \in \mathbb{Z}, n \neq m.$$

Let $0 < r < d\pi$ and define $f_n : (-r, r) \ni x \mapsto e^{i\lambda_n x} \in \mathbb{C}$, $\forall n \in \mathbb{Z}$. Then $\{f_n\}_{n\in\mathbb{Z}}$ is a frame for $\mathscr{L}^2(-r, r)$.

(vii) (cf. Feichtinger and Strohmer (2003)) For $a, b > 0$, define

$$T_a : \mathscr{L}^2(\mathbb{R}) \ni f \mapsto T_a f \in \mathscr{L}^2(\mathbb{R}), \quad T_a f : \mathbb{R} \mapsto (T_a f)(x) \coloneqq f(x - a) \in \mathbb{C}$$

and

$$E_b : \mathscr{L}^2(\mathbb{R}) \ni f \mapsto E_b f \in \mathscr{L}^2(\mathbb{R}), \quad E_b f : \mathbb{R} \mapsto (E_b f)(x) := e^{2\pi i b x} f(x) \in \mathbb{C}.$$

For $c > 0$*, let* $\chi_{[0,c]}$ *be the characteristic function on* $[0,c]$*. Then for* $a \leq c \leq 1$*,* $\{E_m T_{na} \chi_{[0,c]}\}_{n,m \in \mathbb{Z}}$ *is a (Gabor) frame for* $\mathscr{L}^2(\mathbb{R})$ *(this is a particular case of the celebrated abc-problem for Gabor systems (Dai and Sun (2016))).*

(viii) (Janssen and Strohmer (2002)) Let T_a *and* E_b *be the operators in (vii). Let* $g(x) := \cosh(\pi x) = \frac{2}{e^{\pi x} + e^{-\pi x}}$*,* $\forall x \in \mathbb{R}$*. Then, for* $ab < 1$*,* $\{E_{mb} T_{na} g\}_{n,m \in \mathbb{Z}}$ *is a (Gabor) frame for* $\mathscr{L}^2(\mathbb{R})$*.*

(ix) (Janssen (2003)) Let T_a *and* E_b *be the operators in (vii). Let* $g(x) := e^{-|x|}$*,* $\forall x \in \mathbb{R}$*. Then, for* $ab < 1$*,* $\{E_{mb} T_{na} g\}_{n,m \in \mathbb{Z}}$ *is a (Gabor) frame for* $\mathscr{L}^2(\mathbb{R})$*.*

(x) (Janssen (1996)) Let T_a *and* E_b *be the operators in (vii). Let* $g(x) := e^{-x} \chi_{[0,\infty)}(x)$*,* $\forall x \in \mathbb{R}$*. Then* $\{E_{mb} T_{na} g\}_{n,m \in \mathbb{Z}}$ *is a (Gabor) frame for* $\mathscr{L}^2(\mathbb{R})$ *if and only if* $ab \leq 1$*.*

*(xi) (Casazza (1998)) (**Kalton-Casazza theorem**) Every frame is a sum of three orthonormal bases.*

*(xii) (cf. Christensen (2016)) (**Wavelet frame**) Let* $0 < b < 0.0084$*. Define the **Mexican hat** function*

$$\psi(x) := \frac{2}{\sqrt{3}} \pi^{\frac{-1}{4}} (1 - x^2) e^{\frac{-x^2}{2}}, \quad \forall x \in \mathbb{R}.$$

For $j,k \in \mathbb{Z}$*, define*

$$\psi_{j,k}(x) := \psi(2^j x - kb), \quad \forall x \in \mathbb{R}.$$

Then $\{2^{\frac{j}{2}} \psi_{j,k}\}_{j,k \in \mathbb{Z}}$ *is a (wavelet) frame for* $\mathscr{L}^2(\mathbb{R})$*.*

*(xiii) (Benedetto and Kolesar (2006); Strohmer and Heath (2003)) (**Grassmannian frame**) The n-equally spaced lines in* $\mathbb{R}^2$*, namely* $\{(\cos(\pi j/n), \sin(\pi j/n)\}_{j=0}^{n-1}$ *is a (Grassmannian) frame for* $\mathbb{R}^2$*.*

*(xiv) (Benedetto and Fickus (2003)) (**Group frame**) Vertices of each of (five) **Platonic solids** is a tight frame for* $\mathbb{R}^3$*.*

*(xv) (cf. Waldron (2018)) (**Equiangular frame**) For each* $d \in \mathbb{N}$*,* $d+1$ *vertices of the regular simplex in* $\mathbb{R}^d$ *is an (equiangular) frame for* $\mathbb{R}^d$*.*

We now give various examples which are not frames.

Example 1.2.33. *(i) (cf. Christensen (2016)) If* $\{\tau_n\}_{n=1}^\infty$ *is an orthonormal basis for* $\mathscr{H}$*, then* $\{\tau_n+\tau_{n+1}\}_{n=1}^\infty$ *is not a frame for* $\mathscr{H}$*.*

(ii) (cf. Bachman et al. (2000)) If $\{\tau_n\}_{n=1}^\infty$ *is an orthonormal basis for* $\mathscr{H}$*, then* $\{\frac{\tau_n}{n}\}_{n=1}^\infty$ *is not a frame for* $\mathscr{H}$*.*

(iii) (cf. Christensen (2016)) If $\{\tau_n\}_{n=-\infty}^\infty$ *is a Riesz basis for* $\mathscr{H}$*, then* $\{\tau_n+\tau_{n+1}\}_{n=-\infty}^\infty$ *is not a frame for* $\mathscr{H}$*.*

(iv) (cf. Christensen (2016)) For $n\in\mathbb{Z}$*, define* $T_n:\mathscr{L}^2(\mathbb{R})\ni f\mapsto T_nf\in\mathscr{L}^2(\mathbb{R})$*,* $T_nf:\mathbb{R}\ni x\mapsto f(x-n)\in\mathbb{C}$*. Then for any* $\phi\in\mathscr{L}^2(\mathbb{R})$*,* $\{T_n\phi\}_{n\in\mathbb{Z}}$ *is not a frame for* $\mathscr{L}^2(\mathbb{R})$*.*

(v) (Aldroubi and Petrosyan (2017); Christensen et al. (2018)) Let $\mathscr{H}$ *be an infinite dimensional Hilbert space and* $T:\mathscr{H}\to\mathscr{H}$ *be a bounded linear operator which is unitary or compact. Then for every* $\tau\in\mathscr{H}$*,* $\{T^n\tau\}_{n=0}^\infty$ *is not a frame for* $\mathscr{H}$ *(this is a particular case of dynamical sampling (Aldroubi et al. (2017a,b))).*

(vi) (cf. Feichtinger and Strohmer (2003)) Let T_a *and* E_b *be the operators in (vii) of Example 1.2.32. For* $c>0$*, let* $\chi_{[0,c]}$ *be the characteristic function on* $[0,c]$*. Then for* $c\leq a$ *or* $a>1$*,* $\{E_mT_{na}\chi_{[0,c]}\}_{n,m\in\mathbb{Z}}$ *is not a frame for* $\mathscr{L}^2(\mathbb{R})$*.*

There is a simple criterion to check for frames in finite dimensional Hilbert spaces. This reads as follows.

Theorem 1.2.34. *(cf. Han et al. (2007)) A finite set of vectors for a finite dimensional Hilbert space is a frame if and only if it spans the space.*

Remark 1.2.35. *(i) Theorem 1.2.34 gives a very useful algebraic criterion for checking whether a finite set of vectors is a frame for a finite dimensional space rather verifying the analytic condition (frame inequality) which is harder in many cases.*

(ii) Theorem 1.2.34 does not tell that a frame for a finite dimensional Hilbert space is finite. A finite dimensional Hilbert space can have a frame with infinitely many elements. For example, $\{\frac{1}{n}\}_{n=1}^\infty$ *is a tight frame for* $\mathbb{C}$ *(as a vector space over itself), because*

$$\sum_{n=1}^\infty\left|\left\langle h,\frac{1}{n}\right\rangle\right|^2=\sum_{n=1}^\infty\left|\frac{h}{n}\right|^2=\frac{\pi^2}{6}|h|^2,\quad\forall h\in\mathbb{C}.$$

(iii) *Suppose* $\dim(\mathscr{H}) = n$. *From Theorem 1.2.34 we see that a spanning set having at least* $n+1$ *elements is a frame for* $\mathscr{H}$ *but not a Riesz basis for* $\mathscr{H}$.

(iv) *A spanning set need not be a frame. For instance,* $\{n\}_{n=1}^\infty$ *spans* $\mathbb{C}$ *but* $\sum_{n=1}^\infty |\langle 1, n\rangle|^2 = \sum_{n=1}^\infty n^2 = \infty$. *Hence* $\{n\}_{n=1}^\infty$ *is not a frame for* $\mathbb{C}$.

Following theorem is the most important result in the theory of frames.

Theorem 1.2.36. *(Duffin and Schaeffer (1952), Christensen (2016), Han and Larson (2000)) Let* $\{\tau_n\}_n$ *be a frame for* $\mathscr{H}$ *with bounds a and b. Then*

(i) $\overline{\text{span}}\{\tau_n\}_n = \mathscr{H}$.

(ii) *The map* $\theta_\tau : \mathscr{H} \ni h \mapsto \theta_\tau h := \{\langle h, \tau_n\rangle\}_n \in \ell^2(\mathbb{N})$ *is a well-defined bounded linear operator. Further,* $\sqrt{a}\|h\| \leq \|\theta_\tau h\| \leq \sqrt{b}\|h\|, \forall h \in \mathscr{H}$. *In particular,* θ_τ *is injective and its range is closed.*

(iii) *The map* $S_\tau : \mathscr{H} \ni h \mapsto S_\tau h := \sum_{n=1}^\infty \langle h, \tau_n\rangle \tau_n \in \mathscr{H}$ *is a well-defined bounded linear positive invertible operator. Further,*

$$a\|h\|^2 \leq \langle S_\tau h, h\rangle \leq b\|h\|^2, \quad \forall h \in \mathscr{H}, \qquad a\|h\| \leq \|S_\tau h\| \leq b\|h\|, \quad \forall h \in \mathscr{H}.$$

(iv) *(**General Fourier expansion** or **frame decomposition**)*

$$h = \sum_{n=1}^\infty \langle h, \tau_n\rangle S_\tau^{-1}\tau_n = \sum_{n=1}^\infty \langle h, S_\tau^{-1}\tau_n\rangle \tau_n, \quad \forall h \in \mathscr{H}. \tag{1.2.4}$$

(v) $\theta_\tau^*(\{a_n\}_n) = \sum_n a_n \tau_n, \forall \{a_n\}_n \in \ell^2(\mathbb{N})$. *In particular,* $\theta_\tau^* e_n = \tau_n, \forall n \in \mathbb{N}$.

(vi) S_τ *factors as* $S_\tau = \theta_\tau^*\theta_\tau$.

(vii) θ_τ^* *is surjective.*

(viii) $\|S_\tau^{-1}\|^{-1}$ *is the optimal lower frame bound and* $\|S_\tau\| = \|\theta_\tau\|^2$ *is the optimal upper frame bound.*

(ix) $P_\tau := \theta_\tau S_\tau^{-1}\theta_\tau^*$ *is an orthogonal projection onto* $\theta_\tau(\mathscr{H})$.

(x) $\{\tau_n\}_n$ *is Parseval if and only if* θ_τ *is an isometry if and only if* $\theta_\tau\theta_\tau^*$ *is a projection.*

(xi) $\{S_\tau^{-1}\tau_n\}_n$ *is a frame for* $\mathscr{H}$ *with bounds* b^{-1} *and* a^{-1}.

(xii) $\{S_\tau^{-1/2}\tau_n\}_n$ *is a Parseval frame for* $\mathscr{H}$.

(xiii) ***(Best approximation)*** *If $h \in \mathscr{H}$ has representation $h = \sum_{n=1}^{\infty} c_n \tau_n$, for some scalar sequence $\{c_n\}_n \in \ell^2(\mathbb{N})$, then*

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |\langle h, S_\tau^{-1}\tau_n\rangle|^2 + \sum_{n=1}^{\infty} |c_n - \langle h, S_\tau^{-1}\tau_n\rangle|^2.$$

Theorem 1.2.36 says several things. First, it says that every vector in the Hilbert space admits an expansion, called as general Fourier expansion, similar to Fourier expansion coming from an orthonormal basis for a Hilbert space. Second, it says that coefficients in the expansion of a vector need not be unique. This is particularly important in applications, since loss in the information of a vector is less if some of the coefficients are missing. Third, given a frame, it naturally generates other frames. Fourth, a frame gives a bounded linear injective operator from the less known inner product on the Hilbert space $\mathscr{H}$ to the well known standard inner product on the standard separable Hilbert space $\ell^2(\mathbb{N})$. Frame inequality now clearly says that there is a comparison of norms between $\mathscr{H}$ and the standard Hilbert space $\ell^2(\mathbb{N})$. Fifth, a frame embeds $\mathscr{H}$ in $\ell^2(\mathbb{N})$ through the bounded linear operator θ_τ. Sixth, whenever a Hilbert space admits a frame it becomes an image of a surjective operator θ_τ^* from the $\ell^2(\mathbb{N})$ to it. An easy observation from Theorem 1.2.36 is that for an infinite dimensional Hilbert space, a finite collection of vectors can not be a frame.

The operators θ_τ, θ_τ^* and S_τ in Theorem 1.2.36 are called as **analysis operator**, **synthesis operator** and **frame operator**, respectively (cf. Christensen (2016)).

Dilation theory usually tries to extend operator on Hilbert space to larger Hilbert space which are easier to handle as well as well-understood and study the original operator as a slice of it (Arveson (2010); Levy and Shalit (2014); Sz.-Nagy et al. (2010)). As long as frame theory for Hilbert spaces is considered, following theorem is known as Naimark-Han-Larson dilation theorem. This was proved independently by Han and Larson (2000) and by Kashin and Kulikova (2002). History of this theorem is nicely presented in the paper (Czaja (2008)).

Theorem 1.2.37. *(Han and Larson (2000); Kashin and Kulikova (2002))* ***(Naimark-Han-Larson dilation theorem)*** *A collection $\{\tau_n\}_n$ in $\mathscr{H}$ is a*

(i) frame for $\mathscr{H}$ if and only if the exist a Hilbert space $\mathscr{H}_1 \supseteq \mathscr{H}$, a Riesz basis $\{\omega_n\}_n$ for $\mathscr{H}_1$ and a projection $P : \mathscr{H}_1 \to \mathscr{H}$ such that $\tau_n = P\omega_n, \forall n \in \mathbb{N}$.

(ii) Parseval frame for $\mathscr{H}$ if and only if the exist a Hilbert space $\mathscr{H}_1 \supseteq \mathscr{H}$, an orthonormal basis $\{\omega_n\}_n$ for $\mathscr{H}_1$ and an orthogonal projection $P : \mathscr{H}_1 \to \mathscr{H}$ such that $\tau_n = P\omega_n, \forall n \in \mathbb{N}$.

In order to construct an element of the Hilbert space using frames using Equation 1.2.4, we have to first determine inverse of the frame operator which is difficult in general. Thus we seek a way to approximate an element using a sequence which does not involve calculating inverse of frame operator. This is given in the following theorem.

Proposition 1.2.38. *(Duffin and Schaeffer (1952)) (**Frame algorithm**) Let $\{\tau_n\}_{n=1}^{\infty}$ be a frame for $\mathscr{H}$ with bounds a and b. For $h \in \mathscr{H}$ define*

$$h_0 := 0, \quad h_n := h_{n-1} + \frac{2}{a+b} S_\tau (h - h_{n-1}), \quad \forall n \geq 1.$$

Then

$$\|h_n - h\| \leq \left(\frac{b-a}{b+a}\right)^n \|h\|, \quad \forall n \geq 1.$$

In particular, $h_n \to h$ as $n \to \infty$.

Given a collection $\{\tau_n\}_n$, in general, it is difficult to find a and b such that the two inequalities in (1.2.3) hold. Therefore, it is natural to ask whether there is a characterization for frame without using frame bounds. Orthonormal bases are the simplest and easiest sequences we can handle in a Hilbert space, so one can attempt to obtain characterization using orthonormal bases. Since every separable Hilbert space is isometrically isomorphic to the standard Hilbert space $\ell^2(\mathbb{N})$ and the standard unit vectors $\{e_n\}_n$ form an orthonormal basis for $\ell^2(\mathbb{N})$, one can further ask whether frames can be characterized using $\{e_n\}_n$. This question was answered affirmatively by Holub (1994) as follows.

Theorem 1.2.39. *(Holub (1994)) (**Holub's theorem**) A sequence $\{\tau_n\}_n$ in $\mathscr{H}$ is a frame for $\mathscr{H}$ if and only if there exists a surjective bounded linear operator $T : \ell^2(\mathbb{N}) \to \mathscr{H}$ such that $Te_n = \tau_n$, for all $n \in \mathbb{N}$.*

There is a slight variation of Theorem 1.2.39 given by Christensen (2016).

Theorem 1.2.40. *(Christensen (2016)) Let $\{\omega_n\}_n$ be an orthonormal basis for $\mathscr{H}$. Then a sequence $\{\tau_n\}_n$ in $\mathscr{H}$ is a frame for $\mathscr{H}$ if and only if there exists a surjective bounded linear operator $T : \mathscr{H} \to \mathscr{H}$ such that $T\omega_n = \tau_n$, for all $n \in \mathbb{N}$.*

Given a frame $\{\tau_n\}_n$ for $\mathscr{H}$ we now consider the frame $\{S_\tau^{-1}\tau_n\}_n$. This frame satisfies Equation (1.2.4). However, in general there may be other frames satisfying the Equation (1.2.4) like $\{S_\tau^{-1}\tau_n\}_n$. This leads to the notion of dual frames as stated below.

Definition 1.2.41. *(cf. Christensen (2016)) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. A frame $\{\omega_n\}_n$*

for $\mathscr{H}$ is said to be a ***dual*** *frame for $\{\tau_n\}_n$ if*

$$h = \sum_{n=1}^{\infty} \langle h, \omega_n \rangle \tau_n = \sum_{n=1}^{\infty} \langle h, \tau_n \rangle \omega_n, \quad \forall h \in \mathscr{H}.$$

Just like characterization of frames, given a frame, we seek a description of each of its dual frame. This problem was solved by Li (1995) in the following two lemmas and a theorem.

Lemma 1.2.42. *(Li (1995)) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$ and $\{e_n\}_n$ be the standard orthonormal basis for $\ell^2(\mathbb{N})$. Then a frame $\{\omega_n\}_n$ is a dual frame for $\{\tau_n\}_n$ if and only if*

$$\omega_n = Ue_n, \quad \forall n \in \mathbb{N},$$

where $U : \ell^2(\mathbb{N}) \to \mathscr{H}$ is a bounded left-inverse of θ_τ.

Lemma 1.2.43. *(Li (1995)) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. Then $L : \ell^2(\mathbb{N}) \to \mathscr{H}$ is a bounded left-inverse of θ_τ if and only if*

$$L = S_\tau^{-1}\theta_\tau^* + V(I_{\ell^2(\mathbb{N})} - \theta_\tau S_\tau^{-1}\theta_\tau^*),$$

where $V : \ell^2(\mathbb{N}) \to \mathscr{H}$ is a bounded operator.

Theorem 1.2.44. *(Li (1995)) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. Then a frame $\{\omega_n\}_n$ is a dual frame for $\{\tau_n\}_n$ if and only if*

$$\omega_n = S_\tau^{-1}\tau_n + \rho_n - \sum_{k=1}^{\infty} \langle S_\tau^{-1}\tau_n, \tau_k \rangle \rho_k, \quad \forall n \in \mathbb{N},$$

where $\{\rho_n\}_n$ is a sequence in $\mathscr{H}$ such that there exists $b > 0$ satisfying

$$\sum_{n=1}^{\infty} |\langle h, \rho_n \rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathscr{H}.$$

We again consider the frame $\{S_\tau^{-1}\tau_n\}_n$. Note that this frame is obtained by the action of an invertible operator S_τ^{-1} to the original frame $\{\tau_n\}_n$. This leads to the question: what are all the frames which are obtained by operating an invertible operator to the given frame? This naturally brings us to the following definition.

Definition 1.2.45. *(Balan (1999)) Two frames $\{\tau_n\}_n$ and $\{\omega_n\}_n$ for $\mathscr{H}$ are said to be* ***similar*** *or* ***equivalent*** *if there exists a bounded invertible operator $T : \mathscr{H} \to \mathscr{H}$ such* *that*

$$\omega_n = T\tau_n, \quad \forall n \in \mathbb{N}. \tag{1.2.5}$$

Given frames $\{\tau_n\}_n$ and $\{\omega_n\}_n$, it is rather difficult to check whether they are similar because one has to get an invertible operator and verify Equation (1.2.5) for every natural number. Thus it is better if there is a characterization which does not involve natural numbers and involves only operators. Further, it is natural to ask whether there is a formula for the operator T which gives similarity. This was done by Balan (1999) and independently by Han and Larson (2000) which states as follows.

Theorem 1.2.46. *(Balan (1999); Han and Larson (2000)) For two frames $\{\tau_n\}_n$ and $\{\omega_n\}_n$ for $\mathcal{H}$, the following are equivalent.*

(i) $\{\tau_n\}_n$ *and* $\{\omega_n\}_n$ *are similar, i.e., there exists a bounded invertible operator* $T : \mathcal{H} \to \mathcal{H}$ *such that* $\omega_n = T\tau_n$, $\forall n \in \mathbb{N}$.

(ii) $\theta_\omega = \theta_\tau T$*, for some bounded invertible operator* $T : \mathcal{H} \to \mathcal{H}$.

(iii) $P_\omega = P_\tau$.

If one of the above conditions is satisfied, then the invertible operator in (i) and (ii) is unique and is given by $T = S_\tau^{-1}\theta_\tau^*\theta_\omega$.

For a given subset $\mathbb{M}$ of $\mathbb{N}$, set $S_\mathbb{M} : \mathcal{H} \ni h \mapsto \sum_{n\in\mathbb{M}}\langle h, \tau_n\rangle\tau_n \in \mathcal{H}$. Because of Inequalities (1.2.3), $S_\mathbb{M}$ is a well-defined bounded positive operator (which may not be invertible). Let $\mathbb{M}^c$ denote the complement of $\mathbb{M}$ in $\mathbb{N}$. Casazza, Edidin, and Kutyniok derived following identities for frames for Hilbert spaces (Balan et al. (2006, 2005)).

Theorem 1.2.47. *(Balan et al. (2005, 2007)) (**Frame identity**) Let $\{\tau_n\}_n$ be a frame for $\mathcal{H}$. Then for every $\mathbb{M} \subseteq \mathbb{N}$,*

$$\sum_{n\in\mathbb{M}} |\langle h, \tau_n\rangle|^2 - \sum_{n=1}^{\infty} |\langle S_\mathbb{M} h, \tilde{\tau}_n\rangle|^2 = \sum_{n\in\mathbb{M}^c} |\langle h, \tau_n\rangle|^2 - \sum_{n=1}^{\infty} |\langle S_{\mathbb{M}^c} h, \tilde{\tau}_n\rangle|^2, \quad \forall h \in \mathcal{H}.$$

Theorem 1.2.48. *(Balan et al. (2005, 2007)) (**Parseval frame identity**) Let $\{\tau_n\}_n$ be a Parseval frame for $\mathcal{H}$. Then for every $\mathbb{M} \subseteq \mathbb{N}$,*

$$\sum_{n\in\mathbb{M}} |\langle h, \tau_n\rangle|^2 - \left\|\sum_{n\in\mathbb{M}} \langle h, \tau_n\rangle\tau_n\right\|^2 = \sum_{n\in\mathbb{M}^c} |\langle h, \tau_n\rangle|^2 - \left\|\sum_{n\in\mathbb{M}^c} \langle h, \tau_n\rangle\tau_n\right\|^2, \quad \forall h \in \mathcal{H}.$$

Theorem 1.2.48 has applications. It was applied to get the following remarkable lower estimate for Parseval frames.

Theorem 1.2.49. *(Balan et al. (2007); Gavruta (2006)) Let $\{\tau_n\}_n$ be a Parseval frame for $\mathscr{H}$. Then for every $\mathbb{M} \subseteq \mathbb{N}$,*

$$\sum_{n\in\mathbb{M}} |\langle h, \tau_n\rangle|^2 + \left\| \sum_{n\in\mathbb{M}^c} \langle h, \tau_n\rangle \tau_n \right\|^2 = \sum_{n\in\mathbb{M}^c} |\langle h, \tau_n\rangle|^2 + \left\| \sum_{n\in\mathbb{M}} \langle h, \tau_n\rangle \tau_n \right\|^2$$
$$\geq \frac{3}{4}\|h\|^2, \quad \forall h \in \mathscr{H}.$$

Further, the bound 3/4 is optimal.

As another application, Theorem 1.2.49 was used in the study of Parseval frames with finite excesses (Bakic and Beric (2015); Balan et al. (2003)).
Like duality, there is another notion called as orthogonality for frames for Hilbert spaces. This was first introduced by Balan (1998) in his Ph.D. thesis and further studied by Han and Larson (2000).

Definition 1.2.50. *(Balan (1998); Han and Larson (2000)) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. A frame $\{\omega_n\}_n$ for $\mathscr{H}$ is said to be an* ***orthogonal*** *frame for $\{\tau_n\}_n$ if*

$$0 = \sum_{n=1}^{\infty} \langle h, \omega_n\rangle \tau_n = \sum_{n=1}^{\infty} \langle h, \tau_n\rangle \omega_n, \quad \forall h \in \mathscr{H}.$$

Remarkable property of orthogonal frames is that we can interpolate as well as we can take direct sum of them to get new frames. These are illustrated in the following two results.

Proposition 1.2.51. *(Han et al. (2007); Han and Larson (2000)) Let $\{\tau_n\}_n$ and $\{\omega_n\}_n$ be two Parseval frames for $\mathscr{H}$ which are orthogonal. If $C,D \in \mathscr{B}(\mathscr{H})$ are such that $C^*C + D^*D = I_{\mathscr{H}}$, then $\{C\tau_n + D\omega_n\}_n$ is a Parseval frame for $\mathscr{H}$. In particular, if scalars c,d,e,f satisfy $|c|^2 + |d|^2 = 1$, then $\{c\tau_n + d\omega_n\}_n$ is a Parseval frame.*

Proposition 1.2.52. *(Han et al. (2007); Han and Larson (2000)) If $\{\tau_n\}_n$ and $\{\omega_n\}_n$ are orthogonal frames for $\mathscr{H}$, then $\{\tau_n \oplus \omega_n\}_n$ is a frame for $\mathscr{H} \oplus \mathscr{H}$. Further, if both $\{\tau_n\}_n$ and $\{\omega_n\}_n$ are Parseval, then $\{\tau_n \oplus \omega_n\}_n$ is Parseval.*

Recall that Paley-Wiener theorem 1.2.30 says that sequences which are close to orthonormal bases are Riesz bases. Since a frame will also give a series representation, it is natural to ask whether a sequence close to frame is a frame. This was first derived by Christensen (1995a) which showed that sequences which are quadratically close to frames are again frames.

Theorem 1.2.53. *(Christensen (1995a)) (**Christensen's quadratic perturbation**) Let $\{\tau_n\}_{n=1}^\infty$ be a frame for $\mathscr{H}$ with bounds a and b. If $\{\omega_n\}_{n=1}^\infty$ in $\mathscr{H}$ satisfies*

$$c := \sum_{n=1}^\infty \|\tau_n - \omega_n\|^2 < a,$$

then $\{\omega_n\}_{n=1}^\infty$ is a frame for $\mathscr{H}$ with bounds $a\left(1-\sqrt{\frac{c}{a}}\right)^2$ and $b\left(1+\sqrt{\frac{c}{b}}\right)^2$.

Three months later, Christensen generalized Theorem 1.2.53.

Theorem 1.2.54. *(Christensen (1995c)) (**Christensen perturbation**) Let $\{\tau_n\}_{n=1}^\infty$ be a frame for $\mathscr{H}$ with bounds a and b. If $\{\omega_n\}_{n=1}^\infty$ in $\mathscr{H}$ is such that there exist $\alpha, \gamma \geq 0$ with $\alpha + \frac{\gamma}{\sqrt{a}} < 1$ and*

$$\left\|\sum_{n=1}^m c_n(\tau_n - \omega_n)\right\| \leq \alpha \left\|\sum_{n=1}^m c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^m |c_n|^2\right)^{\frac{1}{2}}, \quad \forall c_1, \dots, c_m \in \mathbb{K}, m = 1, 2, \dots,$$

then $\{\omega_n\}_{n=1}^\infty$ is a frame for $\mathscr{H}$ with bounds $a\left(1-(\alpha+\frac{\gamma}{\sqrt{a}})\right)^2$ and $b\left(1+(\alpha+\frac{\gamma}{\sqrt{b}})\right)^2$.

After two years, Casazza and Christensen further extended Theorem 1.2.54.

Theorem 1.2.55. *(Casazza and Christensen (1997)) (**Casazza-Christensen perturbation**) Let $\{\tau_n\}_{n=1}^\infty$ be a frame for $\mathscr{H}$ with bounds a and b. If $\{\omega_n\}_{n=1}^\infty$ in $\mathscr{H}$ is such that there exist $\alpha, \beta, \gamma \geq 0$ with $\max\{\alpha + \frac{\gamma}{\sqrt{a}}, \beta\} < 1$ and*

$$\left\|\sum_{n=1}^m c_n(\tau_n - \omega_n)\right\| \leq \alpha \left\|\sum_{n=1}^m c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^m |c_n|^2\right)^{\frac{1}{2}} + \beta\left\|\sum_{n=1}^m c_n\omega_n\right\|,$$
$$\forall c_1, \dots, c_m \in \mathbb{K}, m = 1, 2, \dots,$$

then $\{\omega_n\}_{n=1}^\infty$ is a frame for $\mathscr{H}$ with bounds $a\left(1-\frac{\alpha+\beta+\frac{\gamma}{\sqrt{a}}}{1+\beta}\right)^2$ and $b\left(1+\frac{\alpha+\beta+\frac{\gamma}{\sqrt{b}}}{1-\beta}\right)^2$.

We next consider Bessel sequences which is next level of generalization of frames.

Definition 1.2.56. *(cf. Christensen (2016)) A collection $\{\tau_n\}_n$ in a Hilbert space $\mathscr{H}$ is said to be a **Bessel sequence** for $\mathscr{H}$ if there exists a real constant $b > 0$ such that*

$$\textit{(\textbf{General Bessel's inequality})} \quad \sum_{n=1}^\infty |\langle h, \tau_n\rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathscr{H}. \tag{1.2.6}$$

Constant b is called as a Bessel bound for $\{\tau_n\}_n$.

Inequality 1.2.3 is stronger than Inequality 1.2.6. Hence every frame is a Bessel sequence. Using Cauchy-Schwarz inequality, it follows that every finite set of vectors is a Bessel sequence (cf. Christensen (2016)). Now using Theorem 1.2.34, we get plenty of Bessel sequences which are not frames (in finite dimensions). As an example in infinite dimensions, we claim that $\{e_2, e_3, \dots\}$ is a Bessel sequence for $\ell^2(\mathbb{N})$ but not a frame. Clearly $\{e_2, e_3, \dots\}$ satisfies Inequality 1.2.6. If this is a frame, let $a > 0$ be such that first inequality in 1.2.3 holds. Then by taking $h = e_1$, we get $a\|e_1\|^2 \leq \sum_{n=2}^{\infty} |\langle e_1, e_n\rangle|^2 = 0 \Rightarrow a = 0$, which is a contradiction.

Example 1.2.57. *(i) If $\{\tau_n\}_n$ is a frame for $\mathcal{H}$, then for every subset $\mathbb{S}$ of $\mathbb{N}$, $\{\tau_n\}_{n\in\mathbb{S}}$ is a Bessel sequence for $\mathcal{H}$, because $\sum_{n\in\mathbb{S}} |\langle h, \tau_n\rangle|^2 \leq \sum_{n=1}^{\infty} |\langle h, \tau_n\rangle|^2, \forall h \in \mathcal{H}$.*

(ii) (cf. Christensen (2016)) Let $g \in \mathcal{L}^2(\mathbb{R})$ be bounded, compactly supported function and $a, b > 0$. Define $f_{n,m} : \mathbb{R} \ni x \mapsto e^{2\pi imbx} g(x - na) \in \mathbb{R}$, $\forall n, m \in \mathbb{Z}$. Then $\{f_{n,m}\}_{n,m\in\mathbb{Z}}$ is a Bessel sequence for $\mathcal{L}^2(\mathbb{R})$.

(iii) (cf. Christensen (2016)) If $\{\tau_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, then $\{\tau_n + \tau_{n+1}\}_{n=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}$ (but not a frame for $\mathcal{H}$).

Theorem 1.2.58. *(cf. Christensen (2016)) A collection $\{\tau_n\}_n$ is a Bessel sequence for $\mathcal{H}$ if and only if the map $\ell^2(\mathbb{N}) \ni \{\tau_n\}_n \mapsto \sum_{n=1}^{\infty} a_n\tau_n \in \mathcal{H}$ is a well-defined bounded-linear operator. Moreover, if $\{\tau_n\}_n$ is a Bessel sequence for $\mathcal{H}$, then the operator $\mathcal{H} \ni h \mapsto \sum_{n=1}^{\infty} \langle h, \tau_n\rangle \tau_n \in \mathcal{H}$ is positive.*

Since a Bessel sequence need not be a frame, it is natural to ask the following question: Given a Bessel sequence, can we add extra elements to it so that the resulting sequence is a frame? Answer is positive. This result was obtained by Li and Sun (2009).

Theorem 1.2.59. *(Li and Sun (2009)) Every Bessel sequence in a Hilbert space can be expanded to a tight frame. Moreover, we can expand a Bessel sequence in infinitely many ways to tight frames.*

Li and Sun (2009) further observed that if a Bessel sequence for a Hilbert space can be expanded finitely to get a tight frame, then the number of elements added can not be small. Precise statement reads as follows.

Theorem 1.2.60. *(Li and Sun (2009)) Let $\{\tau_n\}_n$ be a Bessel sequence for $\mathcal{H}$. If $\{\tau_n\}_n \cup \{\omega_k\}_{k=1}^{N}$ is a λ-tight frame for $\mathcal{H}$, then*

$$N \geq \dim(\lambda I_{\mathcal{H}} - S_\tau)(\mathcal{H}). \tag{1.2.7}$$

Further, the Inequality (1.2.7) can not be improved.

1.3 RIESZ BASES, FRAMES AND BESSEL SEQUENCES FOR BANACH SPACES

Definition of Riesz basis, as given in Definition 1.2.19 requires the notion of inner product. Due to the lack of inner product in a Banach space, Definition 1.2.19 can not be carried over to Banach spaces. However, Theorem 1.2.25 allows to define Riesz basis for Banach spaces as follows.

Definition 1.3.1. *(Aldroubi et al. (2001)) Let $1 < q < \infty$ and $\mathscr{X}$ be a Banach space. A collection $\{\tau_n\}_n$ in $\mathscr{X}$ is said to be a*

(i) ***q-Riesz sequence*** *for $\mathscr{X}$ if there exist $a, b > 0$ such that for every finite subset $\mathbb{S}$ of $\mathbb{N}$,*

$$a\left(\sum_{n\in\mathbb{S}}|c_n|^q\right)^{\frac{1}{q}} \leq \left\|\sum_{n\in\mathbb{S}}c_n\tau_n\right\| \leq b\left(\sum_{n\in\mathbb{S}}|c_n|^q\right)^{\frac{1}{q}}, \quad \forall c_n \in \mathbb{K}. \tag{1.3.1}$$

(ii) ***q-Riesz basis*** *for $\mathscr{X}$ if it is a q-Riesz sequence for $\mathscr{X}$ and $\overline{\operatorname{span}}\{\tau_n\}_n = \mathscr{X}$.*

Example 1.3.2. *Let $\{e_n\}_n$ be the standard Schauder basis for $\ell^p(\mathbb{N})$ and $A : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ be a bounded linear invertible operator. Then it follows that $\{Ae_n\}_n$ is a p-Riesz basis for $\ell^p(\mathbb{N})$.*

Like Theorem 1.2.22, we have a similar result for p-Riesz basis.

Theorem 1.3.3. *(Christensen and Stoeva (2003)) Let $\{f_n\}_n$ be a q-Riesz basis for $\mathscr{X}^*$ and let p be the conjugate index of q. Then there exists a unique p-Riesz basis $\{\tau_n\}_n$ for $\mathscr{X}$ such that*

$$x = \sum_{n=1}^{\infty} f_n(x)\tau_n, \quad \forall x \in \mathscr{X} \quad \text{and} \quad f = \sum_{n=1}^{\infty} f(\tau_n)f_n, \quad \forall f \in \mathscr{X}^*.$$

By realizing that the functional $\mathscr{H} \ni h \mapsto \langle h, \tau_n\rangle \in \mathbb{K}$ is bounded linear, Definition 1.2.31 leads to the following in Banach spaces.

Definition 1.3.4. *(Aldroubi et al. (2001); Christensen and Stoeva (2003)) Let $1 < p < \infty$ and $\mathscr{X}$ be a Banach space.*

(i) A collection $\{f_n\}_n$ of bounded linear functionals in $\mathscr{X}^*$ is said to be a ***p-frame*** *for $\mathscr{X}$ if there exist $a, b > 0$ such that*

$$a\|x\| \leq \left(\sum_{n=1}^{\infty}|f_n(x)|^p\right)^{\frac{1}{p}} \leq b\|x\|, \quad \forall x \in \mathscr{X}.$$

If a can take the value 0, then we say $\{f_n\}_n$ *is a p-Bessel sequence for* $\mathscr{X}$.

(ii) *A collection* $\{\tau_n\}_n$ *in* $\mathscr{X}$ *is said to be a* ***p-frame*** *for* $\mathscr{X}^*$ *if there exist* $a, b > 0$ *such that*

$$a\|f\| \leq \left(\sum_{n=1}^{\infty} |f(\tau_n)|^p\right)^{\frac{1}{p}} \leq b\|f\|, \quad \forall f \in \mathscr{X}^*.$$

Example 1.3.5. (i) *Let* $\{e_n\}_n$ *be the standard Schauder basis for* $\ell^p(\mathbb{N})$, $\{\zeta_n\}_n$ *be the coordinate functionals associated to* $\{e_n\}_n$ *and* $A : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ *be a bounded linear invertible operator. Then it follows that* $\{\zeta_n A\}_n$ *is a p-frame for* $\ell^p(\mathbb{N})$.

(ii) *(Aldroubi et al. (2001)) Let* $1 \leq p < \infty$ *and* $a \in \mathbb{R}$. *Define*

$$T_a : \mathscr{L}^p(\mathbb{R}) \ni f \mapsto T_a f \in \mathscr{L}^p(\mathbb{R}), \quad T_a f : \mathbb{R} \mapsto (T_a f)(x) := f(x-a) \in \mathbb{C}.$$

Define

$$W := \left\{ f : \mathbb{R} \to \mathbb{C} \,\middle|\, \sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |T_k f(x)| < \infty \right\}$$

and for $1 < p < \infty$, $\phi \in W$,

$$S_p := \left\{ \sum_{k\in\mathbb{Z}} c_k T_k \phi \,\middle|\, \{c_k\}_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z}) \right\}.$$

Then S_p *is a closed subspace of* $\mathscr{L}^p(\mathbb{R})$ *and* $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is a p-frame for* $\mathscr{L}^p(\mathbb{R})$.

Like Theorem 1.2.58, we have a similar result for Banach spaces.

Theorem 1.3.6. *(Christensen and Stoeva (2003)) Let* $\mathscr{X}$ *be a Banach space and* $\{f_n\}_n$ *be a sequence in* $\mathscr{X}^*$.

(i) $\{f_n\}_n$ *is a p-Bessel sequence for* $\mathscr{X}$ *with bound b if and only if*

$$T : \ell^q(\mathbb{N}) \ni \{a_n\}_n \mapsto \sum_{n=1}^{\infty} a_n f_n \in \mathscr{X}^* \tag{1.3.2}$$

is a well-defined (hence bounded) linear operator and $\|T\| \leq b$ *(where q is the conjugate index of p).*

(ii) *If* $\mathscr{X}$ *is reflexive, then* $\{f_n\}_n$ *is a p-frame for* $\mathscr{X}$ *if and only if the operator T in (1.3.2) is surjective.*

Rather working on p-frames, one can consider a general notion of frames, which are generalizations of $\ell^p(\mathbb{N})$ spaces. For this, we need the notion of BK-space (Banach scalar valued sequence space or Banach coordinate space).

Definition 1.3.7. *(cf. Banaś and Mursaleen (2014)) A sequence space $\mathscr{X}_d$ is said to be a **BK-space** if it is a Banach space and all the coordinate functionals are continuous, i.e., whenever $\{x_n\}_n$ is a sequence in $\mathscr{X}_d$ converging to $x \in \mathscr{X}_d$, then each coordinate of x_n converges to each coordinate of x.*

Familiar sequence spaces like $\ell^p(\mathbb{N})$, $c(\mathbb{N})$ (space of convergent sequences) and $c_0(\mathbb{N})$ (space of sequences converging to zero) are examples of BK-spaces. We now recall an example of a sequence space which is not a BK-space.

Example 1.3.8. *The space $\mathscr{X}_d := \{\{x_n\}_{n=0}^\infty : x_n \in \mathbb{K}, \forall n \in \mathbb{N} \cup \{0\}\}$ equipped with the metric*

$$d(\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty) := \sum_{n=0}^\infty \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad \forall \{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty \in \mathscr{X}_d$$

is not a BK-space.

Definition 1.3.9. *(Casazza et al. (2005a)) Let $\mathscr{X}$ be a Banach space and $\mathscr{X}_d$ be an associated BK-space. A collection $\{f_n\}_n$ in $\mathscr{X}^*$ is said to be a $\mathscr{X}_d$**-frame** for $\mathscr{X}$ if the following holds.*

(i) $\{f_n(x)\}_n \in \mathscr{X}_d$, for each $x \in \mathscr{X}$.

(ii) There exist $a, b > 0$ such that $a\|x\| \leq \|\{f_n(x)\}_n\| \leq b\|x\|, \forall x \in \mathscr{X}$.

Constants a and b are called as $\mathscr{X}_d$-frame bounds.

Definition 1.3.10. *(Casazza et al. (2005a)) Let $\mathscr{X}$ be a Banach space and $\mathscr{X}_d$ be an associated BK-space. A collection $\{\tau_n\}_n$ in $\mathscr{X}$ is said to be a $\mathscr{X}_d$**-frame** for $\mathscr{X}$ if the following holds.*

(i) $\{f(\tau_n)\}_n \in \mathscr{X}_d$, for each $f \in \mathscr{X}^*$.

(ii) There exist $a, b > 0$ such that $a\|f\| \leq \|\{f(\tau_n)\}_n\| \leq b\|f\|, \forall f \in \mathscr{X}^*$.

Definition 1.3.11. *(Gröchenig (1991)) Let $\mathscr{X}$ be a Banach space and $\mathscr{X}_d$ be an associated BK-space. Let $\{f_n\}_n$ be a collection in $\mathscr{X}^*$ and $S : \mathscr{X}_d \to \mathscr{X}$ be a bounded linear operator. The pair $(\{f_n\}_n, S)$ is said to be a **Banach frame** for $\mathscr{X}$ if the following holds.*

(i) $\{f_n(x)\}_n \in \mathscr{X}_d$, for each $x \in \mathscr{X}$.

(ii) There exist $a,b>0$ such that $a\|x\| \leq \|\{f_n(x)\}_n\| \leq b\|x\|, \forall x \in \mathscr{X}$.

(iii) $S(\{f_n(x)\}_n)=x$, for each $x \in \mathscr{X}$.

*Constants a and b are called as **lower Banach frame bound** and **upper Banach frame bound**, respectively. The operator S is called as **reconstruction operator** and the operator $\theta_f : \mathscr{X} \ni x \mapsto \theta_f(x) \coloneqq \{f_n(x)\}_n \in \mathscr{X}_d$ is called as **analysis operator**.*

Example 1.3.12. *(i) Let $\{\tau_n\}_n$ be a frame for a Hilbert space $\mathscr{H}$ with bounds a and b. Let $f \in \mathscr{H}^*$. Let $h_f \in \mathscr{H}$ be such that $f(h)=\langle h,h_f\rangle$, $\forall h \in \mathscr{H}$ and $\|f\|=\|h_f\|$. Then*

$$a\|f\|^2 = a\|h_f\|^2 \leq \sum_{n=1}^{\infty} |\langle h_f,\tau_n\rangle|^2 = \sum_{n=1}^{\infty} |f(\tau_n)|^2 \leq b\|h_f\|^2 = b\|f\|^2.$$

Therefore $\{\tau_n\}_n$ is an $\ell^2(\mathbb{N})$-frame for $\mathscr{H}$.

(ii) Let $\{\tau_n\}_n$ be a frame for Hilbert space $\mathscr{H}$. We define $f_n(h) \coloneqq \langle h,\tau_n\rangle$, $\forall h \in \mathscr{H}$, $\forall n$ and $S \coloneqq S_\tau^{-1}\theta_\tau^*$. Then $(\{f_n\}_n, S)$ is a Banach frame for $\mathscr{H}$.

(iii) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. We define $f_n(h) \coloneqq \langle h, S_\tau^{-1}\tau_n\rangle$, $\forall h \in \mathscr{H}$, $\forall n$ and $S \coloneqq \theta_\tau^*$. Then $(\{f_n\}_n, S)$ is a Banach frame for $\mathscr{H}$.

(iv) (Casazza et al. (2005a)) Let $\{\tau_n\}_n$ be an orthonormal basis for a Hilbert space $\mathscr{H}$. We define

$$\mathscr{X}_d \coloneqq \{\{\langle h,\tau_n+\tau_{n+1}\rangle\}_n : h \in \mathscr{H}\} = \{\{a_n+a_{n+1}\}_n : \{a\}_n \in \ell^2(\mathbb{N})\}$$

equipped with the norm

$$\|\{a_n+a_{n+1}\}_n\| \coloneqq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}}.$$

Then $\mathscr{X}_d$ is a BK-space. Define $f_n : \mathscr{H} \ni h \mapsto \langle h,\tau_n+\tau_{n+1}\rangle \in \mathbb{K}$, $\forall n$, $T : \mathscr{H} \ni h \mapsto \{f_n(h)\}_n \in \mathscr{X}_d$ and set $S \coloneqq T^{-1}$. Then $(\{f_n\}_n, S)$ is a Banach frame for $\mathscr{H}$. However, $\{\tau_n+\tau_{n+1}\}_n$ is not a frame for $\mathscr{H}$.

For Hilbert spaces it follows immediately that every Hilbert space has a frame because separable spaces have orthonormal bases and an orthonormal basis is a frame. Using Hahn-Banach theorem, the following result was proved in (Casazza et al. (1999)).

Theorem 1.3.13. *(Casazza et al. (1999)) Every separable Banach space admits a Banach frame.*

The notion of atomic decomposition is studied along with the notion of frames for Banach spaces. This is defined as follows.

Definition 1.3.14. *(Gröchenig (1991)) Let $\mathscr{X}$ be a Banach space and $\mathscr{X}_d$ be an associated BK-space. Let $\{f_n\}_n$ be a collection in $\mathscr{X}^*$ and $\{\tau_n\}_n$ be a collection in $\mathscr{X}$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be an* ***atomic decomposition*** *for $\mathscr{X}$ if the following holds.*

(i) $\{f_n(x)\}_n \in \mathscr{X}_d$, *for each* $x \in \mathscr{X}$.

(ii) There exist $a, b > 0$ *such that* $a\|x\| \leq \|\{f_n(x)\}_n\| \leq b\|x\|, \forall x \in \mathscr{X}$.

(iii) $x = \sum_{n=1}^{\infty} f_n(x)\tau_n$, *for each* $x \in \mathscr{X}$.

Constants a and b are called as ***lower atomic bound*** *and* ***upper atomic bound****, respectively.*

Example 1.3.15. *(i) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. By defining $f_n(h) := \langle h, S_\tau^{-1}\tau_n\rangle$, $\forall h \in \mathscr{H}$, $\forall n$, the pair $(\{f_n\}_n, \{\tau_n\}_n)$ satisfies all the conditions of Definition 1.3.14 and hence it is an atomic decomposition for $\mathscr{H}$.*

(ii) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. By defining $f_n(h) := \langle h, \tau_n\rangle$, $\forall h \in \mathscr{H}$, and $\omega_n := S_\tau^{-1}\tau_n$, $\forall n$, the pair $(\{f_n\}_n, \{\omega_n\}_n)$ satisfies all the conditions of Definition 1.3.14 and hence it is an atomic decomposition for $\mathscr{H}$.

Now we make a detailed observation that the notions of atomic decompositions and Banach frames are completely different. Definition 1.3.14 of atomic decomposition demands the expression of every element of a Banach space as a convergent series in the same Banach space using a collection of bounded linear functionals on the space and a collection of elements in the same Banach space. On the other hand, Definition 1.3.11 of Banach frame demands the expression of every element of a Banach space using a bounded linear operator from the BK-space to the Banach space and a collection of bounded linear functionals on the same space without bothering about a converging series expansion. Theorem 1.3.13 guarantees the existence of Banach frame for every separable Banach space. However, the following remarkable result gives a lot of conditions on Banach spaces to admit an atomic decomposition.

Theorem 1.3.16. *(Casazza et al. (1999); Johnson et al. (1971); Pelczynski (1971)) For a Banach space $\mathscr{X}$, the following are equivalent.*

(i) $\mathscr{X}$ has an atomic decomposition.

(ii) $\mathscr{X}$ has a finite dimensional expansion of identity.

(iii) $\mathscr{X}$ *is complemented in a Banach space with a Schauder basis.*

(iv) $\mathscr{X}$ *has bounded approximation property.*

There is a close relationship between atomic decomposition and Banach frames for certain classes of Banach spaces. These are exhibited in the following theorems.

Theorem 1.3.17. *Casazza et al. (1999) Let* $\mathscr{X}$ *be a Banach space,* $\mathscr{X}_d$ *be a BK-space,* $\{f_n\}_n$ *be a collection in* $\mathscr{X}^*$ *and* $S : \mathscr{X}_d \to \mathscr{X}$ *be a bounded linear operator. If the canonical unit vectors* $\{e_n\}_n$ *are in* $\mathscr{X}_d$*, then the following are equivalent.*

(i) $(\{f_n\}_n, S)$ *is a Banach frame for* $\mathscr{X}$ *and* $\{e_n\}_n$ *is a Schauder basis for* $\mathscr{X}_d$*.*

(ii) $(\{f_n\}_n, \{Se_n\}_n)$ *is an atomic decomposition for* $\mathscr{X}$*.*

Theorem 1.3.18. *(Casazza et al. (1999); Johnson et al. (1971); Pelczynski (1971)) Let* $\mathscr{X}$ *be a Banach space and* $\mathscr{X}_d$ *be a BK-space. Let* $\{f_n\}_n$ *be a collection in* $\mathscr{X}^*$ *and* $S : \mathscr{X}_d \to \mathscr{X}$ *be a bounded linear operator. If the canonical unit vectors* $\{e_n\}_n$ *are in* $\mathscr{X}_d$*, then the following are equivalent.*

(i) There exists a BK-space $\mathscr{X}_d$ *such that* $(\{f_n\}_n, \{\tau_n\}_n)$ *is an atomic decomposition for* $\mathscr{X}$*.*

(ii) There exists a BK-space $\mathscr{Y}_d$ *which has canonical unit vectors* $\{e_n\}_n$ *as Schauder basis and a bounded linear operator* $S : \mathscr{Y}_d \to \mathscr{X}$ *such that* $(\{f_n\}_n, S)$ *is a Banach frame for* $\mathscr{X}$*. Further, S can be taken as a projection and* $Se_n = \tau_n$*, for all* $n \in \mathbb{N}$*.*

Next theorem shows that there is a relation between atomic decompositions and projections.

Theorem 1.3.19. *(Casazza et al. (1999)) Let* $\mathscr{X}$ *be a Banach space,* $\{\tau_n\}_n$ *be a collection in* $\mathscr{X}$*,* $\{f_n\}_n$ *be a collection in* $\mathscr{X}^*$ *and* $S : \mathscr{X}_d \to \mathscr{X}$ *be a bounded linear operator. Let* $\{e_n\}_n$ *be the standard unit vectors in* $\mathscr{X}_d$*. Then the following are equivalent.*

(i) There is a BK-space $\mathscr{X}_d$ *such that* $(\{f_n\}_n, \{\tau_n\}_n)$ *is an atomic decomposition for* $\mathscr{X}$*.*

(ii) There is a Banach space $\mathscr{Z}$ *with a Schauder basis* $\{\omega_n\}_n$ *such that* $\mathscr{X} \subseteq \mathscr{Z}$ *and there is a bounded linear projection* $P : \mathscr{Z} \to \mathscr{X}$ *such that* $P\omega_n = \tau_n$*,* $\forall n \in \mathbb{N}$*.*

We mention here before passing that it is known, one can simultaneously construct Banach frames and atomic decompositions for certain classes of Banach spaces such as coorbit spaces (Gröchenig (1991)), α-modulation spaces (Fornasier (2007)), decomposition spaces (Borup and Nielsen (2007)), homogeneous spaces (Dahlke et al. (2007)), weighted coorbit spaces (Dahlke et al. (2004)), generalized coorbit spaces (Dahlke et al. (2008)), inhomogeneous function spaces (Rauhut and Ullrich (2011)) and Bergman spaces (Christensen et al. (2017)).

Using approximation property for Banach spaces, Casazza and Christensen (2008) proved that $\mathscr{X}_d$-frame for a Banach space need not admit representation of every element of the Banach space. With regard to this, the following result gives information when it is possible to express element of the Banach space using series.

Theorem 1.3.20. *(Casazza et al. (2005a)) Let $\mathscr{X}_d$ be a BK-space and $\{f_n\}_n$ be an $\mathscr{X}_d$-frame for $\mathscr{X}$. Let $\theta_f : \mathscr{X} \ni x \mapsto \{f_n(x)\}_n \in \mathscr{X}_d$ (this map is a well-defined linear bounded below operator). The following are equivalent.*

(i) $\theta_f(\mathscr{X})$ is complemented in $\mathscr{X}_d$.

(ii) The operator $\theta_f^{-1} : \theta_f(\mathscr{X}) \to \mathscr{X}$ can be extended to a bounded linear operator $T_f : \mathscr{X}_d \to \mathscr{X}$.

(iii) There exists a bounded linear operator $S : \mathscr{X}_d \to \mathscr{X}$ such that $(\{f_n\}_n, S)$ is a Banach frame for $\mathscr{X}$.

Also, the condition

(iv) There exists a sequence $\{\tau_n\}_n$ in $\mathscr{X}$ such that $\sum_{n=1}^{\infty} a_n\tau_n$ is convergent in $\mathscr{X}$ for all $\{a_n\}_n$ in $\mathscr{X}_d$ and $x = \sum_{n=1}^{\infty} f_n(x)\tau_n, \forall x \in \mathscr{X}$.

implies each of (i)-(iii). If we also assume that the canonical unit vectors $\{e_n\}_n$ form a Schauder basis for $\mathscr{X}_d$, (iv) is equivalent to the above (i)-(iii) and to the following condition (v).

(v) There exists an $\mathscr{X}_d^*$-Bessel sequence $\{\tau_n\}_n \subseteq \mathscr{X} \subseteq \mathscr{X}^{**}$ for $\mathscr{X}^*$ such that $x = \sum_{n=1}^{\infty} f_n(x)\tau_n$, $\forall x \in \mathscr{X}$.

If the canonical unit vectors $\{e_n\}_n$ form a Schauder basis for both $\mathscr{X}_d$ and $\mathscr{X}_d^*$, then (i)-(v) is equivalent to the following condition (vi).

(vi) There exists an $\mathscr{X}_d^*$-Bessel sequence $\{\tau_n\}_n \subseteq \mathscr{X} \subseteq \mathscr{X}^{**}$ for $\mathscr{X}^*$ such that $f = \sum_{n=1}^{\infty} f(\tau_n)f_n$, $\forall f \in \mathscr{X}^*$.

In each of the cases (v) and (vi), $\{\tau_n\}_n$ is actually an $\mathscr{X}_d^*$-frame for $\mathscr{X}^*$.

We end this section by mentioning a perturbation theorem for Banach frames.

Theorem 1.3.21. *(Christensen and Heil (1997)) (**Christensen-Heil perturbation**) Let $(\{f_n\}_n, S)$ be a Banach frame for a Banach space $\mathscr{X}$. Let $\{g_n\}_n$ be a collection in $\mathscr{X}^*$ satisfying the following.*

(i) There exist $\alpha, \gamma \geq 0$ such that

$$\|\{(f_n - g_n)(x)\}_n\| \leq \alpha\|\{f_n(x)\}_n\| + \gamma\|x-y\|, \ \forall x, y \in \mathscr{X}.$$

(ii) $\alpha\|\theta_f\| + \gamma \leq \|S\|^{-1}$.

Then there exists a reconstruction operator T such that $(\{f_n\}_n, T)$ is a Banach frame for $\mathscr{X}$ with bounds $\|S\|^{-1} - (\alpha\|\theta_f\| + \gamma)$ and $\|\theta_f\| + (\alpha\|\theta_f\| + \gamma)$.

1.4 MULTIPLIERS FOR BANACH SPACES

Let $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$ and $\{\tau_n\}_n$, $\{\omega_n\}_n$ be sequences in a Hilbert space $\mathscr{H}$. For $x, y \in \mathscr{H}$, the operator defined by $\mathscr{H} \ni h \mapsto \langle h, y\rangle x \in \mathscr{H}$ is denoted by $x \otimes \overline{y}$.

The study of operators of the form

$$\sum_{n=1}^{\infty} \lambda_n(\tau_n \otimes \overline{\omega_n}) \tag{1.4.1}$$

began with Schatten (1960), in connection with the study of compact operators. Schatten studied the operator in (1.4.1) whenever $\{\tau_n\}_n$ and $\{\omega_n\}_n$ are orthonormal sequences in a Hilbert space $\mathscr{H}$. He showed that if $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$ and $\{\tau_n\}_n$, $\{\omega_n\}_n$ are orthonormal sequences in a Hilbert space $\mathscr{H}$, then the map in (1.4.1) is a well-defined bounded linear operator. Later, operators in (1.4.1) are studied mainly in connection with Gabor analysis (Feichtinger and Nowak (2003), Benedetto and Pfander (2006), Dorfler and Torresani (2010), Gibson et al. (2013), Cordero et al. (2012), Skrettingland (2020)). This was generalized by Balazs (2007) who replaced orthonormal sequences by Bessel sequences.

Let $\{f_n\}_n$ be a sequence in the dual space $\mathscr{X}^*$ of a Banach space $\mathscr{X}$ and $\{\tau_n\}_n$ be a sequence in a Banach space $\mathscr{Y}$. The operator $\tau \otimes f$ is defined by $\tau \otimes f : \mathscr{X} \ni x \mapsto f(x)\tau \in \mathscr{Y}$. It was Rahimi and Balazs (2010) who extended the operator in (1.4.1) from Hilbert spaces to Banach spaces. For a Banach space $\mathscr{X}$ and dual space $\mathscr{X}^*$, they considered the operators (called as **multipliers**) of the form

$$M_{\lambda,f,\tau} := \sum_{n=1}^{\infty} \lambda_n(\tau_n \otimes f_n). \tag{1.4.2}$$

Rahimi and Balazs studied the operator in (1.4.2), whenever $\{\tau_n\}_n$ is a p-Bessel sequence for $\mathscr{X}^*$ and $\{f_n\}_n$ is a q-Bessel sequence for $\mathscr{X}$ (q is the conjugate index of p). Besides theoretical importance, multipliers also play important role in Physics, signal processing, acoustics (Stoeva and Balazs (2020)).

Fundamental result obtained by Rahimi and Balazs (2010) is the following. In this section, q denotes the conjugate index of p.

Theorem 1.4.1. *(Rahimi and Balazs (2010)) Let $\{f_n\}_n$ be a p-Bessel sequence for a Banach space $\mathscr{X}$ with bound b and $\{\tau_n\}_n$ be a q-Bessel sequence for the dual of a Banach space $\mathscr{Y}$ with bound d. If $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$, then the map*

$$T : \mathscr{X} \ni x \mapsto \sum_{n=1}^{\infty} \lambda_n(\tau_n \otimes f_n)x \in \mathscr{Y}$$

is a well-defined bounded linear operator and $\|T\| \leq bd\|\{\lambda_n\}_n\|_\infty$.

By varying only the symbol in a multiplier, we get a bounded linear operator which has the nice property stated in the following theorem.

Proposition 1.4.2. *(Rahimi and Balazs (2010)) Let $\{f_n\}_n$ be a p-Bessel sequence for a Banach space $\mathscr{X}$ with non-zero elements, $\{\tau_n\}_n$ be a q-Riesz sequence for $\mathscr{Y}$ and let $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$. Then the mapping*

$$T : \ell^\infty(\mathbb{N}) \ni \{\lambda_n\}_n \mapsto M_{\lambda,f,\tau} \in \mathscr{B}(\mathscr{X},\mathscr{Y})$$

is a well-defined injective bounded linear operator.

From the spectral theory of compact operators in Hilbert spaces, it easily follows that the symbol of compact operator converges to zero. Following is a general result for Banach spaces.

Proposition 1.4.3. *(Rahimi and Balazs (2010)) Let $\{f_n\}_n$ be a p-Bessel sequence for a Banach space $\mathscr{X}$ with bound b and $\{\tau_n\}_n$ be a q-Bessel sequence for $\mathscr{Y}$ with bound d. If $\{\lambda_n\}_n \in c_0(\mathbb{N})$, then $M_{\lambda,f,\tau}$ is a compact operator.*

Following theorem shows that multipliers behave nicely with respect to change in its parameters. These are known as continuity properties of multipliers in the literature.

Theorem 1.4.4. *(Rahimi and Balazs (2010)) Let $\{f_n\}_n$ be a p-Bessel sequence for $\mathscr{X}$ with bound b, $\{\tau_n\}_n$ be a q-Bessel sequence for $\mathscr{Y}$ with bound d and $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$. Let $k \in \mathbb{N}$ and let $\lambda^{(k)} = \{\lambda_1^{(k)}, \lambda_2^{(k)}, \dots\}$, $\lambda = \{\lambda_1, \lambda_2, \dots\}$, $\tau^{(k)} = \{\tau_1^{(k)}, \tau_2^{(k)}, \dots\}$, $\tau_n^k \in \mathscr{X}$, $\tau = \{\tau_1, \tau_2, \dots\}$. Assume that for each k, $\lambda^{(k)} \in \ell^\infty(\mathbb{N})$ and $\tau^{(k)}$ is a q-Bessel sequence for $\mathscr{Y}$.*

(i) If $\lambda^{(k)} \to \lambda$ as $k \to \infty$ in p-norm, then

$$\|M_{\lambda^{(k)},f,\tau} - M_{\lambda,f,\tau}\| \to 0 \quad as \quad k \to \infty.$$

(ii) If $\{\lambda_n\}_n \in \ell^p(\mathbb{N})$ and $\sum_{n=1}^{\infty} \|\tau_n^{(k)} - \tau_n\|^q \to 0$ as $k \to \infty$, then

$$\|M_{\lambda,f,\tau^{(k)}} - M_{\lambda,f,\tau}\| \to 0 \quad as \quad k \to \infty.$$

1.5 LIPSCHITZ OPERATORS AND LIPSCHITZ COMPACT OPERATORS

We first recall the definition of Lipschitz function.

Definition 1.5.1. *(cf. Weaver (2018)) Let $\mathscr{M}$, $\mathscr{N}$ be metric spaces. A function $f : \mathscr{M} \to \mathscr{N}$ is said to be **Lipschitz** if there exists $b > 0$ such that*

$$d(f(x), f(y)) \leq b\, d(x,y), \quad \forall x, y \in \mathscr{M}.$$

One of the most important results in the study of Lipschitz functions is the following (which is similar to the Banach-Mazur theorem for Banach spaces which shows that every separable Banach space embeds isometrically in the Banach space of continuous functions on $[0,1]$ (cf. Albiac and Kalton (2016))).

Theorem 1.5.2. *(cf. Aharoni (1974); Kalton and Lancien (2008)) (**Aharoni's theorem**) If $\mathscr{M}$ is a separable metric space, then there exists a function $f : \mathscr{M} \to c_0(\mathbb{N})$ and a constant $b > 0$ such that*

$$d(x,y) \leq \|f(x) - f(y)\| \leq b\, d(x,y), \quad \forall x, y \in \mathscr{M}.$$

Definition 1.5.3. *(cf. Weaver (2018)) A metric space $\mathscr{M}$ with a reference point which is usually denoted by 0 is called as a **pointed metric space**. In this case, we write $(\mathscr{M}, 0)$ is a pointed metric space.*

Note that every metric space is a pointed metric space by fixing any point of the space.

Just like norm of linear operator, a reasonable measure of a Lipschitz function can be defined. This is exhibited in the next definition.

Definition 1.5.4. *(cf. Weaver (2018)) Let $\mathscr{X}$ be a Banach space.*

(i) Let $\mathscr{M}$ be a metric space. The collection $\mathrm{Lip}(\mathscr{M},\mathscr{X})$ *is defined as* $\mathrm{Lip}(\mathscr{M},\mathscr{X}) := \{f : f : \mathscr{M} \to \mathscr{X} \text{ is Lipschitz}\}$. *For $f \in \mathrm{Lip}(\mathscr{M},\mathscr{X})$, the **Lipschitz number** is defined as*

$$\mathrm{Lip}(f) := \sup_{x,y\in\mathscr{M},\, x\neq y} \frac{\|f(x)-f(y)\|}{d(x,y)}.$$

(ii) Let $(\mathscr{M},0)$ be a pointed metric space. The collection $\mathrm{Lip}_0(\mathscr{M},\mathscr{X})$ *is defined as* $\mathrm{Lip}_0(\mathscr{M},\mathscr{X}) := \{f : f : \mathscr{M} \to \mathscr{X} \text{ is Lipschitz and } f(0)=0\}$. *For $f \in \mathrm{Lip}_0(\mathscr{M},\mathscr{X})$, the **Lipschitz norm** is defined as*

$$\|f\|_{\mathrm{Lip}_0} := \sup_{x,y\in\mathscr{M}, x\neq y} \frac{\|f(x)-f(y)\|}{d(x,y)}.$$

It is well-known that given two Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, the collection of all bounded linear maps from $\mathscr{X}$ to $\mathscr{Y}$ is a Banach space with respect to operator-norm. A similar result holds for base point preserving Lipschitz maps from pointed metric spaces to Banach spaces.

Theorem 1.5.5. *(cf. Weaver (2018)) Let $\mathscr{X}$ be a Banach space.*

(i) If $\mathscr{M}$ is a metric space, then $\mathrm{Lip}(\mathscr{M},\mathscr{X})$ *is a semi-normed vector space with respect to the semi-norm* $\mathrm{Lip}(\cdot)$.

(ii) If $(\mathscr{M},0)$ is a pointed metric space, then $\mathrm{Lip}_0(\mathscr{M},\mathscr{X})$ *is a Banach space with respect to the norm* $\|\cdot\|_{\mathrm{Lip}_0}$. *Further,* $\mathrm{Lip}_0(\mathscr{X}) := \mathrm{Lip}_0(\mathscr{X},\mathscr{X})$ *is a unital Banach algebra. In particular, if $T \in \mathrm{Lip}_0(\mathscr{X})$ satisfies $\|T - I_{\mathscr{X}}\|_{\mathrm{Lip}_0} < 1$, then T is invertible and $T^{-1} \in \mathrm{Lip}_0(\mathscr{X})$.*

In the study of Lipschitz functions it is natural to shift from metric space to the setting of Banach spaces and use functional analysis tools on Banach spaces. This is achieved through the following theorem.

Theorem 1.5.6. *(cf. Arens and Eells (1956); Kalton (2004); Weaver (2018)) Let $(\mathscr{M},0)$ be a pointed metric space. Then there exists a Banach space $\mathscr{F}(\mathscr{M})$ and an isometric embedding $e : \mathscr{M} \to \mathscr{F}(\mathscr{M})$ satisfying the following universal property: for each Banach space $\mathscr{X}$ and each $f \in \mathrm{Lip}_0(\mathscr{M},\mathscr{X})$, there is a unique bounded linear operator $T_f : \mathscr{F}(\mathscr{M}) \to \mathscr{X}$ such that $T_f e = f$, i.e., the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{M} & & \\ \Big\downarrow e & \overset{f}{\searrow} & \\ \mathscr{F}(\mathscr{M}) & \xrightarrow{T_f} & \mathscr{X} \end{array}$$

Further, $\|T_f\| = \|f\|_{\mathrm{Lip}_0}$. *This property characterizes the pair* $(\mathscr{F}(\mathscr{M}), e)$ *uniquely upto isometric isomorphism. Moreover, the map* $\mathrm{Lip}_0(\mathscr{M}, \mathscr{X}) \ni f \mapsto T_f \in \mathscr{B}(\mathscr{F}(\mathscr{M}), \mathscr{X})$ *is an isometric isomorphism.*

The space $\mathscr{F}(\mathscr{M})$ is known as **Arens-Eells space** or **Lipschitz-free Banach space** (Godefroy (2015)). Theorem 1.5.6 tells that in order to 'find' the space $\mathrm{Lip}_0(\mathscr{M}, \mathscr{X})$, we can find first $\mathscr{F}(\mathscr{M})$ and then $\mathscr{B}(\mathscr{F}(\mathscr{M}), \mathscr{X})$. In particular, $\mathrm{Lip}_0(\mathscr{M}, \mathbb{K})$ is isometrically isomorphic to $\mathscr{F}(\mathscr{M})^*$. For this reason $\mathscr{F}(\mathscr{M})$ is also called as **predual** of metric space $\mathscr{M}$. The bounded linear operator T_f is called as **linearization** of f. We now mention some examples of Lipschitz-free spaces for certain metric spaces.

Example 1.5.7. *(cf. Weaver (2018), Dubei et al. (2009))*

(i) If $\mathbb{R}$ *is considered with usual metric, then* $\mathscr{F}(\mathbb{R}) \cong \mathscr{L}^1(\mathbb{R})$.

(ii) If $\mathscr{M}$ *is any separable metric tree, then* $\mathscr{F}(\mathscr{M}) \cong \mathscr{L}^1([0,1])$.

(iii) If $\mathscr{M}$ *is any set equipped with the metric* $d(x,y) := 2$ *whenever* $x, y \in \mathscr{M}$ *with* $x \neq y$ *and* $d(x,x) := 0$, $\forall x \in \mathscr{M}$, *then* $\mathscr{F}(\mathscr{M}) \cong \ell^1(\mathscr{M})$.

(iv) If $\mathbb{N}$ *is considered with usual metric, then* $\mathscr{F}(\mathbb{N}) \cong \ell^1(\mathbb{N})$.

(v) $\mathscr{F}(\ell^1(\mathbb{N})) \cong \mathscr{L}^1(\mathbb{R})$.

In the theory of bounded linear operators between Banach spaces, an operator is said to be compact if the image of the unit ball under the operator is precompact (Fabian et al. (2011)). Linearity of the operator now gives various characterizations of compactness and plays an important role in rich theories such as theory of integral equations, spectral theory, theory of Fredholm operators, operator algebra (C*-algebra), K-theory, Calkin algebra, (operator) ideal theory, approximation properties of Banach spaces, Schauder basis theory. Lack of linearity is a hurdle when one tries to define compactness of non-linear maps. This hurdle was successfully crossed in the paper (Jiménez-Vargas et al. (2014)) which began the study of Lipschitz compact operators.

Definition 1.5.8. *(Jiménez-Vargas et al. (2014)) If $\mathscr{M}$ is a metric space and $\mathscr{X}$ is a Banach space, then the* ***Lipschitz image*** *of a Lipschitz map (also called as* ***Lipschitz operator****) $f : \mathscr{M} \to \mathscr{X}$ is defined as the set*

$$\left\{ \frac{f(x) - f(y)}{d(x,y)} : x, y \in \mathscr{M}, x \neq y \right\}. \tag{1.5.1}$$

We observe that whenever an operator is linear, the set in (1.5.1) is simply the image of the unit sphere.

Definition 1.5.9. *(Jiménez-Vargas et al. (2014)) If $(\mathscr{M}, 0)$ is a pointed metric space and $\mathscr{X}$ is a Banach space, then a Lipschitz map $f : \mathscr{M} \to \mathscr{X}$ such that $f(0) = 0$ is said to be* ***Lipschitz compact*** *if its Lipschitz image is relatively compact in $\mathscr{X}$, i.e., the closure of the set in (1.5.1) is compact in $\mathscr{X}$.*

As showed in (Jiménez-Vargas et al. (2014)), there is a large collection of Lipschitz compact operators. To state this, first we need a definition.

Definition 1.5.10. *(Chen and Zheng (2012)) Let $(\mathscr{M}, 0)$ be a pointed metric space and $\mathscr{X}$ be a Banach space. A Lipschitz operator $f : \mathscr{M} \to \mathscr{X}$ such that $f(0) = 0$ is said to be* ***strongly Lipschitz p-nuclear*** *$(1 \leq p < \infty)$ if there exist operators $A \in \mathscr{B}(\ell^p(\mathbb{N}), \mathscr{X})$, $g \in \mathrm{Lip}_0(\mathscr{M}, \ell^\infty(\mathbb{N}))$ and a diagonal operator $M_\lambda \in \mathscr{B}(\ell^\infty(\mathbb{N}), \ell^p(\mathbb{N}))$ induced by a sequence $\lambda \in \ell^p(\mathbb{N})$ such that $f = AM_\lambda g$, i.e., the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\;f\;} & \mathscr{X} \\ \Big\downarrow g & & \Big\uparrow A \\ \ell^\infty(\mathbb{N}) & \xrightarrow{\;M_\lambda\;} & \ell^p(\mathbb{N}) \end{array}$$

Proposition 1.5.11. *(Jiménez-Vargas et al. (2014)) Every strongly Lipschitz p-nuclear operator from a pointed metric space to a Banach space is Lipschitz compact.*

Since the image of a linear operator is a subspace, the natural definition of finite rank operator is that image is a finite dimensional subspace. The image of Lipschitz map may not be a subspace. Thus care has to be taken while defining rank of such maps.

Definition 1.5.12. *(Jiménez-Vargas et al. (2014)) If $(\mathscr{M}, 0)$ is a pointed metric space and $\mathscr{X}$ is a Banach space, then a Lipschitz function $f : \mathscr{M} \to \mathscr{X}$ such that $f(0) = 0$ is said to have* ***Lipschitz finite dimensional rank*** *if the linear hull of its Lipschitz image is a finite dimensional subspace of $\mathscr{X}$.*

Definition 1.5.13. *(Jiménez-Vargas et al. (2014)) If $\mathscr{M}$ is a metric space and $\mathscr{X}$ is a Banach space, then a Lipschitz function $f : \mathscr{M} \to \mathscr{X}$ is said to have* ***finite dimensional rank*** *if the linear hull of its image is a finite dimensional subspace of $\mathscr{X}$.*

Next theorem shows that for pointed metric spaces, Definitions 1.5.12 and 1.5.13 are equivalent.

Theorem 1.5.14. *(Achour et al. (2016); Jiménez-Vargas et al. (2014)) Let $(\mathscr{M}, 0)$ be a pointed metric space and $\mathscr{X}$ be a Banach space. For a Lipschitz function $f : \mathscr{M} \to \mathscr{X}$ such that $f(0) = 0$, the following are equivalent.*

(i) f has Lipschitz finite dimensional rank.

(ii) f has finite dimensional rank.

(iii) There exist $f_1, \ldots, f_n$ in $\mathrm{Lip}_0(\mathscr{M}, \mathbb{K})$ and $\tau_1, \ldots, \tau_n$ in $\mathscr{X}$ such that

$$f(x) = \sum_{k=1}^{n} f_k(x)\tau_k, \quad \forall x \in \mathscr{M}.$$

In Hilbert spaces (and not in Banach spaces), every compact operator is approximable by finite rank operators in the operator norm (cf. Fabian et al. (2011)). Following is the definition of approximable operator for Lipschitz maps.

Definition 1.5.15. *(Jiménez-Vargas et al. (2014)) If $(\mathscr{M}, 0)$ is a pointed metric space and $\mathscr{X}$ is a Banach space, then a Lipschitz function $f : \mathscr{M} \to \mathscr{X}$ such that $f(0) = 0$ is said to be* ***Lipschitz approximable*** *if it is the limit in the Lipschitz norm of a sequence of Lipschitz finite rank operators from $\mathscr{M}$ to $\mathscr{X}$.*

Theorem 1.5.16. *(Jiménez-Vargas et al. (2014)) Every Lipschitz approximable operator from pointed metric space $(\mathscr{M}, 0)$ to a Banach space $\mathscr{X}$ is Lipschitz compact.*

1.6 OPERATOR-VALUED ORTHONORMAL BASES, RIESZ BASES, FRAMES AND BESSEL SEQUENCES IN HILBERT SPACES

Through a decade long research, the frame theory for Hilbert spaces was extended to larger extent by Kaftal, Larson and Zhang with the introduction of the notion of operator-valued frame (OVF) in 2009. In the theory of operator-valued frames, the sequence $\{L_n\}_n$ of operators play an important role. These are defined as follows.

Definition 1.6.1. *(Kaftal et al. (2009)) Given $n \in \mathbb{N}$, we define*

$$L_n : \mathscr{H}_0 \ni h \mapsto L_n h := e_n \otimes h \in \ell^2(\mathbb{N}) \otimes \mathscr{H}_0,$$

where $\{e_n\}_n$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$.

Following proposition shows properties of the operator L_n.

Proposition 1.6.2. *(Kaftal et al. (2009)) The operators L_n in Definition 1.6.1 satisfy the following.*

(i) Each L_n is an isometry from $\mathscr{H}_0$ to $\ell^2(\mathbb{N}) \otimes \mathscr{H}_0$, and for $n, m \in \mathbb{N}$ we have

$$L_n^* L_m = \begin{cases} I_{\mathscr{H}_0} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \quad \text{and} \quad \sum_{n=1}^{\infty} L_n L_n^* = I_{\ell^2(\mathbb{N})} \otimes I_{\mathscr{H}_0} \tag{1.6.1}$$

where the convergence is in the strong-operator topology.

(ii) $L_m^*(\{a_n\}_n \otimes y) = a_m y$, $\forall \{a_n\}_n \in \ell^2(\mathbb{N}), \forall y \in \mathscr{H}_0$, for each m in $\mathbb{N}$.

Orthonormal basis for Hilbert spaces are defined in Definition 1.2.8. Considering Definition 1.2.8 and Parseval identity ((iv) in Theorem 1.2.10) for orthonormal basis, Sun (2006) defined the notion of orthonormal basis for operators.

Definition 1.6.3. *(Sun (2006)) A collection $\{F_n\}_n$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is said to be an* ***orthonormal basis*** *or* ***a G-basis*** *in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ if*

$$\langle F_n^* y, F_k^* z \rangle = \delta_{n,k} \langle y, z \rangle, \quad \forall y, z \in \mathscr{H}_0, \ \forall n, k \in \mathbb{N}$$

and

$$\sum_{n=1}^{\infty} \|F_n h\|^2 = \|h\|^2, \quad \forall h \in \mathscr{H}.$$

We observe $\langle F_n^* y, F_k^* z \rangle = \delta_{n,k} \langle y, z \rangle, \forall y, z \in \mathscr{H}_0$, $\forall n, k \in \mathbb{N}$ if and only if $F_n F_k^* = \delta_{n,k} I_{\mathscr{H}_0}$, $\forall n, k \in \mathbb{N}$. Hence if $\{F_n\}_n$ is an orthonormal basis, then $\|F_n\|^2 = \|F_n F_n^*\| = 1, \forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} F_n^* F_n = I_{\mathscr{H}}$. Further, using Proposition 1.6.2 we have

$$\left(\sum_{n=1}^{\infty} A_n^* L_n^* \right) \left(\sum_{k=1}^{\infty} L_k A_k \right) = I_{\mathscr{H}}.$$

Consider the case $\mathscr{H}_0 = \mathbb{K}$. For each $n \in \mathbb{N}$, via Riesz representation theorem (cf. Limaye (2014)), there exists a unique $\tau_n \in \mathscr{H}$ such that $F_n h = \langle h, \tau_n \rangle, \forall h \in \mathscr{H}$. Now first condition in Definition 1.6.3 tells $\langle F_n^* y, F_k^* z \rangle = y\bar{z} \langle \tau_n, \tau_k \rangle = y\bar{z} \delta_{j,k}, \forall j, k \in \mathbb{N}, \forall y, z \in$

$\mathbb{K}$ which shows that $\{\tau_n\}_n$ is orthonormal. Second condition in Definition 1.6.3 says that $\sum_{n=1}^{\infty}|\langle h, \tau_n\rangle|^2 = \|h\|^2, \forall h \in \mathscr{H}$. Theorem 1.2.10 now tells that $\{\tau_n\}_n$ is an orthonormal basis for $\mathscr{H}$. Hence Definition 1.6.3 generalizes the definition of orthonormal basis.

Example 1.6.4. *(i) (Sun (2006)) Let $\{\tau_n\}_n$ be an orthonormal basis for $\mathscr{H}$. Define $F_n : \mathscr{H} \ni h \mapsto \langle h, \tau_n\rangle \in \mathbb{K}$, for each $n \in \mathbb{N}$. Then $\{F_n\}_n$ is an operator-valued orthonormal basis in $\mathscr{B}(\mathscr{H}, \mathbb{K})$.*

(ii) If $U : \mathscr{H} \to \mathscr{H}_0$ is unitary, then $\{U\}$ is an orthonormal basis in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$.

(iii) Let $n \geq 2$ and $A_1, \dots, A_n$ be n isometries on $\ell^2(\mathbb{N})$ such that $A_1A_1^* + \cdots + A_nA_n^* = I_{\ell^2(\mathbb{N})}$ (these are Cuntz algebra generators (Cuntz (1977))). We then have

$$(A_jA_j^*)^2 = A_j(A_j^*A_j)A_j^* = A_jI_{\mathscr{H}}A_j^* = A_jA_j^*.$$

Hence $A_jA_j^*$'s are projections. Further $A_j^*A_k = 0$, $\forall j \neq k$. Therefore $\{A_j^*\}_{j=1}^n$ is an operator-valued orthonormal basis in $\mathscr{B}(\ell^2(\mathbb{N}))$.

(iv) Equation (1.6.1) says that $\{L_n^*\}_n$ is an orthonormal basis in $\mathscr{B}(\ell^2(\mathbb{N}) \otimes \mathscr{H}_0, \mathscr{H}_0)$.

Using Theorem 1.2.25, Sun (2006) defined the notion of Riesz basis for operators.

Definition 1.6.5. *(Sun (2006)) A collection $\{A_n\}_n$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is said to be an* ***operator-valued Riesz basis*** *or* ***G-Riesz basis*** *in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ if it satisfies the following.*

(i) $\{h \in \mathscr{H} : A_nh = 0, \forall n \in \mathbb{N}\} = \{0\}$.

(ii) There exist $a, b > 0$ such that for every finite subset $\mathbb{S}$ of $\mathbb{N}$,

$$a\sum_{n\in\mathbb{S}}\|h_n\|^2 \leq \left\|\sum_{n\in\mathbb{S}}A_n^*h_n\right\|^2 \leq b\sum_{n\in\mathbb{S}}\|h_n\|^2, \quad \forall h_n \in \mathscr{H}_0.$$

Example 1.6.6. *(i) (Sun (2006)) Let $\{\tau_n\}_n$ be a Riesz basis for $\mathscr{H}$. Define $A_n : \mathscr{H} \ni h \mapsto \langle h, \tau_n\rangle \in \mathbb{K}$, for each $n \in \mathbb{N}$. Then $A_n^*y = y\tau_n, \forall y \in \mathbb{K}$. Now from Theorem 1.2.25, $\{A_n\}_n$ is an operator-valued Riesz basis in $\mathscr{B}(\mathscr{H}, \mathbb{K})$.*

(ii) If $U : \mathscr{H} \to \mathscr{H}_0$ is invertible, then $\{U\}$ is an operator-valued Riesz basis in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$.

(iii) Let $A_1, \dots, A_n$ be as in (iii) of Example 1.6.4 and let $A, B \in \mathscr{B}(\mathscr{H})$ be invertible. Then $\{AA_j^*B\}_{j=1}^n$ is an operator-valued Riesz basis in $\mathscr{B}(\mathscr{H})$. In fact, if $AA_j^*Bh =$

$0, \forall j$, then $A_j^*Bh = 0, \forall j$ which gives

$$h = B^{-1}Bh = B^{-1}\left(\sum_{k=1}^{n} A_k A_k^* Bh\right) = 0$$

and every finite subset $\mathbb{S}$ of $\mathbb{N}$,

$$(\|B^{-1}\|\|A^{-1}\|)^{-1}\sum_{j\in\mathbb{S}}\|h_j\|^2 \leq \left\|\sum_{j\in\mathbb{S}} B^*A_jA^*h_j\right\|^2 \leq \|B\|\|A\|\sum_{j\in\mathbb{S}}\|h_j\|^2,$$

for all $h_j \in \mathscr{H}$.

Sun (2006) derived the following result which tells that we can define the notion of operator-valued Riesz basis in a manner similar to Definition 1.2.19.

Theorem 1.6.7. *(Sun (2006)) A collection $\{A_n\}_n$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is an operator-valued Riesz basis if and only if there exist an operator-valued orthonormal basis $\{F_n\}_n$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ and an invertible $T \in \mathscr{B}(\mathscr{H})$ such that $A_n = F_nU, \forall n$.*

Historically, many generalizations of frames for Hilbert spaces are proposed such as frames for subspaces (Casazza and Kutyniok (2004)), fusion frames (Casazza et al. (2008b)), outer frames (Aldroubi et al. (2004)), oblique frames (Christensen and Eldar (2004)), pseudo frames (Li and Ogawa (2004)), quasi-projectors (Fornasier (2004)). It was in 2006, when Sun gave the definition of G-frame which unified all these notions of frames for Hilbert spaces.

Definition 1.6.8. *(Sun (2006)) A collection $\{A_n\}_n$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is said to be a **G-frame** in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ if there exist $a, b > 0$ such that*

$$a\|h\|^2 \leq \sum_{n=1}^{\infty}\|A_nh\|^2 \leq b\|h\|^2, \quad \forall h \in \mathscr{H}.$$

Basic idea for the notion of OVF is the following. Definition 1.2.31 can be written in an equivalent form as

$$\text{the map } S_\tau : \mathscr{H} \ni h \mapsto \sum_{n=1}^{\infty}\langle h, \tau_n\rangle\tau_n \in \mathscr{H} \text{ is a well-defined bounded positive invertible operator.} \tag{1.6.2}$$

If we now define $A_n : \mathscr{H} \ni h \mapsto \langle h, x_n\rangle \in \mathbb{K}$, for each $n \in \mathbb{N}$, then Statement (1.6.2) can be rewritten as

$$\sum_{n=1}^{\infty} A_n^* A_n \text{ converges in the strong-operator topology on } \mathscr{B}(\mathscr{H}) \text{ to a bounded positive invertible operator.} \tag{1.6.3}$$

Now Statement (1.6.3) leads to

Definition 1.6.9. *(Kaftal et al. (2009)) A collection $\{A_n\}_n$ in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ is said to be an* ***operator-valued frame*** *(OVF) in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ if the series*

$$\textbf{\textit{(Operator-valued frame operator)}} \quad S_A := \sum_{n=1}^{\infty} A_n^* A_n$$

converges in the strong-operator topology on $\mathscr{B}(\mathscr{H})$ to a bounded invertible operator. Constants a and b are called as lower and upper frame bounds, respectively. Supremum (resp. infimum) of the set of all lower (resp. upper) frame bounds is called optimal lower (resp. upper) frame bound. If the optimal frame bounds are equal, then the frame is called as tight operator-valued frame. A tight operator-valued frame whose optimal frame bound is one is termed as Parseval operator-valued frame.

Before proceeding, we first show that Definitions 1.6.8 and 1.6.9 are equivalent. For this, we first need a result.

Theorem 1.6.10. *(Sun (2006)) If $\{A_n\}_n$ in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ is a G-frame, then the map $S_A : \mathscr{H} \ni h \mapsto \sum_{n=1}^{\infty} A_n^* A_n h \in \mathscr{H}$ is a well-defined bounded linear invertible operator.*

Following theorem will reflect the equivalence of notions of OVF and G-frame.

Theorem 1.6.11. *A collection $\{A_n\}_n$ in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ is a G-frame in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ if and only if $\{A_n\}_n$ is an operator-valued frame in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$.*

Proof. $(\Rightarrow)$ Theorem 1.6.10 says that S_A is a bounded linear invertible operator. Since $A_n^* A_n \geq 0, \forall n \in \mathbb{N}$, S_A is positive. Hence $\{A_n\}_n$ is an OVF.
$(\Leftarrow)$ Since $\sum_{n=1}^{\infty} A_n^* A_n$ is positive invertible, there are $a, b > 0$ such that $aI_{\mathscr{H}} \leq \sum_{n=1}^{\infty} A_n^* A_n \leq bI_{\mathscr{H}}$. This implies $a\|h\|^2 \leq \langle \sum_{n=1}^{\infty} A_n^* A_n h, h\rangle = \sum_{n=1}^{\infty} \|A_n h\|^2 \leq b\|h\|^2, \forall h \in \mathscr{H}$. Hence $\{A_n\}_n$ is a G-frame. $\square$

Remark 1.6.12. *Even though Sun's paper (Sun (2006)) published earlier than the paper by Kaftal et al. (2009), it is mentioned in the introduction of paper (Kaftal et al. (2009)) that authors of paper (Kaftal et al. (2009)) started the work in January 1999.*

Example 1.6.13. *(i) (Sun (2006)) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. Define $A_n : \mathscr{H} \ni h \mapsto \langle h, \tau_n\rangle \in \mathbb{K}$, for each $n \in \mathbb{N}$. Then $A_n^*y = y\tau_n, \forall y \in \mathbb{K}$. Now from Theorem 1.2.36, the map $\mathscr{H} \ni h \mapsto \sum_{n=1}^\infty A_n^*A_nh = \sum_{n=1}^\infty \langle h, \tau_n\rangle \tau_n \in \mathscr{H}$ is a well-defined positive invertible operator. Hence $\{A_n\}_n$ is an operator-valued frame in $\mathscr{B}(\mathscr{H}, \mathbb{K})$.*

(ii) If $A : \mathscr{H} \to \mathscr{H}_0$ is a bounded below linear operator, then $\{A\}$ is an operator-valued frame in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$.

(iii) Let $A_1, \dots, A_n$ be as in (iii) of Example 1.6.4 and let $A, B \in \mathscr{B}(\mathscr{H})$ be bounded below. Then $\{AA_j^*B\}_{j=1}^n$ is an operator-valued frame in $\mathscr{B}(\mathscr{H})$.

The fundamental tool used in the study of OVF is the factorization of frame operator S_A. This and other important properties of OVFs are stated in the following theorem.

Theorem 1.6.14. *(Kaftal et al. (2009), Sun (2006)) Let $\{A_n\}_n$ be an OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. Then*

(i) $\overline{\operatorname{span}} \cup_{n=1}^\infty A_n^*(\mathscr{H}_0) = \mathscr{H}$.

(ii) The analysis operator

$$\theta_A : \mathscr{H} \ni h \mapsto \theta_A h := \sum_{n=1}^\infty L_nA_nh \in \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$$

is a well-defined bounded linear operator. Further, $\sqrt{a}\|h\| \leq \|\theta_A h\| \leq \sqrt{b}\|h\|, \forall h \in \mathscr{H}$. In particular, θ_A is injective and its range is closed.

(iii) We have

$$a\|h\|^2 \leq \langle S_Ah, h\rangle \leq b\|h\|^2, \forall h \in \mathscr{H} \text{ and } a\|h\| \leq \|S_Ah\| \leq b\|h\|, \forall h \in \mathscr{H}.$$

(iv) $h = \sum_{n=1}^\infty (A_nS_A^{-1})^*A_nh = \sum_{n=1}^\infty A_n^*(A_nS_A^{-1})h, \forall h \in \mathscr{H}$.

(v) $\theta_A^*z = \sum\limits_{n=1}^\infty A_n^*L_n^*z$, $\forall z \in \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$.

(vi) Frame operator factors as $S_A = \theta_A^*\theta_A$.

(vii) θ_A^* is surjective.

(viii) $\|S_A^{-1}\|^{-1}$ is the optimal lower frame bound and $\|S_A\| = \|\theta_A\|^2$ is the optimal upper frame bound.

(ix) $P_A := \theta_AS_A^{-1}\theta_A^* : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ is an orthogonal projection onto $\theta_A(\mathscr{H})$.

(x) $\{A_n\}_n$ *is Parseval if and only if* θ_A *is an isometry if and only if* $\theta_A\theta_A^*$ *is a projection.*

(xi) $\{A_nS_A^{-1}\}_n$ *is an OVF in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *with bounds* b^{-1} *and* a^{-1}.

(xii) $\{A_nS_A^{-1/2}\}_n$ *is a Parseval OVF in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$.

(xiii) ***(Best approximation)*** *If* $h\in\mathscr{H}$ *has representation* $h=\sum_{n=1}^\infty A_n^*y_n$, *for some sequence* $\{y_n\}_n$ *in* $\mathscr{H}_0$, *then*

$$\sum_{n=1}^\infty\|y_n\|^2=\sum_{n=1}^\infty\|A_nS_A^{-1}h\|^2+\sum_{n=1}^\infty\|y_n-A_nS_A^{-1}h\|^2.$$

Similar to the notion of duality, orthogonality and similarity for frames in Hilbert spaces, there are similar notions for operator-valued frames. We now recall these notions and mention some results.

Definition 1.6.15. *(Kaftal et al. (2009)) An OVF* $\{B_n\}_n$ *in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *is said to be a* ***dual*** *for an OVF* $\{A_n\}_n$ *in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *if* $\sum_{n=1}^\infty B_n^*A_n=I_{\mathscr{H}}$.

Definition 1.6.16. *(Kaftal et al. (2009)) An OVF* $\{B_n\}_n$ *in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *is said to be* ***orthogonal*** *to an OVF* $\{A_n\}_n$ *in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *if* $\sum_{n=1}^\infty B_n^*A_n=0$.

Proposition 1.6.17. *(Kaftal et al. (2009)) Let* $\{A_n\}_n$ *and* $\{B_n\}_n$ *be two Parseval OVFs in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *which are orthogonal. If* $C,D,E,F\in\mathscr{B}(\mathscr{H})$ *are such that* $C^*C+D^*D=I_{\mathscr{H}}$, *then* $\{A_nC+B_nD\}_n$ *is a Parseval OVF in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$. *In particular, if scalars* c,d, *satisfy* $|c|^2+|d|^2=1$, *then* $\{cA_n+dB_n\}_n$ *is a Parseval OVF.*

Proposition 1.6.18. *(Kaftal et al. (2009)) If* $\{A_n\}_n$, *and* $\{B_n\}_n$ *are orthogonal OVFs in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$, *then* $\{A_n\oplus B_n\}_n$ *is an OVF in* $\mathscr{B}(\mathscr{H}\oplus\mathscr{H},\mathscr{H}_0)$. *Further, if both* $\{A_n\}_n$ *and* $\{B_n\}_n$ *are Parseval, then* $\{A_n\oplus B_n\}_n$ *is Parseval.*

Definition 1.6.19. *(Kaftal et al. (2009)) An OVF* $\{B_n\}_n$ *in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *is said to be* ***similar*** *or* ***equivalent*** *to an OVF* $\{A_n\}_n$ *in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *if there exists a bounded invertible* $R_{A,B}\in\mathscr{B}(\mathscr{H})$ *such that* $B_n=A_nR_{A,B},\forall n\in\mathbb{N}$.

Similar frames share some nice properties that knowing analysis, synthesis and frame operators of one give that of another.

Lemma 1.6.20. *(Kaftal et al. (2009)) Let* $\{A_n\}_n$ *and* $\{B_n\}_n$ *be similar OVFs and* $B_n=A_nR_{A,B},\forall n\in\mathbb{N}$, *for some invertible* $R_{A,B}\in\mathscr{B}(\mathscr{H})$. *Then*

(i) $\theta_B=\theta_AR_{A,B}$.

(ii) $S_B = R_{A,B}^* S_A R_{A,B}$.

(iii) $P_B = P_A$.

There is a complete classification of similarity using operators.

Theorem 1.6.21. *(Kaftal et al. (2009)) For two OVFs $\{A_n\}_n$ and $\{B_n\}_n$, the following are equivalent.*

(i) $B_n = A_n R_{A,B}, \forall n \in \mathbb{N},$ *for some invertible* $R_{A,B} \in \mathscr{B}(\mathscr{H})$.

(ii) $\theta_B = \theta_A R_{A,B}$ *for some invertible* $R_{A,B} \in \mathscr{B}(\mathscr{H})$.

(iii) $P_B = P_A$.

If one of the above conditions is satisfied, then invertible operators in (i) *and* (ii) *are unique and are given by $R_{A,B} = S_A^{-1}\theta_A^*\theta_B$. In the case that $\{A_n\}_n$ is Parseval, then $\{B_n\}_n$ is Parseval if and only if $R_{A,B}$ is unitary.*

In the study of frames, rather indexing with natural numbers or other indexing sets, it is more useful in some cases to study frames indexed by groups and generated by a single operator.

Let G be a discrete topological group and $\{\chi_g\}_{g\in G}$ be the standard orthonormal basis for $\ell^2(G)$. Let λ be the left regular representation of G defined by $\lambda_g\chi_q(r) = \chi_q(g^{-1}r)$, $\forall g,q,r \in G$ and ρ be the right regular representation of G defined by $\rho_g\chi_q(r) = \chi_q(rg)$, $\forall g,q,r \in G$. By $\mathcal{L}(G)$, we mean the von Neumann algebra generated by unitaries $\{\lambda_g\}_{g\in G}$ in $\mathscr{B}(\ell^2(G))$. Similarly $\mathcal{R}(G)$ denotes the von Neumann algebra generated by $\{\rho_g\}_{g\in G}$ in $\mathscr{B}(\ell^2(G))$. We recall that $\mathcal{L}(G)' = \mathcal{R}(G)$, $\mathcal{R}(G)' = \mathcal{L}(G)$ (cf. Conway (2000)), where $\mathscr{A}'$ denotes the commutant of $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$.

Definition 1.6.22. *(Kaftal et al. (2009)) Let π be a unitary representation of a discrete group G on a Hilbert space $\mathscr{H}$. An operator A in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ is called a **operator frame generator** (resp. a Parseval frame generator) w.r.t. an operator Ψ in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ if $\{A_g := A\pi_{g^{-1}}\}_{g\in G}$ is a factorable weak OVF (resp. Parseval) in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$. In this case, we say A is an operator frame generator for π.*

Frames generated by groups have the remarkable property that the frame operator belongs to the commutant of $\pi(G)$. These and other properties are given in the following proposition.

Proposition 1.6.23. *(Kaftal et al. (2009)) Let A and B be operator frame generators in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ for a unitary representation π of G on $\mathscr{H}$. Then* *(i)* $\theta_A\pi_g = (\lambda_g \otimes I_{\mathscr{H}_0})\theta_A, \theta_B\pi_g = (\lambda_g \otimes I_{\mathscr{H}_0})\theta_B, \forall g \in G.$

(ii) $\theta_A^*\theta_B$ *is in the commutant* $\pi(G)'$ *of* $\pi(G)''$. *Further,* $S_A \in \pi(G)'$.

(iii) $\theta_A T\theta_A^*, \theta_A T\theta_B^* \in \mathcal{R}(G)\otimes\mathscr{B}(\mathscr{H}_0), \forall T \in \pi(G)'$. *In particular,* $P_A \in \mathcal{R}(G)\otimes\mathscr{B}(\mathscr{H}_0)$.

Following theorem gives a characterization of frames generated by unitary representation without using representation.

Theorem 1.6.24. *(Kaftal et al. (2009)) Let G be a discrete group, e be the identity of G and* $\{A_g\}_{g\in G}$ *be a Parseval OVF in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$. *Then there is a unitary representation* π *of G on* $\mathscr{H}$ *for which*

$$A_g = A_e\pi_{g^{-1}}, \quad \forall g \in G$$

if and only if

$$A_{gp}A_{gq}^* = A_pA_q^*, \quad \forall g,p,q \in G.$$

One of the most important results obtained by Kaftal et al. (2009) is the connectedness of the set of all generators of operator-valued frames generated by groups.

Theorem 1.6.25. *(Kaftal et al. (2009)) Let* π *be a unitary representation of a discrete group G on* $\mathscr{H}$. *Suppose*

$$\emptyset \neq \mathcal{F}_G := \{A \in \mathscr{B}(\mathscr{H},\mathscr{H}_0) : \{A\pi_{g^{-1}}\}_{g\in G} \text{ is an operator-valued frame in } \mathscr{B}(\mathscr{H},\mathscr{H}_0)\}.$$

(i) If $\dim\mathscr{H}_0 < \infty$, *then* $\mathcal{F}_G$ *is* ***path-connected*** *in the* ***operator-norm topology*** *on* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$.

(ii) If $\dim\mathscr{H}_0 = \infty$, *then* $\mathcal{F}_G$ *is path-connected in the operator-norm topology on* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *if and only if the von Neumann algebra* $\mathcal{R}(G)$ *generated by the right regular representations of G is* ***diffuse*** *(i.e.,* $\mathcal{R}(G)$ *has no nonzero minimal projections).*

Stability result for OVFs is due to Sun (2007).

Theorem 1.6.26. *(Sun (2007)) Let* $\{A_n\}_n$ *be an OVF in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *with frame bounds a and b. Suppose* $\{B_n\}_n$ *in* $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ *is such that there exist* $\alpha,\beta,\gamma \geq 0$ *with* $\max\{\alpha+\frac{\gamma}{\sqrt{a}},\beta\} < 1$ *and for all* $m = 1,2,\dots,$

$$\left\|\sum_{n=1}^{m}(A_n^*-B_n^*)y_n\right\| \leq \alpha\left\|\sum_{n=1}^{m}A_n^*y_n\right\| + \beta\left\|\sum_{n=1}^{m}B_n^*y_n\right\| + \gamma\left(\sum_{n=1}^{m}\|y_n\|^2\right)^{\frac{1}{2}}, \quad \forall y_n \in \mathscr{H}_0.$$

Then $\{B_n\}_n$ is an operator-valued frame in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ with frame bounds

$$a\left(1-\frac{\alpha+\beta+\frac{\gamma}{\sqrt{a}}}{1+\beta}\right)^2 \text{ and } b\left(1+\frac{\alpha+\beta+\frac{\gamma}{\sqrt{b}}}{1-\beta}\right)^2.$$

Like Bessel sequences for Hilbert spaces, there is a similar notion for operators.

Definition 1.6.27. *(Sun (2006)) A collection $\{A_n\}_n$ in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ is said to be an* ***operator-valued Bessel sequence*** *(or* **G-Bessel sequence***) if there exists $b>0$ such that*

$$(\textbf{Operator-valued Bessel's inequality}) \quad \sum_{n=1}^{\infty}\|A_nh\|^2 \leq b\|h\|^2, \quad \forall h \in \mathscr{H}.$$

Constant b is called as a Bessel bound for $\{A_n\}_n$.

Similar to Theorem 1.6.11 we have the following result.

Theorem 1.6.28. *A collection $\{A_n\}_n$ in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ is an operator-valued Bessel sequence if and only if the series $\sum_n A_n^*A_n$ converges in the strong-operator topology on $\mathscr{B}(\mathscr{H})$ to a bounded operator.*

Following are some examples of operator-valued Bessel sequences.

Example 1.6.29. *(i) (Sun (2006)) Let $\{\tau_n\}_n$ be a Bessel sequence for $\mathscr{H}$. Define $A_n : \mathscr{H} \ni h \mapsto \langle h, \tau_n\rangle \in \mathbb{K}$, for each $n \in \mathbb{N}$. Then $A_n^*y = y\tau_n, \forall y \in \mathbb{K}$. Now from Theorem 1.2.58, the map $\mathscr{H} \ni h \mapsto \sum_{n=1}^\infty A_n^*A_nh = \sum_{n=1}^\infty \langle h, \tau_n\rangle\tau_n \in \mathscr{H}$ is a well-defined positive operator. Hence $\{A_n\}_n$ is an operator-valued Bessel sequence in $\mathscr{B}(\mathscr{H},\mathbb{K})$.*

(ii) From operator-norm inequality, we see that any finite collection of operators is an operator-valued Bessel sequence.

CHAPTER 2

FRAMES FOR METRIC SPACES

2.1 BASIC PROPERTIES

In this chapter, we define frames for metric spaces and derive several fundamental properties.

Definition 2.1.1. ***(p-frame for metric space)*** *Let $\mathcal{M}$ be a metric space and $p \in [1,\infty)$. A collection $\{f_n\}_n$ of Lipschitz functions in* $\operatorname{Lip}(\mathcal{M},\mathbb{K})$ *is said to be a* ***metric p-frame*** *or* ***Lipschitz p-frame*** *for $\mathcal{M}$ if there exist $a,b>0$ such that*

$$a\,d(x,y) \leq \left(\sum_{n=1}^{\infty}|f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} \leq b\,d(x,y), \quad \forall x,y \in \mathcal{M}.$$

If we do not demand the first inequality, then we say $\{f_n\}_n$ is a metric p-Bessel sequence for $\mathcal{M}$.

We now see that whenever $\mathcal{M}$ is a Banach space and f_n's are linear functionals, then Definition 2.1.1 reduces to Definition 1.3.4. We now give various examples.

Example 2.1.2. *Let $\{f_n\}_n$ be a p-frame for a Banach space $\mathcal{X}$. Choose any bi-Lipschitz function $A:\mathcal{X}\to\mathcal{X}$. Then it follows that $\{f_nA\}_n$ is a metric p-frame for $\mathcal{X}$.*

Example 2.1.3. *Let $1<a<\infty$. Let us take $\mathcal{M}:=[a,\infty)$ and define $f_n:\mathcal{M}\to\mathbb{R}$ by*

$$f_0(x) := 1, \quad \forall x \in \mathcal{M}$$
$$f_n(x) := \frac{(\log x)^n}{n!}, \quad \forall x \in \mathcal{M}, \forall n \geq 1.$$

Then $f_n'(x)=\frac{(\log x)^{(n-1)}}{(n-1)!x}$, $\forall x \in \mathcal{M}, \forall n \geq 1$. Since f_n' is bounded on $\mathcal{M}$, $\forall n \geq 1$, it follows *that f_n is Lipschitz on $\mathscr{M}$, $\forall n \geq 1$. For $x, y \in \mathscr{M}$ with $x < y$, we now see that*

$$\sum_{n=0}^{\infty} |f_n(x) - f_n(y)| = \sum_{n=0}^{\infty} \left| \frac{(\log x)^n}{n!} - \frac{(\log y)^n}{n!} \right| = \sum_{n=0}^{\infty} \frac{(\log y)^n}{n!} - \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!}$$
$$= e^{\log y} - e^{\log x} = y - x = |x - y|.$$

Hence $\{f_n\}_n$ is a metric 1-frame for $\mathscr{M}$.

Example 2.1.4. *Let $1 < a < b < \infty$. We take $\mathscr{M} := [\frac{1}{1-a}, \frac{1}{1-b}]$ and define $f_n : \mathscr{M} \to \mathbb{R}$ by*

$$f_n(x) := \left(1 - \frac{1}{x}\right)^n, \quad \forall x \in \mathscr{M}, \forall n \geq 0.$$

Then $f_n'(x) = \frac{n}{-x^2}\left(1 - \frac{1}{x}\right)^{n-1}$, $\forall x \in \mathscr{M}, \forall n \geq 1$. Therefore f_n is a Lipschitz function, for each $n \geq 1$. We now see that $\{f_n\}_n$ is a metric 1-frame for $\mathscr{M}$. In fact, for $x, y \in \mathscr{M}$, with $x < y$, we have

$$\sum_{n=0}^{\infty} |f_n(x) - f_n(y)| = \sum_{n=0}^{\infty} \left| \left(1 - \frac{1}{x}\right)^n - \left(1 - \frac{1}{y}\right)^n \right| = \sum_{n=0}^{\infty} \left(1 - \frac{1}{y}\right)^n - \sum_{n=0}^{\infty} \left(1 - \frac{1}{x}\right)^n$$
$$= y - x = |x - y|.$$

Example 2.1.5. *Let $\{f_n\}_n$ be a p-frame for a Banach space $\mathscr{X}$. Let $\phi : \mathbb{K} \to \mathbb{K}$ be bi-Lipschitz and define $g_n(x) := \phi(f_n(x))$, $\forall x \in \mathscr{X}$, $\forall n \in \mathbb{N}$. It then follows that $\{g_n\}_n$ is a metric p-frame for $\mathscr{X}$.*

By looking at Theorem 1.3.6 we can ask whether there is a result similar for metric p-frames. We answer this partially through the following theorem.

Theorem 2.1.6. *Let $(\mathscr{M}, 0)$ be a pointed metric space and $\{f_n\}_n$ be a sequence in $\operatorname{Lip}_0(\mathscr{M}, \mathbb{K})$. Then $\{f_n\}_n$ is a metric p-Bessel sequence for $\mathscr{M}$ with bound b if and only if*

$$T : \ell^q(\mathbb{N}) \ni \{a_n\}_n \mapsto T\{a_n\}_n \in \operatorname{Lip}_0(\mathscr{M} \times \mathscr{M}, \mathbb{K}), \tag{2.1.1}$$
$$T\{a_n\}_n : \mathscr{M} \times \mathscr{M} \ni (x, y) \mapsto \sum_{n=1}^{\infty} a_n(f_n(x) - f_n(y)) \in \mathbb{K}$$

is a well-defined (hence bounded) linear operator and $\|T\| \leq b$ (where q is the conjugate index of p).

Proof. ($\Rightarrow$) Let $\{a_n\}_n \in \ell^q(\mathbb{N})$ and $n, m \in \mathbb{N}$ with $n < m$. First we have to show that the series in (2.1.1) is convergent. For all $x,y \in \mathscr{M}$,

$$\left|\sum_{k=n}^{m} a_k(f_k(x)-f_k(y))\right| \leq \left(\sum_{k=n}^{m} |a_k|^q\right)^{\frac{1}{q}} \left(\sum_{k=n}^{m} |f_k(x)-f_k(y)|^p\right)^{\frac{1}{p}}$$
$$\leq b\left(\sum_{k=n}^{m} |a_k|^q\right)^{\frac{1}{q}} d(x,y).$$

Therefore the series in (2.1.1) converges. We next show that the map $T\{a_n\}_n$ is Lipschitz. Consider

$$\begin{aligned}
\|T\{a_n\}_n\|_{\mathrm{Lip}_0} &= \sup_{(x,y),(u,v)\in\mathscr{M}\times\mathscr{M},(x,y)\neq(u,v)} \frac{|T\{a_n\}_n(x,y)-T\{a_n\}_n(u,v)|}{d(x,u)+d(y,v)}\\
&= \sup_{(x,y),(u,v)\in\mathscr{M}\times\mathscr{M},(x,y)\neq(u,v)} \frac{|\sum_{n=1}^{\infty} a_n(f_n(x)-f_n(u))-\sum_{n=1}^{\infty} a_n(f_n(y)-f_n(v))|}{d(x,u)+d(y,v)}\\
&\leq \sup_{(x,y),(u,v)\in\mathscr{M}\times\mathscr{M},(x,y)\neq(u,v)} \frac{|\sum_{n=1}^{\infty} a_n(f_n(x)-f_n(u))|+|\sum_{n=1}^{\infty} a_n(f_n(y)-f_n(v))|}{d(x,u)+d(y,v)}\\
&\leq b \sup_{(x,y),(u,v)\in\mathscr{M}\times\mathscr{M},(x,y)\neq(u,v)} \frac{(\sum_{n=1}^{\infty} |a_n|^q)^{\frac{1}{q}} d(x,u)+(\sum_{n=1}^{\infty} |a_n|^q)^{\frac{1}{q}} d(y,v)}{d(x,u)+d(y,v)}\\
&= b\left(\sum_{n=1}^{\infty} |a_n|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence T is well-defined. Clearly T is linear. Boundedness of T with bound b will follow from previous calculation.
($\Leftarrow$) From the definition of T, it is bounded by Banach-Steinhaus. Given $x,y \in \mathscr{M}$, we define a map

$$\Phi_{x,y} : \ell^q(\mathbb{N}) \ni \{a_n\}_n \mapsto \Phi_{x,y}\{a_n\}_n := \sum_{n=1}^{\infty} a_n(f_n(x)-f_n(y)) \in \mathbb{K}$$

which is a bounded linear functional. Hence $\{f_n(x)-f_n(y)\}_n \in \ell^p(\mathbb{N})$. Let $\{e_n\}_n$ be the standard Schauder basis for $\ell^p(\mathbb{N})$. Then

$$\|\Phi_{x,y}\| = \left(\sum_{n=1}^{\infty} |\Phi_{x,y}\{e_n\}_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}}.$$

Now

$$b\left(\sum_{n=1}^{\infty}|a_n|^q\right)^{\frac{1}{q}} = b\|\{a_n\}_n\| \geq \|T\{a_n\}_n\|_{\text{Lip}_0}$$
$$\geq \sup_{(x,0),(y,0)\in\mathscr{M}\times\mathscr{M},(x,0)\neq(y,0)} \frac{|T\{a_n\}_n(x,0)-T\{a_n\}_n(y,0)|}{d(x,y)}$$
$$= \sup_{(x,0),(y,0)\in\mathscr{M}\times\mathscr{M},(x,0)\neq(y,0)} \frac{|\sum_{n=1}^{\infty}a_n(f_n(x)-f_n(y))|}{d(x,y)}$$
$$= \sup_{(x,0),(y,0)\in\mathscr{M}\times\mathscr{M},(x,0)\neq(y,0)} \frac{|\Phi_{x,y}\{a_n\}_n|}{d(x,y)}$$

which implies

$$|\Phi_{x,y}\{a_n\}_n| \leq b\left(\sum_{n=1}^{\infty}|a_n|^q\right)^{\frac{1}{q}} d(x,y), \quad \forall x,y\in\mathscr{M}.$$

Using all these, we finally get

$$\left(\sum_{n=1}^{\infty}|f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} = \|\Phi_{x,y}\| \leq b\,d(x,y), \quad \forall x,y\in\mathscr{M}.$$

Hence $\{f_n\}_n$ is a metric p-Bessel sequence for $\mathscr{M}$ with bound b. □

In the spirit of definition of $\mathscr{X}_d$-frame, Definition 2.1.1 can be generalized.

Definition 2.1.7. *Let $\mathscr{M}$ be a metric space and $\mathscr{M}_d$ be an associated BK-space. Let $\{f_n\}_n$ be a sequence in* $\text{Lip}(\mathscr{M},\mathbb{K})$. *We say that $\{f_n\}_n$ is a **metric $\mathscr{M}_d$-frame** (or **Lipschitz $\mathscr{M}_d$-frame**) for $\mathscr{M}$ if the following conditions hold.*

(i) $\{f_n(x)\}_n \in \mathscr{M}_d$, *for each* $x\in\mathscr{M}$,

(ii) There exist positive a,b such that $a\,d(x,y) \leq \|\{f_n(x)-f_n(y)\}_n\|_{\mathscr{M}_d} \leq b\,d(x,y)$, $\forall x,y\in\mathscr{M}$.

*We say constant a as **lower metric $\mathscr{M}_d$-frame bound** and constant b as **upper metric $\mathscr{M}_d$-frame bound**. If we do not demand the first inequality, then we say $\{f_n\}_n$ is a metric **$\mathscr{M}_d$-Bessel sequence**.*

An easier way of producing metric $\mathscr{M}_d$-frame is the following. Let $\mathscr{M}_d$ be a BK-space which admits a Schauder basis $\{\tau_n\}_n$. Let $\{f_n\}_n$ be the coefficient functionals associated with $\{\tau_n\}_n$. Let $\mathscr{M}$ be a metric space and $A : \mathscr{M} \to \mathscr{M}_d$ be bi-Lipschitz with bounds a and b. Define $g_n := f_nA, \forall n$. Then g_n is a Lipschitz function for all n. Now

$$\begin{aligned} a\,d(x,y) &\leq \|Ax - Ay\|_{\mathscr{M}_d} = \|\{f_n(Ax - Ay)\}_n\|_{\mathscr{M}_d} \\ &= \|\{f_n(Ax) - f_n(Ay)\}_n\|_{\mathscr{M}_d} = \|\{g_n(x) - g_n(y)\}_n\|_{\mathscr{M}_d} \leq b\,d(x,y), \quad \forall x,y \in \mathscr{M}. \end{aligned}$$

Hence $\{g_n\}_n$ is a metric $\mathscr{M}_d$-frame for $\mathscr{M}$.
Following result ensures that metric frames are universal in nature.

Theorem 2.1.8. *Every separable metric space $\mathscr{M}$ admits a metric $\mathscr{M}_d$-frame.*

Proof. From Theorem 1.5.2 it follows that there exists a bi-Lipschitz function $f : \mathscr{M} \to c_0(\mathbb{N})$. Let $p_n : c_0(\mathbb{N}) \to \mathbb{K}$ be the coordinate projection, for each n. If we now set $f_n := p_n f$, for each n, then $\{f_n\}_n$ is a metric $c_0(\mathbb{N})$-frame for $\mathscr{M}$. □

Given metric $\mathscr{M}_d$-frames $\{f_n\}_n$, $\{g_n\}_n$ and a nonzero scalar λ, one can naturally ask whether we can scale and add them to get new frames? i.e., whether $\{f_n + \lambda g_n\}_n$ is a frame? In the case of Hilbert spaces, a use of Minkowski's inequality shows that whenever $\{\tau_n\}_n$ and $\{\omega_n\}_n$ are frames for a Hilbert space $\mathscr{H}$, then $\{\tau_n + \lambda\omega_n\}_n$ is a Bessel sequence for $\mathscr{H}$. In general, this sequence need not be a frame for $\mathscr{H}$. Thus we have to impose extra conditions to ensure the existence of lower frame bound. For Hilbert spaces this is done by Favier and Zalik (1995). We now obtain similar results for metric spaces.

Theorem 2.1.9. *Let $\{f_n\}_n$ be a metric $\mathscr{M}_d$-frame for metric space $\mathscr{M}$ with bounds a and b. Let λ be a non-zero scalar. Then*

(i) $\{\lambda f_n\}_n$ is a metric $\mathscr{M}_d$-frame for $\mathscr{M}$ with bounds $a\lambda$ and $b\lambda$.

(ii) If $A : \mathscr{M} \to \mathscr{M}$ is bi-Lipschitz with bounds c and d, then $\{f_nA\}_n$ is a metric $\mathscr{M}_d$-frame for $\mathscr{M}$ with bounds ac and bd.

(iii) If $\{g_n\}_n$ is a metric $\mathscr{M}_d$-Bessel sequence for $\mathscr{M}$ with bound d and $|\lambda| < \frac{a}{d}$, then $\{f_n + \lambda g_n\}_n$ is a metric $\mathscr{M}_d$-frame for $\mathscr{M}$ with bounds $a - |\lambda|d$ and $b + |\lambda|d$.

Proof. First two conclusions follow easily. For the upper frame bound of $\{f_n + \lambda g_n\}_n$ we use triangle inequality. Now for lower frame bound, using reverse triangle inequality, we get

$$\begin{aligned} \|\{(f_n + \lambda g_n)(x) - (f_n + \lambda g_n)(y)\}_n\|_{\mathscr{M}_d} &= \|\{f_n(x) - f_n(y) + \lambda(g_n(x) - g_n(y))\}_n\|_{\mathscr{M}_d} \\ &\geq \|\{f_n(x) - f_n(y)\}_n\|_{\mathscr{M}_d} - \|\{\lambda(g_n(x) - g_n(y))\}_n\|_{\mathscr{M}_d} \end{aligned}$$

$$\geq a\,d(x,y) - |\lambda|\,d(x,y) = (a - |\lambda|)\,d(x,y), \quad \forall x,y \in \mathscr{M}.$$

$\square$

We next define "metric frame" which is stronger than Definition 2.1.7 in light of definition of Banach frame.

Definition 2.1.10. *Let $\mathscr{M}$ be a metric space and $\mathscr{M}_d$ be an associated BK-space. Let $\{f_n\}_n$ be a sequence in $\operatorname{Lip}(\mathscr{M},\mathbb{K})$ and $S : \mathscr{M}_d \to \mathscr{M}$. We say that $(\{f_n\}_n, S)$ is a* ***metric frame*** *or* ***Lipschitz metric frame*** *for $\mathscr{M}$ if the following conditions hold.*

(i) $\{f_n(x)\}_n \in \mathscr{M}_d$, for each $x \in \mathscr{M}$,

(ii) There exist positive a,b such that $a\,d(x,y) \leq \|\{f_n(x) - f_n(y)\}_n\|_{\mathscr{M}_d} \leq b\,d(x,y)$, $\forall x,y \in \mathscr{M}$,

(iii) S is Lipschitz and $S(\{f_n(x)\}_n) = x$, for each $x \in \mathscr{M}$.

Mapping S is called as Lipschitz reconstruction operator. We say constant a as ***lower frame bound*** *and constant b as* ***upper frame bound****. If we do not demand the first inequality, then we say $(\{f_n\}_n, S)$ is a* ***metric Bessel sequence****.*

We observe that if $(\{f_n\}_n, S)$ is a metric frame for $\mathscr{M}$, then condition (i) in Definition 2.1.10 tells that the mapping (we call as analysis map)

$$\theta_f : \mathscr{M} \ni x \mapsto \theta_f x := \{f_n(x)\}_n \in \mathscr{M}_d$$

is well-defined and condition (ii) in Definition 2.1.10 tells that θ_f satisfies

$$a\,d(x,y) \leq \|\theta_f x - \theta_f y\| \leq b\,d(x,y), \quad \forall x,y \in \mathscr{M}.$$

Hence θ_f is bi-Lipschitz and injective. Thus a metric frame puts the space into $\mathscr{M}_d$ via θ_f and reconstructs every element by using reconstruction operator S. Now note that $S\theta_f = I_{\mathscr{M}}$. This operator description helps us to derive the following propositions easily.

Proposition 2.1.11. *If $(\{f_n\}_n, S)$ is a metric frame for $\mathscr{M}$, then $P_{f,S} := \theta_f S : \mathscr{M}_d \to \mathscr{M}_d$ is idempotent and $P_{f,S}(\mathscr{M}_d) = \theta_f(\mathscr{M}_d)$.*

Proposition 2.1.12. *Let $\{f_n\}_n$ be a $\mathscr{M}_d$-frame for $\mathscr{M}$ and $S : \mathscr{M}_d \to \mathscr{M}$ be Lipschitz. Then $(\{f_n\}_n, S)$ is a metric frame for $\mathscr{M}$ if and only if S is a left-Lipschitz inverse of θ_f if and only if θ_f is a right-Lipschitz inverse of S.*

We now give some explicit examples of metric frames.

Example 2.1.13. *Let $\mathcal{M}$, $\{f_n\}_n$ be as in Example 2.1.3 and let $a=1$. Take $\mathcal{M}_d := \ell^1(\{0\}\cup\mathbb{N})$ and define*

$$S:\mathcal{M}_d \ni \{a_n\}_n \mapsto S\{a_n\}_n := 1+\left|\sum_{n=1}^{\infty} a_n\right| \in \mathcal{M}.$$

Then

$$\begin{aligned}|S\{a_n\}_n - S\{b_n\}_n| &= \left| |\sum_{n=1}^{\infty} a_n| - |\sum_{n=1}^{\infty} b_n| \right| \leq \left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n\right| \\ &= \left|\sum_{n=1}^{\infty}(a_n-b_n)\right| \leq \sum_{n=1}^{\infty}|a_n-b_n| \leq \sum_{n=0}^{\infty}|a_n-b_n| \\ &= \|\{a_n\}_n - \{b_n\}_n\|, \quad \forall \{a_n\}_n,\{b_n\}_n \in \ell^1(\{0\}\cup\mathbb{N}).\end{aligned}$$

Thus S is Lipschitz. Further,

$$S(\{f_n(x)\}_n) = 1+\left|\sum_{n=1}^{\infty} f_n(x)\right| = 1+\sum_{n=1}^{\infty}\frac{(\log x)^n}{n!} = x, \quad \forall x \in \mathcal{M}.$$

Hence $(\{f_n\}_n, S)$ is a metric frame for $\mathcal{M}$. Note that if we define

$$T:\mathcal{M}_d \ni \{a_n\}_n \mapsto S\{a_n\}_n := 1+\sum_{n=1}^{\infty}|a_n| \in \mathcal{M},$$

then

$$\begin{aligned}|T\{a_n\}_n - T\{b_n\}_n| &= \left|\sum_{n=1}^{\infty}|a_n| - \sum_{n=1}^{\infty}|b_n|\right| = \left|\sum_{n=1}^{\infty}(|a_n|-|b_n|)\right| \\ &\leq \sum_{n=1}^{\infty}\left||a_n|-|b_n|\right| \leq \sum_{n=1}^{\infty}|a_n-b_n| \leq \sum_{n=0}^{\infty}|a_n-b_n| \\ &= \|\{a_n\}_n - \{b_n\}_n\|, \quad \forall \{a_n\}_n,\{b_n\}_n \in \ell^1(\{0\}\cup\mathbb{N}).\end{aligned}$$

Thus T is Lipschitz. Hence $(\{f_n\}_n, T)$ is also a metric frame for $\mathcal{M}$.

Example 2.1.14. *Let $f_1:\mathbb{K}\to\mathbb{K}$ be bi-Lipschitz and let $f_2,\dots,f_m:\mathbb{K}\to\mathbb{K}$ be m Lipschitz maps such that*

$$f_1(x)+\cdots+f_m(x) = x, \quad \forall x \in \mathbb{K}.$$

We now define $S : \mathbb{K}^m \ni (x_1, \dots, x_m) \mapsto \sum_{j=1}^m x_j \in \mathbb{K}$. Then $(\{f_n\}_n, S)$ is a metric frame for $\mathbb{K}$. Note that the operator S is linear.

After the definition of metric frame, the first question which comes is the existence. In Theorem 1.3.13 it was proved that every separable Banach space admits a Banach frame. Even though this result is not known in metric space settings, two results are derived one is close to the definition of metric frame and another gives existence under certain assumptions. To do this we prove a result which we derive from Mc-Shane extension theorem.

Theorem 2.1.15. ***(Mc-Shane extension theorem)*** *(cf. Weaver (2018)) Let $\mathscr{M}$ be a metric space and $\mathscr{M}_0$ be a nonempty subset of $\mathscr{M}$. If $f_0 : \mathscr{M}_0 \to \mathbb{R}$ is Lipschitz, then there exists a Lipschitz function $f : \mathscr{M} \to \mathbb{R}$ such that $f|\mathscr{M}_0 = f_0$ and $\operatorname{Lip}(f) = \operatorname{Lip}(f_0)$.*

Using Theorem 2.1.15 we derive the following.

Corollary 2.1.16. *If $(\mathscr{M}, 0)$ is a pointed metric space, then for every $x \in \mathscr{M}$, there exists a Lipschitz function $f : \mathscr{M} \to \mathbb{R}$ such that $f(x) = d(x,0)$, $f(0) = 0$ and $\operatorname{Lip}(f) = 1$.*

Proof. Case (i) : $x \neq 0$. Define $\mathscr{M}_0 := \{0, x\}$ and $f_0(0) = 0$, $f_0(x) = d(x,0)$. Then $|f_0(x) - f_0(0)| = d(x,0)$ and hence f_0 is Lipschitz. Application of Theorem 2.1.15 now completes the proof.
Case (ii) : $x = 0$. Take a non-zero point $y \in \mathscr{M}$. We use the argument in case (i) by replacing y in the place of x. □

Theorem 2.1.17. *Let $\mathscr{M}$ be a separable metric space. Then there exist a BK-space $\mathscr{M}_d$, a sequence $\{f_n\}_n$ in $\operatorname{Lip}_0(\mathscr{M}, \mathbb{R})$ and a function $S : \mathscr{M}_d \to \mathscr{M}$ such that*

(i) $\{f_n(x)\}_n \in \mathscr{M}_d$, for each $x \in \mathscr{M}$,

(ii) $\|\{f_n(x) - f_n(y)\}_n\|_{\mathscr{M}_d} \leq d(x,y), \forall x, y \in \mathscr{M}$,

(iii) $S(\{f_n(x)\}_n) = x$, for each $x \in \mathscr{M}$.

Proof. Let $\{x_n\}_n$ be a dense set in $\mathscr{M}$. Then for each $n \in \mathbb{N}$, from Corollary 2.1.16 there exists a Lipschitz function $f_n : \mathscr{M} \to \mathbb{R}$ such that $f_n(x_n) = d(x_n, 0)$, $f_n(0) = 0$ and $\operatorname{Lip}(f_n) = 1$. Let $x \in \mathscr{M}$ be fixed. Now for each $n \in \mathbb{N}$,

$$|f_n(x)| = |f_n(x) - f_n(0)| \leq \|f_n\|_{\operatorname{Lip}_0} d(x,0) = d(x,0)$$

which gives $\sup_{n \in \mathbb{N}} |f_n(x)| \leq d(x,0)$. Since $\{x_n\}_n$ is dense, there exists a sub sequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $x_{n_k} \to x$ as $n \to \infty$. From the inequality

$$|d(y,z) - d(y,w)| \leq d(z,w), \quad \forall y, z, w \in \mathscr{M}$$

we see then that $d(x_{n_k},0) \to d(x,0)$ as $n \to \infty$. Consider

$$\begin{aligned} d(x_{n_k},0) &= f_{n_k}(x_{n_k}) \leq |f_{n_k}(x_{n_k}) - f_{n_k}(x)| + |f_{n_k}(x)| \\ &\leq 1.d(x_{n_k},x) + |f_{n_k}(x)|, \quad \forall k \in \mathbb{N} \\ \implies \lim_{k\to\infty}&(d(x_{n_k},0) - d(x_{n_k},x)) \leq \sup_{k\in\mathbb{N}}(d(x_{n_k},0) - d(x_{n_k},x)) \leq \sup_{k\in\mathbb{N}}|f_{n_k}(x)|. \end{aligned}$$

Therefore

$$\begin{aligned} \sup_{n\in\mathbb{N}}|f_n(x)| \leq d(x,0) &= \lim_{k\to\infty} d(x_{n_k},0) = \lim_{k\to\infty}(d(x_{n_k},0) - d(x_{n_k},x)) \\ &\leq \sup_{k\in\mathbb{N}}|f_{n_k}(x)| \leq \sup_{n\in\mathbb{N}}|f_n(x)|. \end{aligned}$$

So we proved that

$$d(x,0) = \sup_{n\in\mathbb{N}}|f_n(x)|, \quad \forall x \in \mathscr{M}. \tag{2.1.2}$$

Define $\mathscr{M}_d^0 := \{\{f_n(x)\}_n : x \in \mathscr{M}\}$. Equality (2.1.2) then tells that $\mathscr{M}_d^0$ is a subset of $\ell^\infty(\mathbb{N})$. Now we define $S_0 : \mathscr{M}_d^0 \ni \{f_n(x)\}_n \mapsto x \in \mathscr{M}$. Then from Equality (2.1.2),

$$\begin{aligned} d(S_0(\{f_n(x)\}_n), S_0(\{f_n(y)\}_n)) &= d(x,y) \leq d(x,0) + d(0,y) \\ &= \sup_{n\in\mathbb{N}}|f_n(x)| + \sup_{n\in\mathbb{N}}|f_n(y)| \\ &= \|\{f_n(x)\}_n\| + \|\{f_n(y)\}_n\|, \quad \forall x,y \in \mathscr{M}. \end{aligned}$$

We also have

$$\begin{aligned} \|\{f_n(x) - f_n(y)\}_n\|_{\mathscr{M}_d} &= \sup_{n\in\mathbb{N}}|f_n(x) - f_n(y)| \\ &\leq \sup_{n\in\mathbb{N}}\|f_n\|_{\mathrm{Lip}_0} d(x,y) = d(x,y), \quad \forall x,y \in \mathscr{M}. \end{aligned}$$

We can now take S as Lipschitz extension of S_0 to $\ell^\infty(\mathbb{N})$ and $\mathscr{M}_d = \ell^\infty(\mathbb{N})$ which completes the proof. □

Theorem 2.1.18. *If $A : \mathscr{M} \to \mathscr{M}_d$ is bi-Lipschitz and there is a Lipschitz projection $P : \mathscr{M}_d \to A(\mathscr{M})$, then $\mathscr{M}$ admits a metric frame.*

Proof. Let $\{h_n\}_n$ be the sequence of coordinate functionals associated with $\mathscr{M}_d$. Define $f_n := h_nA$ and $S := A^{-1}P$. Then

$$S(\{f_n(x)\}_n) = A^{-1}P(\{h_n(Ax)\}_n) = A^{-1}PAx = A^{-1}Ax = x, \quad \forall x \in \mathscr{M}.$$

Hence $(\{f_n\}_n, S)$ is a metric frame for $\mathscr{M}$. $\square$

It is well-known that Mc-Schane extension theorem fails for complex valued Lipschitz functions. Thus we may ask whether we can take a complex sequence space in Theorem 2.1.17. It is possible for certain metric spaces due to the following theorem.

Theorem 2.1.19. *(**Kirszbraun extension theorem**) (cf. Valentine (1945)) Let $\mathscr{H}$ be a Hilbert space and $\mathscr{M}_0$ be a nonempty subset of $\mathscr{H}$. If $f_0 : \mathscr{M}_0 \to \mathbb{K}$ is Lipschitz, then there exists a Lipschitz function $f : \mathscr{H} \to \mathbb{K}$ such that $f|\mathscr{M}_0 = f_0$ and $\operatorname{Lip}(f) = \operatorname{Lip}(f_0)$.*

Following proposition shows that given a metric frame, we can generate more metric frames.

Proposition 2.1.20. *Let $(\{f_n\}_n, S)$ be a metric frame for $\mathscr{M}$. If maps $A, B : \mathscr{M} \to \mathscr{M}$ are such that A is bi-Lipschitz, B is Lipschitz and $BA = I_{\mathscr{M}}$, then $(\{f_nA\}_n, BS)$ is a metric frame for $\mathscr{M}$. In particular, if $A : \mathscr{M} \to \mathscr{M}$ is bi-Lipschitz invertible, then $(\{f_nA\}_n, A^{-1}S)$ is a metric frame for $\mathscr{M}$.*

Proof. Bi-Lipschitzness of A tells that condition (ii) in Definition 2.1.10 holds. Now by using $BA = I_{\mathscr{M}}$ we get $BS(\{f_nAx\}_n) = BAx = x, \forall x \in \mathscr{M}$. $\square$

Previous proposition not only helps to generate metric frames from metric frames but also from Banach frames. Since there are large number of examples of Banach frames for a variety of Banach spaces, just by operating with bi-Lipschitz invertible functions on subsets, it produces metric frames for that subset. Next we characterize metric frames using Lipschitz functions. The following theorem precisely says when an $\mathscr{M}_d$-frame can be converted into a metric frame.

Theorem 2.1.21. *Let $\{f_n\}_n$ be a metric $\mathscr{M}_d$-frame for $\mathscr{M}$. Then the following are equivalent.*

(i) There exists a Lipschitz projection $P : \mathscr{M}_d \to \theta_f(\mathscr{M})$.

(ii) There exists a Lipschitz map $V : \mathscr{M}_d \to \mathscr{M}$ such that $V|_{\theta_f(\mathscr{M})} = \theta_f^{-1}$.

(iii) There exists a Lipschitz map $S : \mathscr{M}_d \to \mathscr{M}$ such that $(\{f_n\}_n, S)$ is a metric frame for $\mathscr{M}$.

Proof. (i) $\Rightarrow$ (ii) Define $V := \theta_f^{-1}P$. Then for $y = \theta_f(x), x \in \mathscr{M}$ we get $Vy = V\theta_f(x) = \theta_f^{-1}P\theta_f(x) = \theta_f^{-1}\theta_f(x) = \theta_f^{-1}y$.
(ii) $\Rightarrow$ (i) Set $P := \theta_f V$. Now $P^2 = \theta_f V \theta_f V = \theta_f I_{\mathscr{M}} V = P$.
(ii) $\Rightarrow$ (iii) Define $S := V$. Then $S\{f_n(x)\}_n = S\theta_f(x) = V\theta_f(x) = x$, for all $x \in \mathscr{M}$. Hence $(\{f_n\}_n, S)$ is a metric frame for $\mathscr{M}$.
(iii) $\Rightarrow$ (ii) Define $V := S$. Then $V\theta_f(x) = S\theta_f(x) = S\{f_n(x)\}_n = x$, for all $x \in \mathscr{M}$. $\square$

2.2 METRIC FRAMES FOR BANACH SPACES

Now we turn onto the representation of elements using metric frames. Naturally, to deal with sums we must look in to Banach space structure. Following theorem can be compared with Theorem 1.3.20.

Theorem 2.2.1. *Let $\{f_n\}_n$ be a metric p-frame for a Banach space $\mathscr{X}$. Assume that $f_n(0) = 0$ for all n. Then the following are equivalent.*

(i) There exists a bounded linear map $V : \mathscr{M}_d \to \mathscr{X}$ such that $V|_{\theta_f(\mathscr{M})} = \theta_f^{-1}$.

(ii) There exists a bounded linear map $S : \mathscr{M}_d \to \mathscr{X}$ such that $(\{f_n\}_n, S)$ is a metric p-frame for $\mathscr{X}$.

(iii) There exists a sequence $\{\tau_n\}_n$ in $\mathscr{X}$ such that $\sum_{n=1}^{\infty} c_n\tau_n$ converges for all $\{c_n\}_n \in \ell^p(\mathbb{N})$ and $x = \sum_{n=1}^{\infty} f_n(x)\tau_n, \forall x \in \mathscr{X}$.

(iv) There exists a q-Bessel sequence $\{\tau_n\}_n$ in $\mathscr{X} \subseteq \mathscr{X}^{**}$ such that $x = \sum_{n=1}^{\infty} f_n(x)\tau_n$, $\forall x \in \mathscr{X}$.

(v) There exists a q-Bessel sequence $\{\tau_n\}_n$ in $\mathscr{X} \subseteq \mathscr{X}^{**}$ such that $f = \sum_{n=1}^{\infty} f(\tau_n)f_n$, $\forall f \in \mathscr{X}^*$.

In each of the cases (iv) and (v), $\{\tau_n\}_n$ is actually a q-frame for $\mathscr{X}^*$.

Proof. Proof of (i) $\iff$ (ii) is similar to the proof of (ii) $\iff$ (iii) in Theorem 2.1.21.
(iii) $\Rightarrow$ (i) Given information tells that the map

$$V : \ell^p(\mathbb{N}) \ni \{c_n\}_n \to \sum_{n=1}^{\infty} c_n\tau_n \in \mathscr{X}$$

is well-defined. Banach-Steinhaus theorem now asserts that V is bounded. Now for $y = \theta_f(x), x \in \mathscr{X}$ we get

$$Vy = V\theta_f(x) = V(\{f_n(x)\}_n) = \sum_{n=1}^{\infty} f_n(x)\tau_n = x = \theta_f^{-1}\theta_f(x) = \theta_f^{-1}y.$$

(i) $\Rightarrow$ (iii) Let $\{e_n\}_n$ be the standard Schauder basis for $\ell^p(\mathbb{N})$ and define $\tau_n := Ve_n$, for all n. Since V is bounded linear and $\sum_{n=1}^{\infty} c_n e_n$ converges for all $\{c_n\}_n \in \ell^p(\mathbb{N})$, it follows that $\sum_{n=1}^{\infty} c_n\tau_n$ converges for all $\{c_n\}_n \in \ell^p(\mathbb{N})$. Moreover,

$$x = V\theta_f(x) = V(\{f_n(x)\}_n) = \sum_{n=1}^{\infty} f_n(x)\tau_n, \quad \forall x \in \mathscr{X}.$$

(iii) $\iff$ (iv) By considering τ_n in $\mathscr{X}^{**}$ through James embedding and using Theorem 1.3.20 we get that $\{\tau_n\}_n$ is a q-Bessel sequence in $\mathscr{X}$ if and only if $\sum_{n=1}^{\infty} c_n \tau_n$ converges for all $\{c_n\}_n \in \ell^p(\mathbb{N})$.

(iv) $\Rightarrow$ (v) Let b be a Bessel bound for $\{\tau_n\}_n$. Then for all $f \in \mathscr{X}^*$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\left\| f - \sum_{k=1}^{n} f(\tau_k) f_k \right\|_{\mathrm{Lip}_0} &= \sup_{x,y\in\mathscr{X},\, x\neq y} \frac{\left|\left(f - \sum_{k=1}^{n} f(\tau_k) f_k\right)(x) - \left(f - \sum_{k=1}^{n} f(\tau_k) f_k\right)(y)\right|}{\|x-y\|} \\
&= \sup_{x,y\in\mathscr{X},\, x\neq y} \frac{\left|f\left(\sum_{k=1}^{\infty} f_k(x)\tau_k\right) - f\left(\sum_{k=1}^{\infty} f_k(y)\tau_k\right) - \sum_{k=1}^{\infty} f(\tau_k)(f_k(x)-f_k(y))\right|}{\|x-y\|} \\
&= \sup_{x,y\in\mathscr{X},\, x\neq y} \frac{\left|\sum_{k=1}^{n} f(\tau_k)(f_k(x)-f_k(y)) - \sum_{k=1}^{\infty} f(\tau_k)(f_k(x)-f_k(y))\right|}{\|x-y\|} \\
&= \sup_{x,y\in\mathscr{X},\, x\neq y} \frac{\left|\sum_{k=n+1}^{\infty} f(\tau_k)(f_k(x)-f_k(y))\right|}{\|x-y\|} \\
&\leq \sup_{x,y\in\mathscr{X},\, x\neq y} \frac{\left(\sum_{k=n+1}^{\infty} |f(\tau_k)|^q\right)^{\frac{1}{q}} \left(\sum_{k=n+1}^{\infty} |f_k(x)-f_k(y)|^p\right)^{\frac{1}{p}}}{\|x-y\|} \\
&\leq b\left(\sum_{k=n+1}^{\infty} |f_k(x)-f_k(y)|^p\right)^{\frac{1}{p}} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

(v) $\Rightarrow$ (iv) Let b be a Bessel bound for $\{\tau_n\}_n$. Let $x \in \mathscr{X}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
\left\| x - \sum_{k=1}^{n} f_k(x)\tau_k \right\| &= \sup_{f\in\mathscr{X}^*, \|f\|=1} \left| f(x) - \sum_{k=1}^{n} f_k(x) f(\tau_k) \right| \\
&= \sup_{f\in\mathscr{X}^*, \|f\|=1} \left| \left(\sum_{k=1}^{\infty} f(\tau_k) f_k\right)(x) - \sum_{k=1}^{n} f_k(x) f(\tau_k) \right| \\
&= \sup_{f\in\mathscr{X}^*, \|f\|=1} \left| \sum_{k=n+1}^{\infty} f_k(x) f(\tau_k) \right| \\
&\leq \left(\sum_{k=n+1}^{\infty} |f(\tau_k)|^q\right)^{\frac{1}{q}} \left(\sum_{k=n+1}^{\infty} |f_k(x)-f_k(0)|^p\right)^{\frac{1}{p}} \\
&\leq b\left(\sum_{k=n+1}^{\infty} |f_k(x)|^p\right)^{\frac{1}{p}} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Now we are left with proving that $\{\tau_n\}_n$ is a q-frame for $\mathscr{X}$. Assume (iv). Let $f \in \mathscr{X}^*$. Then

$$\|f\| = \sup_{x\in\mathscr{X}, \|x\|=1} |f(x)| = \sup_{x\in\mathscr{X}, \|x\|=1} \left| f\left(\sum_{n=1}^{\infty} f_n(x)\tau_n\right) \right|$$

$$= \sup_{x\in \mathscr{X}, \|x\|=1} \left| \sum_{n=1}^{\infty} f_n(x) f(\tau_n) \right| \leq b \left(\sum_{n=1}^{\infty} |f_n(x)|^p \right)^{\frac{1}{p}}.$$

Since f was arbitrary, the conclusion follows. □

Theorem 1.3.20 and Theorem 2.2.1 suggest the following question. For which metric spaces and BK-spaces, does Theorem 2.2.1 hold? We next present a result which demands only reconstruction of elements using Lipschitz functions on a Banach space and not frame conditions. First we see a result for this purpose.

Lemma 2.2.2. *(cf. Casazza et al. (2005a)) Given a Banach space $\mathscr{X}$ and a sequence $\{\tau_n\}_n$ of non-zero elements in $\mathscr{X}$, let*

$$\mathscr{Y}_d := \left\{ \{a_n\}_n : \sum_{n=1}^{\infty} a_n \tau_n \text{ converges in } \mathscr{X} \right\}.$$

Then $\mathscr{Y}_d$ is a Banach space with respect to the norm

$$\|\{a_n\}_n\| := \sup_m \left\| \sum_{n=1}^{m} a_n \tau_n \right\|.$$

Further, the canonical unit vectors form a Schauder basis for $\mathscr{Y}_d$.

Theorem 2.2.3. *Let $\mathscr{X}$ be a Banach space and $\{f_n\}_n$ be a sequence in $\operatorname{Lip}_0(\mathscr{X},\mathbb{K})$. Then the following are equivalent.*

(i) There exists a sequence $\{\tau_n\}_n$ in $\mathscr{X}$ such that $x = \sum_{n=1}^{\infty} f_n(x)\tau_n, \forall x \in \mathscr{X}$.

(ii) Let $\{\tau_n\}_n$ be a sequence in $\mathscr{X}$ and define $S_n(x) := \sum_{k=1}^{n} f_k(x)\tau_k$, $\forall x \in \mathscr{X}$, for each $n \in \mathbb{N}$. Then $\sup_{n\in\mathbb{N}} \|S_n\|_{\operatorname{Lip}_0} < \infty$ and there exist a BK-space $\mathscr{M}_d$ and a bounded linear map $S : \mathscr{M}_d \to \mathscr{M}$ such that $(\{f_n\}_n, S)$ is a metric frame for $\mathscr{X}$.

Further, a choice for τ_n is $\tau_n = Se_n$ for each $n \in \mathbb{N}$, where $\{e_n\}_n$ is the standard Schauder basis for $\ell^p(\mathbb{N})$.

Proof. (ii) $\Rightarrow$ (i) This follows from Theorem 2.2.1.
(i) ($\Rightarrow$) (ii) We give an argument which is similar to the arguments given in Casazza et al. (2005a). Define $A := \{n \in \mathbb{N} : \tau_n = 0\}$ and $B := \mathbb{N} \setminus A$. Let $c_0(A)$ be the space of sequences converging to zero, indexed by A, equipped with sup-norm. Let $\{e_n\}_{n\in A}$ be the canonical Schauder basis for $c_0(A)$. Since the norm is sup-norm, it easily follows

that $\{\frac{1}{n(\|f_n\|_{\text{Lip}_0}+1)}e_n\}_{n\in A}$ is also a Schauder basis for $c_0(A)$. Define

$$\mathscr{Z}_d := \left\{ \{c_n\}_{n\in A} : \sum_{n\in A} \frac{c_n}{n(\|f_n\|_{\text{Lip}_0}+1)} e_n \text{ converges in } A \right\}.$$

We equip $\mathscr{Z}_d$ with the norm

$$\|\{c_n\}_{n\in A}\|_{\mathscr{Z}_d} := \left\| \frac{c_n}{n(\|f_n\|_{\text{Lip}_0}+1)} \right\|_{c_0(A)} = \sup_{n\in A} \left| \frac{c_n}{n(\|f_n\|_{\text{Lip}_0}+1)} \right|.$$

Then $\{e_n\}_{n\in A}$ is a Schauder basis for $\mathscr{Z}_d$. Clearly $\mathscr{Z}_d$ is a BK-space. Let $\mathscr{Y}_d$ be as defined in Lemma 2.2.2, for the index set B. Now set $\mathscr{M}_d := \mathscr{Y}_d \oplus \mathscr{Z}_d$ equipped with norm $\|y\oplus z\|_{\mathscr{M}_d} := \|y\|_{\mathscr{Y}_d} + \|z\|_{\mathscr{Z}_d}$. It then follows that, for each $x\in\mathscr{X}$, $\{f_n(x)\}_{n\in B} \oplus \{f_n(x)\}_{n\in A} \in \mathscr{M}_d$. We next show that $\{f_n\}_n$ is a metric $\mathscr{M}_d$-frame for $\mathscr{X}$. Let $x,y\in\mathscr{X}$. Then

$$\begin{aligned}
\|x-y\| &= \left\| \sum_{n=1}^{\infty} (f_n(x)-f_n(y))\tau_n \right\| = \lim_{n\to\infty} \left\| \sum_{k=1}^{n} (f_k(x)-f_k(y))\tau_k \right\| \\
&\leq \sup_{n\in\mathbb{N}} \left\| \sum_{k=1}^{n} (f_k(x)-f_k(y))\tau_k \right\| = \sup_{n\in B} \left\| \sum_{k=1}^{n} (f_k(x)-f_k(y))\tau_k \right\| \\
&= \|\{f_n(x)-f_n(y)\}_{n\in B}\|_{\mathscr{Y}_d} \\
&\leq \|\{f_n(x)-f_n(y)\}_{n\in B}\|_{\mathscr{Y}_d} + \|\{f_n(x)-f_n(y)\}_{n\in A}\|_{\mathscr{Z}_d} \\
&= \|\{f_n(x)-f_n(y)\}_{n\in B} \oplus \{f_n(x)-f_n(y)\}_{n\in A}\|_{\mathscr{M}_d}
\end{aligned}$$

and

$$\begin{aligned}
&\|\{f_n(x)-f_n(y)\}_{n\in B} \oplus \{f_n(x)-f_n(y)\}_{n\in A}\|_{\mathscr{M}_d} \\
&= \|\{f_n(x)-f_n(y)\}_{n\in B}\|_{\mathscr{Y}_d} + \|\{f_n(x)-f_n(y)\}_{n\in A}\|_{\mathscr{Z}_d} \\
&= \sup_{n\in B} \left\| \sum_{k=1}^{n} (f_k(x)-f_k(y))\tau_k \right\| + \sup_{n\in A} \left| \frac{f_n(x)-f_n(y)}{n(\|f_n\|_{\text{Lip}_0}+1)} \right| \\
&= \sup_{n\in B} \|S_n(x)-S_n(y)\| + \sup_{n\in A} \left| \frac{f_n(x)-f_n(y)}{n(\|f_n\|_{\text{Lip}_0}+1)} \right| \\
&\leq \sup_{n\in B} \|S_n\|_{\text{Lip}_0} \|x-y\| + \sup_{n\in A} \frac{\|f_n\|_{\text{Lip}_0}\|x-y\|}{n(\|f_n\|_{\text{Lip}_0}+1)} \\
&\leq \left(\sup_{n\in B} \|S_n\|_{\text{Lip}_0} + 1 \right) \|x-y\|.
\end{aligned}$$

We now define

$$S : \mathscr{M}_d \ni \{a_n\}_{n\in B} \oplus \{b_n\}_{n\in A} \mapsto \sum_{n\in B} a_n\tau_n \in \mathscr{X}.$$

Clearly S is linear. Boundedness of S follows from the following calculation.

$$\begin{aligned}\|S(\{a_n\}_{n\in B} \oplus \{b_n\}_{n\in A})\| &= \left\|\sum_{n\in B} a_n\tau_n\right\| \leq \sup_{n\in B}\left\|\sum_{k=1}^{n} a_k\tau_k\right\| \\ &= \|\{a_n\}_{n\in B}\|_{\mathscr{Y}_d} \leq \|\{a_n\}_{n\in B} \oplus \{b_n\}_{n\in A}\|_{\mathscr{M}_d}.\end{aligned}$$

□

Using Theorem 1.5.6 we derive the following result which tells that given a metric frame for a metric space we can get a metric frame using linear functionals for a subset of the Banach space.

Theorem 2.2.4. *Let $\{f_n\}_n$ be a sequence in $\mathrm{Lip}_0(\mathscr{M}, \mathbb{K})$. For each $n \in \mathbb{N}$, let T_{f_n} be linearization of f_n. Let e and $\mathscr{F}(\mathscr{M})$ be as in Theorem 1.5.6. Then $\{f_n\}_n$ is a metric frame for $\mathscr{M}$ with bounds a and b if and only if $\{T_{f_n}\}_n$ is a metric frame for $e(\mathscr{M})$ with bounds a and b. In particular, $(\{f_n\}_n, S)$ is a metric frame for $\mathscr{M}$ if and only if $(\{T_{f_n}\}_n, eS)$ is a metric frame for $e(\mathscr{M})$.*

Proof. ($\Rightarrow$) Let $u, v \in e(\mathscr{M})$. Then $u = e(x), v = e(y)$, for some $x, y \in \mathscr{M}$. Now using the fact that e is an isometry,

$$\begin{aligned}a\|u-v\| = a\|e(x)-e(y)\| = a\,d(x,y) &\leq \|\{f_n(x) - f_n(y)\}_n\| \\ &= \|\{(T_{f_n}e)(x) - (T_{f_n}e)(y)\}_n\| = \|\{T_{f_n}(e(x)) - T_{f_n}(e(y))\}_n\| \\ &= \|\{T_{f_n}(u) - T_{f_n}(v)\}_n\| \leq b\,d(x,y) = b\|e(x)-e(y)\| = b\|u-v\|.\end{aligned}$$

($\Leftarrow$) Let $x, y \in \mathscr{M}$. Then $e(x), e(y) \in e(\mathscr{M})$. Hence

$$\begin{aligned}a\,d(x,y) = a\|e(x)-e(y)\| &\leq \|\{T_{f_n}(e(x)) - T_{f_n}(e(y))\}_n\| \\ &= \|\{f_n(x) - f_n(y)\}_n\| \leq b\|e(x)-e(y)\| = b\,d(x,y).\end{aligned}$$

Since x, y were arbitrary, the result follows. □

Remark 2.2.5. *We can not use Theorem 2.2.4 to view metric frames as Banach frames. The reason is that $e(\mathscr{M})$ is just a subset of $\mathscr{F}(\mathscr{M})$ and need not be a vector space. Moreover, the map eS is Lipschitz and may not be linear.*

2.3 PERTURBATIONS

Here we present some stability results. These are important as they say that sequences which are close to metric frames are again metric frames. On the other hand, it asserts that if we perturb a metric frame we again get a metric frame.

Theorem 2.3.1. *Let $\{f_n\}_n$ be a p-metric frame for $\mathscr{M}$ with bounds a and b. Let $\{g_n\}_n$ be a sequence in* $\operatorname{Lip}(\mathscr{M},\mathbb{K})$ *satisfying the following.*

(i) There exist $\alpha, \beta, \gamma \geq 0$ such that $\beta < 1$, $\alpha < 1$, $\gamma < (1-\alpha)a$.

(ii) For all $x, y \in \mathscr{M}$, and $m = 1, 2, \dots$,

$$\left(\sum_{n=1}^{m} |(f_n-g_n)(x)-(f_n-g_n)(y)|^p\right)^{\frac{1}{p}} \leq \alpha \left(\sum_{n=1}^{m} |f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} + \beta \left(\sum_{n=1}^{m} |g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y). \tag{2.3.1}$$

Then $\{g_n\}_n$ is a p-metric frame for $\mathscr{M}$ with bounds $\frac{((1-\alpha)a-\gamma)}{1+\beta}$ and $\frac{((1+\alpha)b+\gamma)}{1-\beta}$.

Proof. Using Minkowski's inequality and Inequality (2.3.1), we get, for all $x, y \in \mathscr{M}$ and $m \in \mathbb{N}$,

$$\begin{aligned}
&\left(\sum_{n=1}^{m} |g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{n=1}^{m} |(f_n-g_n)(x)-(f_n-g_n)(y)|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{m} |f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} \\
&\leq (1+\alpha)\left(\sum_{n=1}^{m} |f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} + \beta \left(\sum_{n=1}^{m} |g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y)
\end{aligned}$$

which implies

$$(1-\beta)\left(\sum_{n=1}^{m} |g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} \leq (1+\alpha)\left(\sum_{n=1}^{m} |f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y),$$

for all $x, y \in \mathscr{M}$. Since the sum $\sum_{n=1}^{\infty} |f_n(x)-f_n(y)|^p$ converges, $\sum_{n=1}^{\infty} |g_n(x)-g_n(y)|^p$

will also converge. Inequality (2.3.1) now gives

$$\left(\sum_{n=1}^{\infty}|(f_n-g_n)(x)-(f_n-g_n)(y)|^p\right)^{\frac{1}{p}} \leq \alpha\left(\sum_{n=1}^{\infty}|f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} + \beta\left(\sum_{n=1}^{\infty}|g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y). \tag{2.3.2}$$

By doing a similar calculation and using Inequality (2.3.2) we get for all $x,y \in \mathscr{M}$,

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty}|g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{n=1}^{\infty}|(f_n-g_n)(x)-(f_n-g_n)(y)|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}|f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} \\
&\leq (1+\alpha)\left(\sum_{n=1}^{\infty}|f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} + \beta\left(\sum_{n=1}^{\infty}|g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y) \\
&\leq (1+\alpha)b\,d(x,y) + \beta\left(\sum_{n=1}^{\infty}|g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y) \\
&= ((1+\alpha)b+\gamma)\,d(x,y) + \beta\left(\sum_{n=1}^{\infty}|g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}}
\end{aligned}$$

which gives

$$\begin{aligned}
(1-\beta)\left(\sum_{n=1}^{\infty}|g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} &\leq ((1+\alpha)b+\gamma)\,d(x,y), \quad \forall x,y \in \mathscr{M} \\
\text{i.e., }\left(\sum_{n=1}^{\infty}|g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} &\leq \frac{((1+\alpha)b+\gamma)}{1-\beta}\,d(x,y), \quad \forall x,y \in \mathscr{M}.
\end{aligned}$$

Hence we obtained upper frame bound for $\{g_n\}_n$. For lower frame bound, let $x,y \in \mathscr{M}$. Then

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty}|f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{n=1}^{\infty}|(f_n-g_n)(x)-(f_n-g_n)(y)|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}|g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leq \alpha \left(\sum_{n=1}^{\infty} |f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} + (1+\beta)\left(\sum_{n=1}^{\infty} |g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y)$$

which implies

$$\begin{aligned}(1-\alpha)ad(x,y) &\leq (1-\alpha)\left(\sum_{n=1}^{\infty} |f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} \\ &\leq (1+\beta)\left(\sum_{n=1}^{\infty} |g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y), \quad \forall x,y\in \mathscr{M}\end{aligned}$$

i.e., $\dfrac{((1-\alpha)a-\gamma)}{1+\beta} \leq \left(\sum_{n=1}^{\infty} |g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}}, \quad \forall x,y\in \mathscr{M}.$

□

Using Theorem 2.3.1 we obtain the following result.

Corollary 2.3.2. *Let $\{f_n\}_n$ be a p-metric frame for $\mathscr{M}$ with bounds a and b. Let $\{g_n\}_n$ be a sequence in* $\operatorname{Lip}(\mathscr{M},\mathbb{K})$ *such that*

$$r := \left(\sum_{n=1}^{\infty} \operatorname{Lip}(f_n-g_n)^p\right)^{\frac{1}{p}} < a.$$

Then $\{g_n\}_n$ is a p-metric frame for $\mathscr{M}$ with bounds $a-r$ and $b+r$.

Proof. Define $\alpha := 0$, $\beta := 0$ and $\gamma := r$. Then for all $x,y\in\mathscr{M}$,

$$\begin{aligned}\left(\sum_{n=1}^{\infty} |(f_n-g_n)(x)-(f_n-g_n)(y)|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{n=1}^{\infty} \operatorname{Lip}(f_n-g_n)^p d(x,y)^p\right)^{\frac{1}{p}} \\ = \left(\sum_{n=1}^{\infty} \operatorname{Lip}(f_n-g_n)^p\right)^{\frac{1}{p}} d(x,y) &= r d(x,y) \\ = \alpha\left(\sum_{n=1}^{\infty} |f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} &+ \beta\left(\sum_{n=1}^{\infty} |g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y).\end{aligned}$$

Thus the hypothesis in Theorem 2.3.1 holds. Hence the corollary. □

Corollary 2.3.3. *Let $\{f_n\}_n$ be a p-metric Bessel sequence for $\mathscr{M}$ with bound b. Let $\{g_n\}_n$ be a sequence in* $\operatorname{Lip}(\mathscr{M},\mathbb{K})$ *satisfying the following.*

(i) There exist $\alpha,\beta,\gamma\geq 0$ such that $\beta<1$.

(ii) For all $x, y \in \mathscr{M}$, and $m = 1, 2, \ldots$,

$$\left(\sum_{n=1}^{m}|(f_n-g_n)(x)-(f_n-g_n)(y)|^p\right)^{\frac{1}{p}} \leq \alpha\left(\sum_{n=1}^{m}|f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} + \beta\left(\sum_{n=1}^{m}|g_n(x)-g_n(y)|^p\right)^{\frac{1}{p}} + \gamma d(x,y).$$

Then $\{g_n\}_n$ is a p-metric Bessel sequence for $\mathscr{M}$ with bound $\frac{((1+\alpha)b+\gamma)}{1-\beta}$.

We next derive a stability result in which we perturb the Lipschitz functions and then derive the existence of reconstruction operator. This is motivated from Theorem 1.3.21.

Theorem 2.3.4. *Let $(\{f_n\}_n, S)$ be a metric frame for a Banach space $\mathscr{X}$. Assume that $f_n(0) = 0$, for all $n \in \mathbb{N}$, and $S(0) = 0$. Let $\{g_n\}_n$ be a collection in $\operatorname{Lip}_0(\mathscr{X}, \mathbb{K})$ satisfying the following.*

(i) There exist $\alpha, \gamma \geq 0$ such that

$$\|\{(f_n-g_n)(x)-(f_n-g_n)(y)\}_n\| \leq \alpha\|\{f_n(x)-f_n(y)\}_n\| + \gamma\|x-y\|, \ \forall x, y \in \mathscr{X}. \tag{2.3.3}$$

(ii) $\alpha\|\theta_f\|_{\operatorname{Lip}_0} + \gamma \leq \|S\|^{-1}_{\operatorname{Lip}_0}$.

Then there exists a reconstruction Lipschitz operator T such that $(\{f_n\}_n, T)$ is a metric frame for $\mathscr{X}$ with bounds $\|S\|^{-1}_{\operatorname{Lip}_0} - (\alpha\|\theta_f\|_{\operatorname{Lip}_0} + \gamma)$ and $\|\theta_f\|_{\operatorname{Lip}_0} + (\alpha\|\theta_f\|_{\operatorname{Lip}_0} + \gamma)$.

Proof. Let $x \in \mathscr{X}$. Since $g_n(0) = 0$ and $f_n(0) = 0$ for all $n \in \mathbb{N}$, using Inequality (2.3.3),

$$\begin{aligned}\|\{g_n(x)\}_n\| &\leq \|\{(f_n-g_n)(x)\}_n\| + \|\{f_n(x)\}_n\| \\ &\leq (\alpha+1)\|\{f_n(x)\}_n\| + \gamma\|x\|.\end{aligned}$$

Therefore if we define $\theta_g : \mathscr{X} \ni x \mapsto \{g_n(x)\}_n \in \mathscr{M}_d$, then this map is well-defined. Again using Inequality (2.3.3), we show that θ_g is Lipschitz. For $x, y \in \mathscr{X}$,

$$\begin{aligned}\|\theta_g x - \theta_g y\| &= \|\{g_n(x)-g_n(y)\}_n\| = \|\{-g_n(x)+g_n(y)\}_n\| \\ &\leq \|\{(f_n-g_n)(x)-(f_n-g_n)(y)\}_n\| + \|\{f_n(x)-f_n(y)\}_n\| \\ &\leq (1+\alpha)\|\{f_n(x)-f_n(y)\}_n\| + \gamma\|x-y\| = (1+\alpha)\|\theta_f x - \theta_f y\| + \gamma\|x-y\| \\ &\leq (1+\alpha)\|\theta_f\|_{\operatorname{Lip}_0}\|x-y\| + \gamma\|x-y\| = ((1+\alpha)\|\theta_f\|_{\operatorname{Lip}_0} + \gamma)\|x-y\|.\end{aligned}$$

Thus $\|\theta_g\|_{\text{Lip}_0} \leq (1+\alpha)\|\theta_f\|_{\text{Lip}_0} + \gamma$. Previous calculation also tells that upper frame bound is $((1+\alpha)\|\theta_f\|_{\text{Lip}_0} + \gamma)$. We see further that Inequality (2.3.3) can be written as

$$\begin{aligned}\|(\theta_f - \theta_g)x - (\theta_f - \theta_g)y\| &\leq \alpha\|\theta_f x - \theta_f y\| + \gamma\|x-y\| \\ &\leq (\alpha\|\theta_f\|_{\text{Lip}_0} + \gamma)\|x-y\|, \quad \forall x,y \in \mathscr{X}. \end{aligned} \tag{2.3.4}$$

Now noting $S\theta_f = I_{\mathscr{X}}$ and using Inequality (2.3.4) we see that

$$\begin{aligned}\|I_{\mathscr{X}} - S\theta_g\|_{\text{Lip}_0} &= \|S\theta_f - S\theta_g\|_{\text{Lip}_0} \\ &\leq \|S\|_{\text{Lip}_0}\|\theta_f - \theta_g\|_{\text{Lip}_0} \\ &\leq \|S\|_{\text{Lip}_0}(\alpha\|\theta_f\|_{\text{Lip}_0} + \gamma) < 1.\end{aligned}$$

Since $\text{Lip}_0(\mathscr{X})$ is a unital Banach algebra (Theorem 1.5.5), last inequality tells that $S\theta_g$ is invertible and its inverse is also a Lipschitz operator and

$$\|(S\theta_g)^{-1}\|_{\text{Lip}_0} \leq \frac{1}{1 - \|S\|_{\text{Lip}_0}(\alpha\|\theta_f\|_{\text{Lip}_0} + \gamma)}.$$

Define $T := (S\theta_g)^{-1}S$. Then $T\theta_g = I_{\mathscr{X}}$ and

$$\begin{aligned}\|x-y\| = \|T\theta_g x - T\theta_g y\| &\leq \|T\|_{\text{Lip}_0}\|\theta_g x - \theta_g y\| \\ &\leq \frac{1}{1 - \|S\|_{\text{Lip}_0}(\alpha\|\theta_f\|_{\text{Lip}_0} + \gamma)}\|\theta_g x - \theta_g y\|, \quad \forall x,y \in \mathscr{X}\end{aligned}$$

which gives the lower bound stated in the theorem. □

Corollary 2.3.5. *Let $(\{f_n\}_n, S)$ be a metric Bessel sequence for a Banach space $\mathscr{X}$. Assume that $f_n(0) = 0$, for all $n \in \mathbb{N}$, and $S(0) = 0$. Let $\{g_n\}_n$ be a collection in $\text{Lip}_0(\mathscr{X}, \mathbb{K})$ satisfying the following. There exist $\alpha, \gamma \geq 0$ such that*

$$\|\{(f_n - g_n)(x) - (f_n - g_n)(y)\}_n\| \leq \alpha\|\{f_n(x) - f_n(y)\}_n\| + \gamma\|x-y\|, \quad \forall x,y \in \mathscr{X}.$$

Then there exists a reconstruction Lipschitz operator T such that $(\{f_n\}_n, T)$ is a metric Bessel sequence for $\mathscr{X}$ with bound $\|\theta_f\|_{\text{Lip}_0} + (\alpha\|\theta_f\|_{\text{Lip}_0} + \gamma)$.

CHAPTER 3

MULTIPLIERS FOR METRIC SPACES

3.1 DEFINITION AND BASIC PROPERTIES OF MULTIPLIERS

In this chapter, we introduce and study multipliers for metric spaces. We use the following notation in this chapter. Let $\mathscr{M}$ be a metric space and $\mathscr{X}$ be a Banach space. Given $f \in \operatorname{Lip}(\mathscr{M}, \mathbb{K})$ and $\tau \in \mathscr{X}$, define

$$\tau \otimes f : \mathscr{M} \ni x \mapsto (\tau \otimes f)(x) := f(x)\tau \in \mathscr{X}.$$

Then it follows that $\tau \otimes f$ is a Lipschitz operator and $\operatorname{Lip}(\tau \otimes f) = \|\tau\| \operatorname{Lip}(f)$.

We first derive a result which allows us to define multipliers for metric spaces. In the sequel, $1 < p < \infty$ and q denotes the conjugate index of p.

Theorem 3.1.1. *Let $\{f_n\}_n$ in $\operatorname{Lip}_0(\mathscr{M}, \mathbb{K})$ be a Lipschitz p-Bessel sequence for a pointed metric space $(\mathscr{M}, 0)$ with bound b and $\{\tau_n\}_n$ in a Banach space $\mathscr{X}$ be a Lipschitz q-Bessel sequence for $\operatorname{Lip}_0(\mathscr{X}, \mathbb{K})$ with bound d. If $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$, then the map*

$$T : \mathscr{M} \ni x \mapsto \sum_{n=1}^{\infty} \lambda_n (\tau_n \otimes f_n)x \in \mathscr{X}$$

is a well-defined Lipschitz operator such that $T(0) = 0$ with Lipschitz norm at most $bd\|\{\lambda_n\}_n\|_\infty$.

Proof. Let $n, m \in \mathbb{N}$ with $n \leq m$. Then for each $x \in \mathscr{M}$, using Holder's inequality,

$$\left\|\sum_{k=n}^{m} \lambda_k (\tau_k \otimes f_k)(x)\right\| = \left\|\sum_{k=n}^{m} \lambda_k f_k(x)\tau_k\right\| = \sup_{\phi \in \mathscr{X}^*, \|\phi\| \leq 1} \left|\phi\left(\sum_{k=n}^{m} \lambda_k f_k(x)\tau_k\right)\right|$$
$$= \sup_{\phi \in \mathscr{X}^*, \|\phi\| \leq 1} \left|\sum_{k=n}^{m} \lambda_k f_k(x)\phi(\tau_k)\right|$$

$$\leq \sup_{\phi\in\mathscr{X}^*, \|\phi\|\leq 1} \sum_{k=n}^{m} |\lambda_k||f_k(x)||\phi(\tau_k)|$$

$$\leq \sup_{n\in\mathbb{N}} |\lambda_n| \sup_{\phi\in\mathscr{X}^*, \|\phi\|\leq 1} \sum_{k=n}^{m} |f_k(x)||\phi(\tau_k)|$$

$$\leq \sup_{n\in\mathbb{N}} |\lambda_n| \sup_{\phi\in\mathscr{X}^*, \|\phi\|\leq 1} \left(\sum_{k=n}^{m} |f_k(x)|^p\right)^{\frac{1}{p}} \left(\sum_{k=n}^{m} |\phi(\tau_k)|^q\right)^{\frac{1}{q}}$$

$$\leq \sup_{n\in\mathbb{N}} |\lambda_n| \sup_{\phi\in\mathscr{X}^*, \|\phi\|\leq 1} \left(\sum_{k=n}^{m} |f_k(x)|^p\right)^{\frac{1}{p}} d\|\phi\|$$

$$= d \sup_{n\in\mathbb{N}} |\lambda_n| \left(\sum_{k=n}^{m} |f_k(x)|^p\right)^{\frac{1}{p}}.$$

Since $(\sum_{k=1}^{\infty} |f_k(x)|^p)^{\frac{1}{p}}$ converges, $\sum_{k=1}^{\infty} \lambda_k(\tau_k \otimes f_k)(x)$ also converges. Now for all $x, y \in \mathscr{M}$,

$$\|Tx - Ty\| = \left\|\sum_{n=1}^{\infty} \lambda_n f_n(x)\tau_n - \sum_{n=1}^{\infty} \lambda_n f_n(y)\tau_n\right\| = \left\|\sum_{n=1}^{\infty} \lambda_n (f_n(x) - f_n(y))\tau_n\right\|$$

$$= \sup_{\phi\in\mathscr{X}^*, \|\phi\|\leq 1} \left|\phi\left(\sum_{n=1}^{\infty} \lambda_n (f_n(x) - f_n(y))\tau_k\right)\right|$$

$$= \sup_{\phi\in\mathscr{X}^*, \|\phi\|\leq 1} \left|\sum_{n=1}^{\infty} \lambda_n (f_n(x) - f_n(y))\phi(\tau_k)\right|$$

$$\leq \sup_{n\in\mathbb{N}} |\lambda_n| \sup_{\phi\in\mathscr{X}^*, \|\phi\|\leq 1} \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |\phi(\tau_n)|^q\right)^{\frac{1}{q}}$$

$$\leq \sup_{n\in\mathbb{N}} |\lambda_n| \sup_{\phi\in\mathscr{X}^*, \|\phi\|\leq 1} \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} d\|\phi\|$$

$$= d \sup_{n\in\mathbb{N}} |\lambda_n| \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \leq bd \sup_{n\in\mathbb{N}} |\lambda_n| d(x, y).$$

Hence

$$\|T\|_{\mathrm{Lip}_0} = \sup_{x,y\in\mathscr{M},\ x\neq y} \frac{\|Tx - Ty\|}{d(x, y)} \leq bd \sup_{n\in\mathbb{N}} |\lambda_n|.$$

□

Corollary 3.1.2. *Let* $\{f_n\}_n$ *in* $\mathrm{Lip}(\mathscr{M}, \mathbb{K})$ *be a Lipschitz p-Bessel sequence for a metric space* $\mathscr{M}$ *with bound b and* $\{\tau_n\}_n$ *in a Banach space* $\mathscr{X}$ *be a Lipschitz q-Bessel* *sequence for* $\operatorname{Lip}_0(\mathscr{X},\mathbb{K})$ *with bound d. If* $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$*, then for fixed* $z \in \mathscr{M}$*, the map*

$$T : \mathscr{M} \ni x \mapsto \sum_{n=1}^{\infty} \lambda_n(\tau_n \otimes (f_n - f(z)))x \in \mathscr{X}$$

is a well-defined Lipschitz operator with Lipschitz number at most $bd\|\{\lambda_n\}_n\|_\infty$.

Proof. Define $g_n := f_n - f(z), \forall n \in \mathbb{N}$. Then for all $x, y \in \mathscr{M}$,

$$\left(\sum_{n=1}^{\infty} |g_n(x) - g_n(y)|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \leq b\,d(x,y).$$

Hence $\{g_n\}_n$ is a Lipschitz p-Bessel sequence for the pointed metric space $(\mathscr{M}, z)$ and we apply Theorem 3.1.1 to $\{g_n\}_n$ which gives the result. □

Definition 3.1.3. *Let* $\{f_n\}_n$ *in* $\operatorname{Lip}_0(\mathscr{M},\mathbb{K})$ *be a Lipschitz p-Bessel sequence for a pointed metric space* $(\mathscr{M}, 0)$ *and* $\{\tau_n\}_n$ *in a Banach space* $\mathscr{X}$ *be a Lipschitz q-Bessel sequence for* $\operatorname{Lip}_0(\mathscr{X},\mathbb{K})$*. Let* $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$*. The Lipschitz operator*

$$M_{\lambda,f,\tau} := \sum_{n=1}^{\infty} \lambda_n(\tau_n \otimes f_n)$$

is called as the ***Lipschitz*** (p,q)***-Bessel multiplier****. The sequence* $\{\lambda_n\}_n$ *is called as* ***symbol*** *for* $M_{\lambda,f,\tau}$.

We easily see that Definition 3.1.3 generalizes Definition 3.2 in (Rahimi and Balazs (2010)). By varying the symbol and fixing other parameters in the multiplier we get map from $\ell^\infty(\mathbb{N})$ to $\operatorname{Lip}_0(\mathscr{M}, \mathscr{X})$. Property of this map for Hilbert space was derived by Balazs (Lemma 7.1 in (Balazs (2007))) and for Banach spaces it is due to Rahimi and Balazs (Proposition 3.3 in (Rahimi and Balazs (2010))). In the next proposition we study it in the context of metric spaces.

Proposition 3.1.4. *Let* $\{f_n\}_n$ *in* $\operatorname{Lip}_0(\mathscr{M},\mathbb{K})$ *be a Lipschitz p-Bessel sequence for* $(\mathscr{M}, 0)$ *with non-zero elements,* $\{\tau_n\}_n$ *in* $\mathscr{X}$ *be a q-Riesz sequence for* $\operatorname{Lip}_0(\mathscr{X},\mathbb{K})$ *and* $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$*. Then the mapping*

$$T : \ell^\infty(\mathbb{N}) \ni \{\lambda_n\}_n \mapsto M_{\lambda,f,\tau} \in \operatorname{Lip}_0(\mathscr{M}, \mathscr{X})$$

is a well-defined injective bounded linear operator.

Proof. From the norm estimate of $M_{\lambda,f,\tau}$, we see that T is a well-defined bounded linear operator. Let $\{\lambda_n\}_n, \{\mu_n\}_n \in \ell^\infty(\mathbb{N})$ be such that $M_{\lambda,f,\tau} = T\{\lambda_n\}_n = T\{\mu_n\}_n = M_{\mu,f,\tau}$.

Then $\sum_{n=1}^\infty \lambda_n f_n(x)\tau_n = M_{\lambda,f,\tau}x = M_{\mu,f,\tau}x = \sum_{n=1}^\infty \mu_n f_n(x)\tau_n$, $\forall x \in \mathscr{M} \Rightarrow \sum_{n=1}^\infty(\lambda_n - \mu_n)f_n(x)\tau_n = 0$, $\forall x \in \mathscr{M}$. Now using Inequality (1.3.1),

$$\begin{aligned} &a\left(\sum_{n=1}^\infty |(\lambda_n-\mu_n)f_n(x)|^q\right)^{\frac{1}{q}} \leq \left\|\sum_{n=1}^\infty (\lambda_n-\mu_n)f_n(x)\tau_n\right\| = 0, \quad \forall x \in \mathscr{M} \\ &\implies (\lambda_n-\mu_n)f_n(x) = 0, \quad \forall n \in \mathbb{N}, \forall x \in \mathscr{M}. \end{aligned}$$

Let $n \in \mathbb{N}$ be fixed. Since $f_n \neq 0$, there exists $x \in \mathscr{M}$ such that $f_n(x) \neq 0$. Therefore we get $\lambda_n - \mu_n = 0$. By varying $n \in \mathbb{N}$ we arrive at $\lambda_n = \mu_n$, $\forall n \in \mathbb{N}$. Hence T is injective. □

3.2 CONTINUITY PROPERTIES OF MULTIPLIERS

In Proposition 1.4.3, it was obtained that whenever the symbol is in $c_0(\mathbb{N})$, then the multiplier is compact. Using the notion of Lipschitz compact operator (Definition 1.5.9), we derive non linear analogue of Proposition 1.4.3.

Proposition 3.2.1. *Let $\{f_n\}_n$ in $\operatorname{Lip}_0(\mathscr{M},\mathbb{K})$ be a Lipschitz p-Bessel sequence for $(\mathscr{M},0)$ with bound b and $\{\tau_n\}_n$ in $\mathscr{X}$ be a Lipschitz q-Bessel sequence for $\operatorname{Lip}_0(\mathscr{X},\mathbb{K})$ with bound d. If $\{\lambda_n\}_n \in c_0(\mathbb{N})$, then $M_{\lambda,f,\tau}$ is a Lipschitz compact operator.*

Proof. For each $m \in \mathbb{N}$, define $M_{\lambda_m,f,\tau} := \sum_{n=1}^m \lambda_n(\tau_n \otimes f_n)$. Then $M_{\lambda_m,f,\tau}$ is a Lipschitz finite rank operator (Theorem 1.5.14). Now

$$\begin{aligned} \|M_{\lambda_m,f,\tau} - M_{\lambda,f,\tau}\|_{\operatorname{Lip}_0} &= \sup_{x,y\in\mathscr{M},\ x\neq y} \frac{\|(M_{\lambda_m,f,\tau}-M_{\lambda,f,\tau})x-(M_{\lambda_m,f,\tau}-M_{\lambda,f,\tau})y\|}{d(x,y)} \\ &= \sup_{x,y\in\mathscr{M},\ x\neq y} \frac{\left\|\sum_{n=m+1}^\infty \lambda_n f_n(x)\tau_n - \sum_{n=m+1}^\infty \lambda_n f_n(y)\tau_n\right\|}{d(x,y)} \\ &= \sup_{x,y\in\mathscr{M},\ x\neq y} \frac{\left\|\sum_{n=m+1}^\infty \lambda_n (f_n(x)-f_n(y))\tau_n\right\|}{d(x,y)} \\ &\leq bd \sup_{m+1\leq n<\infty} |\lambda_n| \to 0 \text{ as } m\to\infty. \end{aligned}$$

Hence $M_{\lambda,f,\tau}$ is the limit of a sequence of Lipschitz finite rank operators $\{M_{\lambda_m,f,\tau}\}_{m=1}^\infty$ with respect to the Lipschitz norm. Thus $M_{\lambda,f,\tau}$ is Lipschitz approximable and from Theorem 1.5.16 it follows that $M_{\lambda,f,\tau}$ is Lipschitz compact. □

We now study the properties of multiplier by changing its parameters. Following result extends Theorem 1.4.4.

Theorem 3.2.2. *Let $\{f_n\}_n$ in $\mathrm{Lip}_0(\mathscr{M},\mathbb{K})$ be a Lipschitz p-Bessel sequence for $\mathscr{M}$ with bound b and $\{\tau_n\}_n$ in $\mathscr{X}$ be a Lipschitz q-Bessel sequence for $\mathrm{Lip}_0(\mathscr{X},\mathbb{K})$ with bound d and $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$. Let $k \in \mathbb{N}$ and let $\lambda^{(k)} = \{\lambda_1^{(k)}, \lambda_2^{(k)}, \dots\}$, $\lambda = \{\lambda_1, \lambda_2, \dots\}$, $\tau^{(k)} = \{\tau_1^{(k)}, \tau_2^{(k)}, \dots\}$, $\tau_n^k \in \mathscr{X}$, $\tau = \{\tau_1, \tau_2, \dots\}$. Assume that for each k, $\lambda^{(k)} \in \ell^\infty(\mathbb{N})$ and $\tau^{(k)}$ is a pointed Lipschitz q-Bessel sequence for $\mathrm{Lip}_0(\mathscr{X},\mathbb{K})$.*

(i) If $\lambda^{(k)} \to \lambda$ as $k \to \infty$ in p-norm, then

$$\|M_{\lambda^{(k)},f,\tau} - M_{\lambda,f,\tau}\|_{\mathrm{Lip}_0} \to 0 \text{ as } k \to \infty.$$

(ii) If $\{\lambda_n\}_n \in \ell^p(\mathbb{N})$ and $\sum_{n=1}^\infty \|\tau_n^{(k)} - \tau_n\|^q \to 0$ as $k \to \infty$, then

$$\|M_{\lambda,f,\tau^{(k)}} - M_{\lambda,f,\tau}\|_{\mathrm{Lip}_0} \to 0 \text{ as } k \to \infty.$$

Proof. (i) Using Theorem 3.1.1,

$$\begin{aligned}
&\|M_{\lambda^{(k)},f,\tau} - M_{\lambda,f,\tau}\|_{\mathrm{Lip}_0} \\
&= \sup_{x,y\in\mathscr{M},\, x\neq y} \frac{\|(M_{\lambda^{(k)},f,\tau} - M_{\lambda,f,\tau})x - (M_{\lambda^{(k)},f,\tau} - M_{\lambda,f,\tau})y\|}{d(x,y)} \\
&= \sup_{x,y\in\mathscr{M},\, x\neq y} \frac{\left\|\sum_{n=1}^\infty (\lambda_n^{(k)} - \lambda_n) f_n(x)\tau_n - \sum_{n=1}^\infty (\lambda_n^{(k)} - \lambda_n) f_n(y)\tau_n\right\|}{d(x,y)} \\
&= \sup_{x,y\in\mathscr{M},\, x\neq y} \frac{\left\|\sum_{n=1}^\infty (\lambda_n^{(k)} - \lambda_n)(f_n(x) - f_n(y))\tau_n\right\|}{d(x,y)} \\
&\leq bd \sup_{n\in\mathbb{N}} |\lambda_n^{(k)} - \lambda_n| = bd\|\{\lambda_n^{(k)} - \lambda_n\}_n\|_\infty \\
&\leq bd\|\{\lambda_n^{(k)} - \lambda_n\}_n\|_p \to 0 \text{ as } k \to \infty.
\end{aligned}$$

(ii) Using Holder's inequality,

$$\begin{aligned}
&\|M_{\lambda,f,\tau^{(k)}} - M_{\lambda,f,\tau}\|_{\mathrm{Lip}_0} \\
&= \sup_{x,y\in\mathscr{M},\, x\neq y} \frac{\|(M_{\lambda,f,\tau^{(k)}} - M_{\lambda,f,\tau})x - (M_{\lambda,f,\tau^{(k)}} - M_{\lambda,f,\tau})y\|}{d(x,y)} \\
&= \sup_{x,y\in\mathscr{M},\, x\neq y} \frac{\left\|\sum_{n=1}^\infty \lambda_n f_n(x)(\tau_n^{(k)} - \tau_n) - \sum_{n=1}^\infty \lambda_n f_n(y)(\tau_n^{(k)} - \tau_n)\right\|}{d(x,y)} \\
&= \sup_{x,y\in\mathscr{M},\, x\neq y} \frac{\left\|\sum_{n=1}^\infty \lambda_n (f_n(x) - f_n(y))(\tau_n^{(k)} - \tau_n)\right\|}{d(x,y)}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{x,y\in\mathscr{M},\, x\neq y} \; \sup_{\phi\in\mathscr{X}^*,\|\phi\|\leq 1} \frac{\left|\sum_{n=1}^{\infty} \lambda_n (f_n(x)-f_n(y))\phi(\tau_n^{(k)}-\tau_n)\right|}{d(x,y)} \\
&\leq \sup_{x,y\in\mathscr{M},\, x\neq y} \; \sup_{\phi\in\mathscr{X}^*,\|\phi\|\leq 1} \frac{\left(\sum_{n=1}^{\infty} |\lambda_n (f_n(x)-f_n(y))|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |\phi(\tau_n^{(k)}-\tau_n)|^q\right)^{\frac{1}{q}}}{d(x,y)} \\
&\leq \sup_{x,y\in\mathscr{M},\, x\neq y} \; \sup_{\phi\in\mathscr{X}^*,\|\phi\|\leq 1} \\
&\quad \left\{ \frac{\left(\sum_{n=1}^{\infty} |\lambda_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |f_n(x)-f_n(y)|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |\phi(\tau_n^{(k)}-\tau_n)|^q\right)^{\frac{1}{q}}}{d(x,y)} \right\} \\
&\leq b\|\{\lambda_n\}_n\|_p \left(\sum_{n=1}^{\infty} \|\tau_n^{(k)}-\tau_n\|^q\right)^{\frac{1}{q}} \to 0 \text{ as } k\to\infty.
\end{aligned}
$$

□

CHAPTER 4

p-APPROXIMATE SCHAUDER FRAMES FOR BANACH SPACES

4.1 p-APPROXIMATE SCHAUDER FRAMES

Let $\mathcal{X}$ be a separable Banach space and $\mathcal{X}^*$ be its dual. Equation (1.2.4) motivated Casazza, Dilworth, Odell, Schlumprecht, and Zsak, to define the notion of Schauder frame for $\mathcal{X}$ in 2008.

Definition 4.1.1. *(Casazza et al. (2008a)) Let $\{\tau_n\}_n$ be a sequence in $\mathcal{X}$ and $\{f_n\}_n$ be a sequence in $\mathcal{X}^*$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be a* ***Schauder frame*** *for $\mathcal{X}$ if*

$$x = \sum_{n=1}^{\infty} f_n(x)\tau_n, \quad \forall x \in \mathcal{X}. \tag{4.1.1}$$

Definition 4.1.1 was generalized for $\mathbb{R}^n$ by Thomas in her Master's thesis and later to Banach spaces by Freeman, Odell, Schlumprecht, and Zsak.

Definition 4.1.2. *(Freeman et al. (2014); Thomas (2012)) Let $\{\tau_n\}_n$ be a sequence in $\mathcal{X}$ and $\{f_n\}_n$ be a sequence in $\mathcal{X}^*$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be an* ***approximate Schauder frame*** *(ASF) for $\mathcal{X}$ if*

$$\textit{(Frame operator)} \quad S_{f,\tau} : \mathcal{X} \ni x \mapsto S_{f,\tau}x := \sum_{n=1}^{\infty} f_n(x)\tau_n \in \mathcal{X} \tag{4.1.2}$$

is a well-defined bounded linear, invertible operator.

Note that whenever $S_{f,\tau} = I_{\mathcal{X}}$, the identity operator on $\mathcal{X}$, Definition 4.1.2 reduces to Definition 4.1.1. Since $S_{f,\tau}$ is invertible, it follows that there are $a, b > 0$ such that

$$a\|x\| \leq \left\|\sum_{n=1}^{\infty} f_n(x)\tau_n\right\| \leq b\|x\|, \quad \forall x \in \mathcal{X}.$$

We call a as **lower ASF bound** and b as **upper ASF bound**. Supremum (resp. infimum) of the set of all lower (resp. upper) ASF bounds is called **optimal lower** (resp. **optimal upper**) ASF bound. From the theory of bounded linear operators between Banach spaces, one sees that optimal lower frame bound is $\|S_{f,\tau}^{-1}\|^{-1}$ and optimal upper frame bound is $\|S_{f,\tau}\|$. Advantage of ASF over Schauder frame is that it is more easier to get the operator in (4.1.2) as invertible than obtaining Equation (4.1.1).

Example 4.1.3. *(Freeman et al. (2014)) Let $2 < p < \infty$ and $\{\lambda_n\}_n$ be an unbounded sequence of scalars. For $a \in \mathbb{R}$, define*

$$T_a : \mathscr{L}^p(\mathbb{R}) \ni f \mapsto T_a f \in \mathscr{L}^p(\mathbb{R}); \quad T_a f : \mathbb{R} \ni x \mapsto (T_a f)(x) := f(x-a) \in \mathbb{C}.$$

Then there exist $\phi \in \mathscr{L}^p(\mathbb{R})$ and a sequence $\{f_n\}_n$, $f_n \in (\mathscr{L}^p(\mathbb{R}))^*$, $\forall n \in \mathbb{N}$ such that $(\{f_n\}_n, \{T_{\lambda_n}\phi\}_n)$ is an ASF for $\mathscr{L}^p(\mathbb{R})$.

Definition 4.1.4. *An ASF $(\{f_n\}_n, \{\tau_n\}_n)$ for $\mathscr{X}$ is said to be a* ***p-approximate Schauder frame*** *(p-ASF), $p \in [1,\infty)$ if both the maps*

$$\textit{(Analysis operator)} \quad \theta_f : \mathscr{X} \ni x \mapsto \theta_f x := \{f_n(x)\}_n \in \ell^p(\mathbb{N}) \text{ and} \tag{4.1.3}$$

$$\textit{(Synthesis operator)} \quad \theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \theta_\tau \{a_n\}_n := \sum_{n=1}^{\infty} a_n \tau_n \in \mathscr{X} \tag{4.1.4}$$

are well-defined bounded linear operators. A Schauder frame which is a p-ASF is called as a ***simple p-ASF*** *or* ***Parseval p-ASF****.*

It can be easily observed that a p-approximate Schauder frame is an approximate Schauder frame and a Schauder frame is an approximate Schauder frame. We now give an example to show that the set of all p-approximate Schauder frames is strictly smaller than the set of all approximate Schauder frames. Let $\mathscr{X} = \mathbb{K}$. Define $\tau_n := \frac{1}{n^2}$, $f_n(x) = x, \forall x \in \mathbb{K}, \forall n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} f_n(x)\tau_n = \frac{\pi^2}{6}x, \forall x \in \mathbb{K}$. Therefore $(\{f_n\}_n, \{\tau_n\}_n)$ is an approximate Schauder frame for $\mathscr{X}$. Let $x \in \mathbb{K}$ be non zero. Then for every $p \in [1,\infty)$,

$$\sum_{n=1}^{m} |f_n(x)|^p = m|x|^p \to \infty \quad \text{as} \quad m \to \infty.$$

Thus $\{f_n(x)\}_n \notin \ell^p(\mathbb{N})$ for any $p \in [1,\infty)$ and hence $(\{f_n\}_n, \{\tau_n\}_n)$ is not a p-ASF for any $p \in [1,\infty)$. We next note that there is a bijection between the set of approximate Schauder frames and the set of all Schauder frames (Lemma 3.1 in (Freeman et al. (2014))). We observe that, in terms of inequalities, (4.1.3) and (4.1.4) say that there exist $c,d>0$, such that

$$\left(\sum_{n=1}^{\infty}|f_n(x)|^p\right)^{\frac{1}{p}} \leq c\|x\|, \quad \forall x \in \mathscr{X} \text{ and} \tag{4.1.5}$$

$$\left\|\sum_{n=1}^{\infty}a_n\tau_n\right\| \leq d\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{\frac{1}{p}}, \quad \forall \{a_n\}_n \in \ell^p(\mathbb{N}). \tag{4.1.6}$$

We now give various examples of p-ASFs.

Example 4.1.5. *Let $p \in [1,\infty)$ and $U : \mathscr{X} \to \ell^p(\mathbb{N})$, $V : \ell^p(\mathbb{N}) \to \mathscr{X}$ be bounded linear operators such that VU is bounded invertible. Let $\{e_n\}_n$ denote the standard Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Define*

$$f_n := \zeta_n U, \quad \tau_n := Ve_n, \quad \forall n \in \mathbb{N}.$$

Then $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-ASF for $\mathscr{X}$. In particular, if $U : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ is bounded invertible, then $(\{\zeta_n U\}_n, \{U^{-1}e_n\}_n)$ is a p-ASF for $\ell^p(\mathbb{N})$.

Example 4.1.6. *Let $p \in [1,\infty)$ and $\{\tau_n\}_{n=1}^m$ be a basis for a finite dimensional Banach space $\mathscr{X}$. Choose any basis $\{f_n\}_{n=1}^m$ for $\mathscr{X}^*$. We claim that $(\{f_n\}_{n=1}^m, \{\tau_n\}_{n=1}^m)$ is a p-ASF for $\mathscr{X}$. To prove this, since $\mathscr{X}$ is finite dimensional, it suffices to prove that the map $\mathscr{X} \ni x \mapsto \sum_{n=1}^m f_n(x)\tau_n \in \mathscr{X}$ is injective. Let $x \in \mathscr{X}$ be such that $\sum_{n=1}^m f_n(x)\tau_n = 0$. Since $\{\tau_n\}_{n=1}^m$ is a basis for $\mathscr{X}$, we then have $f_1(x) = \cdots = f_n(x) = 0$. We then have $f(x) = 0, \forall f \in \mathscr{X}^*$. Hahn-Banach theorem now says that $x = 0$. Hence the claim holds and consequently $(\{f_n\}_{n=1}^m, \{\tau_n\}_{n=1}^m)$ is a p-ASF for $\mathscr{X}$.*

Example 4.1.7. *Recall that a spanning set is a frame for a finite dimensional Hilbert space (Han et al. (2007)). We now generalize this for p-ASFs. Let $p \in [1,\infty)$, $\mathscr{X}$ be a finite dimensional Banach space and $\{\tau_n\}_{n=1}^m$ be a spanning set for $\mathscr{X}$. We claim that there exists a collection $\{f_n\}_{n=1}^m$ in $\mathscr{X}^*$ such that $(\{f_n\}_{n=1}^m, \{\tau_n\}_{n=1}^m)$ is a p-ASF for $\mathscr{X}$. Since $\{\tau_n\}_{n=1}^m$ spans $\mathscr{X}$, there exists a basis in the collection $\{\tau_n\}_{n=1}^m$. By rearranging, if necessary, we may assume that $\{\tau_n\}_{n=1}^r$ is a basis for $\mathscr{X}$. Let $\{f_n\}_{n=1}^r$ be the dual basis for $\{\tau_n\}_{n=1}^r$. Choose linear operators $U,V : \mathscr{X} \to \mathscr{X}$ such that VU is injective or surjective. If we now set $f_{r+1} = \cdots = f_n = 0$, it then follows that $(\{f_nU\}_{n=1}^m, \{V\tau_n\}_{n=1}^m)$ is a p-ASF for $\mathscr{X}$.*

Example 4.1.8. *Let $\mathscr{X}$ be a Banach space which admits a Schauder basis $\{\omega_n\}_n$ and let $\{g_n\}_n$ be the coordinate functionals associated with $\{e_n\}_n$. Let $U,V : \mathscr{X} \to \mathscr{X}$ be* *bounded linear operators such that VU is invertible. Define*

$$f_n := g_n U, \quad \tau_n := V\omega_n, \quad \forall n \in \mathbb{N}.$$

Then $(\{f_n\}_n, \{\tau_n\}_n)$ is an approximate Schauder frame for $\mathscr{X}$. If $VU = I_{\mathscr{X}}$, then $(\{f_n\}_n, \{\tau_n\}_n)$ is a Schauder frame for $\mathscr{X}$.

Example 4.1.9. *Let $p \in [1,\infty)$ and $U : \mathscr{X} \to \ell^p(\mathbb{N})$, $V : \ell^p(\mathbb{N}) \to \mathscr{X}$ be bounded linear operators such that VU is invertible. Let $\{e_n\}_n$ denote the canonical Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Define*

$$f_n := \zeta_n U, \quad \tau_n := Ve_n, \quad \forall n \in \mathbb{N}.$$

Then $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-ASF for $\mathscr{X}$.

Now we have Banach space analogous of Theorem 1.2.36.

Theorem 4.1.10. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Then*

(i) We have

$$x = \sum_{n=1}^{\infty} (f_n S_{f,\tau}^{-1})(x)\tau_n = \sum_{n=1}^{\infty} f_n(x) S_{f,\tau}^{-1}\tau_n, \quad \forall x \in \mathscr{X}. \tag{4.1.7}$$

(ii) $(\{f_n S_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ is a p-ASF for $\mathscr{X}$.

(iii) The analysis operator $\theta_f : \mathscr{X} \ni x \mapsto \{f_n(x)\}_n \in \ell^p(\mathbb{N})$ is injective.

(iv) The synthesis operator $\theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \sum_{n=1}^{\infty} a_n\tau_n \in \mathscr{X}$ is surjective.

(v) Frame operator splits as $S_{f,\tau} = \theta_\tau\theta_f$.

(vi) $P_{f,\tau} := \theta_f S_{f,\tau}^{-1}\theta_\tau : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ is a projection onto $\theta_f(\mathscr{X})$.

Proof. First follows from the continuity and linearity of $S_{f,\tau}^{-1}$. Because $S_{f,\tau}$ is invertible, we have (ii). Again invertibility of $S_{f,\tau}$ makes θ_f injective and θ_τ surjective. (v) and (vi) are routine calculations. □

Now we can derive a generalization of Theorem 1.2.39 for Banach spaces.

Theorem 4.1.11. *Let $\{e_n\}_n$ denote the standard Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-ASF for $\mathscr{X}$ if and only if*

$$f_n = \zeta_n U, \quad \tau_n = Ve_n, \quad \forall n \in \mathbb{N},$$

where $U : \mathscr{X} \to \ell^p(\mathbb{N})$, $V : \ell^p(\mathbb{N}) \to \mathscr{X}$ are bounded linear operators such that VU is bounded invertible.

Proof. ($\Leftarrow$) Clearly θ_f and θ_τ are bounded linear operators. Now let $x \in \mathscr{X}$. Then

$$S_{f,\tau}x = \sum_{n=1}^{\infty} f_n(x)\tau_n = \sum_{n=1}^{\infty} \zeta_n(Ux)Ve_n = V\left(\sum_{n=1}^{\infty} \zeta_n(Ux)e_n\right) = VUx. \tag{4.1.8}$$

Hence $S_{f,\tau}$ is bounded invertible.
($\Rightarrow$) Define $U := \theta_f$, $V := \theta_\tau$. Then $\zeta_n Ux = \zeta_n \theta_f x = \zeta_n(\{f_k(x)\}_k) = f_n(x)$, $\forall x \in \mathscr{X}$, $Ve_n = \theta_\tau e_n = \tau_n$, $\forall n \in \mathbb{N}$ and $VU = \theta_\tau\theta_f = S_{f,\tau}$ which is bounded invertible. □

Note that Theorem 4.1.11 generalizes Theorem 1.2.39. In fact, in the case of Hilbert spaces, Theorem 4.1.11 reads as "A sequence $\{\tau_n\}_n$ in $\mathscr{H}$ is a frame for $\mathscr{H}$ if and only if there exists a bounded linear operator $T : \ell^2(\mathbb{N}) \to \mathscr{H}$ such that $Te_n = \tau_n$, for all $n \in \mathbb{N}$ and TT^* is invertible". Now we know that TT^* is invertible if and only if T is surjective.
Since every separable Hilbert space admits an orthonormal basis, the existence of orthonormal basis in Theorem 1.2.40 is automatic. On the other hand, Enflo showed that there are separable Banach spaces which do not have Schauder basis (Enflo (1973); James (1982)). Thus to obtain analogous of Theorem 1.2.40 for Banach spaces, we need to impose condition on $\mathscr{X}$.

Theorem 4.1.12. *Assume that $\mathscr{X}$ admits a Schauder basis $\{\omega_n\}_n$. Let $\{g_n\}_n$ denote the coordinate functionals associated with $\{\omega_n\}_n$. Assume that*

$$\{g_n(x)\}_n \in \ell^p(\mathbb{N}), \quad \forall x \in \mathscr{X}. \tag{4.1.9}$$

Then a pair $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-ASF for $\mathscr{X}$ if and only if

$$f_n = g_nU, \quad \tau_n = V\omega_n, \quad \forall n \in \mathbb{N},$$

where $U,V : \mathscr{X} \to \mathscr{X}$ are bounded linear operators such that VU is bounded invertible.

Proof. ($\Leftarrow$) This is similar to the calculation done in (4.1.8).
($\Rightarrow$) Let T be the map defined by

$$T : \mathscr{X} \ni \sum_{n=1}^{\infty} a_n\omega_n \mapsto \sum_{n=1}^{\infty} a_ne_n \in \ell^p(\mathbb{N}).$$

Assumption (4.1.9) then says that T is a bounded invertible operator with inverse $T^{-1} : \ell^p(\mathbb{N}) \ni \sum_{n=1}^{\infty} b_ne_n \mapsto \sum_{n=1}^{\infty} b_n\omega_n \in \mathscr{X}$. Define $U := T^{-1}\theta_f$ and $V := \theta_\tau T$. Then U,V

are bounded such that $VU = (\theta_\tau T)(T^{-1}\theta_f) = \theta_\tau\theta_f = S_{f,\tau}$ is invertible and for $x \in \mathcal{X}$ we have

$$(g_nU)(x) = g_n(T^{-1}\theta_f x) = g_n(T^{-1}(\{f_k(x)\}_k)) = g_n\left(\sum_{k=1}^{\infty} f_k(x)T^{-1}e_k\right)$$
$$= g_n\left(\sum_{k=1}^{\infty} f_k(x)\omega_k\right) = \sum_{k=1}^{\infty} f_k(x)g_n(\omega_k) = f_n(x), \quad \forall x \in \mathcal{X}$$

and $V\omega_n = \theta_\tau T\omega_n = \theta_\tau e_n = \tau_n, \forall n \in \mathbb{N}$. □

4.2 DUAL FRAMES FOR p-APPROXIMATE SCHAUDER FRAMES

Equation (4.1.7) motivates us to define the notion of dual frame as follows.

Definition 4.2.1. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. A p-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{X}$ is a **dual p-ASF** for $(\{f_n\}_n, \{\tau_n\}_n)$ if*

$$x = \sum_{n=1}^{\infty} g_n(x)\tau_n = \sum_{n=1}^{\infty} f_n(x)\omega_n, \quad \forall x \in \mathcal{X}.$$

Note that dual frames always exist. In fact, the Equation (4.1.7) shows that the frame $(\{f_nS_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ is a dual for $(\{f_n\}_n, \{\tau_n\}_n)$. We call the frame $(\{f_nS_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ as the **canonical dual** for $(\{f_n\}_n, \{\tau_n\}_n)$. With this notion, the following theorem follows easily.

Theorem 4.2.2. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$ with frame bounds a and b. Then*

(i) The canonical dual p-ASF for the canonical dual p-ASF for $(\{f_n\}_n, \{\tau_n\}_n)$ is itself.

(ii) $\frac{1}{b}, \frac{1}{a}$ are frame bounds for the canonical dual for $(\{f_n\}_n, \{\tau_n\}_n)$.

(iii) If a, b are optimal frame bounds for $(\{f_n\}_n, \{\tau_n\}_n)$, then $\frac{1}{b}, \frac{1}{a}$ are optimal frame bounds for its canonical dual.

One can naturally ask when a p-ASF has unique dual? An affirmative answer is in the following result.

Proposition 4.2.3. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. If $\{\tau_n\}_n$ is a Schauder basis for $\mathcal{X}$ and $f_k(\tau_n) = \delta_{k,n}, \forall k, n \in \mathbb{N}$, then $(\{f_n\}_n, \{\tau_n\}_n)$ has unique dual.*

Proof. Let $(\{g_n\}_n, \{\omega_n\}_n)$ and $(\{u_n\}_n, \{\rho_n\}_n)$ be two dual p-ASFs for $(\{f_n\}_n, \{\tau_n\}_n)$. Then

$$\sum_{n=1}^{\infty}(g_n(x) - u_n(x))\tau_n = 0 = \sum_{n=1}^{\infty} f_n(x)(\omega_n - \rho_n), \quad \forall x \in \mathscr{X}.$$

First equality gives $g_n = u_n, \forall n \in \mathbb{N}$ and evaluating second equality at a fixed τ_k gives $\omega_k = \rho_k$. Since k was arbitrary, proposition follows. □

We now characterize dual frames by using analysis and synthesis operators.

Proposition 4.2.4. *For two p-ASFs $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathscr{X}$, the following are equivalent.*

(i) $(\{g_n\}_n, \{\omega_n\}_n)$ *is a dual for* $(\{f_n\}_n, \{\tau_n\}_n)$.

(ii) $\theta_\tau \theta_g = \theta_\omega \theta_f = I_{\mathscr{X}}$.

Like Lemmas 1.2.42, 1.2.43 and Theorem 1.2.44, we now characterize dual frames using standard Schauder basis for $\ell^p(\mathbb{N})$.

Lemma 4.2.5. *Let $\{e_n\}_n$ denote the standard Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Then a p-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathscr{X}$ is a dual for $(\{f_n\}_n, \{\tau_n\}_n)$ if and only if*

$$g_n = \zeta_n U, \quad \omega_n = V e_n, \quad \forall n \in \mathbb{N},$$

where $U : \mathscr{X} \to \ell^p(\mathbb{N})$ is a bounded right-inverse of θ_τ, and $V : \ell^p(\mathbb{N}) \to \mathscr{X}$ is a bounded left-inverse of θ_f such that VU is bounded invertible.

Proof. $(\Leftarrow)$ From the 'if' part of proof of Theorem 4.1.11, we get that $(\{g_n\}_n, \{\omega_n\}_n)$ is a p-ASF for $\mathscr{X}$. We have to check for duality of $(\{g_n\}_n, \{\omega_n\}_n)$. Also, we have $\theta_\tau \theta_g = \theta_\tau U = I_{\mathscr{X}}$, $\theta_\omega \theta_f = V\theta_f = I_{\mathscr{X}}$.
$(\Rightarrow)$ Let $(\{g_n\}_n, \{\omega_n\}_n)$ be a dual p-ASF for $(\{f_n\}_n, \{\tau_n\}_n)$. Then $\theta_\tau \theta_g = I_{\mathscr{X}}$, $\theta_\omega \theta_f = I_{\mathscr{X}}$. Define $U := \theta_g, V := \theta_\omega$. Then $U : \mathscr{X} \to \ell^p(\mathbb{N})$ is a bounded right-inverse of θ_τ, and $V : \ell^p(\mathbb{N}) \to \mathscr{X}$ is a bounded left-inverse of θ_f such that the operator $VU = \theta_\omega \theta_g = S_{g,\omega}$ is invertible. Further,

$$(\zeta_n U)x = \zeta_n \left(\sum_{k=1}^{\infty} g_k(x) e_k\right) = \sum_{k=1}^{\infty} g_k(x)\zeta_n(e_k) = g_n(x), \quad \forall x \in \mathscr{X}$$

and $Ve_n = \theta_\omega e_n = \omega_n, \forall n \in \mathbb{N}$. □

Lemma 4.2.6. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Then*

(i) $R : \mathscr{X} \to \ell^p(\mathbb{N})$ is a bounded right-inverse of θ_τ if and only if

$$R = \theta_f S_{f,\tau}^{-1} + (I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau)U$$

where $U : \mathscr{X} \to \ell^p(\mathbb{N})$ is a bounded linear operator.

(ii) $L : \ell^p(\mathbb{N}) \to \mathscr{X}$ is a bounded left-inverse of θ_f if and only if

$$L = S_{f,\tau}^{-1}\theta_\tau + V(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau),$$

where $V : \ell^p(\mathbb{N}) \to \mathscr{X}$ is a bounded linear operator.

Proof. (i) $(\Leftarrow)$ $\theta_\tau(\theta_f S_{f,\tau}^{-1} + (I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau)U) = I_{\mathscr{X}} + \theta_\tau U - I_{\mathscr{X}}\theta_\tau U = I_{\mathscr{X}}$. Therefore $\theta_f S_{f,\tau}^{-1} + (I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau)U$ is a bounded right-inverse of θ_τ.

$(\Rightarrow)$ Define $U := R$. Then $\theta_f S_{f,\tau}^{-1} + (I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau)U = \theta_f S_{f,\tau}^{-1} + (I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau)R = \theta_f S_{f,\tau}^{-1} + R - \theta_f S_{f,\tau}^{-1} = R$.

(ii) $(\Leftarrow)$ $(S_{f,\tau}^{-1}\theta_\tau + V(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau))\theta_f = I_{\mathscr{X}} + V\theta_f - V\theta_f I_{\mathscr{X}} = I_{\mathscr{X}}$. Therefore $S_{f,\tau}^{-1}\theta_\tau + V(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau)$ is a bounded left-inverse of θ_f.

$(\Rightarrow)$ Define $V := L$. Then $S_{f,\tau}^{-1}\theta_\tau + V(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau) = S_{f,\tau}^{-1}\theta_\tau + L(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau) = S_{f,\tau}^{-1}\theta_\tau + L - S_{f,\tau}^{-1}\theta_\tau = L$.

□

Theorem 4.2.7. *Let $\{e_n\}_n$ denote the standard Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Then a p-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathscr{X}$ is a dual for $(\{f_n\}_n, \{\tau_n\}_n)$ if and only if*

$$g_n = f_n S_{f,\tau}^{-1} + \zeta_n U - f_n S_{f,\tau}^{-1}\theta_\tau U,$$
$$\omega_n = S_{f,\tau}^{-1}\tau_n + V e_n - V\theta_f S_{f,\tau}^{-1}\tau_n, \quad \forall n \in \mathbb{N}$$

such that the operator

$$S_{f,\tau}^{-1} + VU - V\theta_f S_{f,\tau}^{-1}\theta_\tau U$$

is bounded invertible, where $U : \mathscr{X} \to \ell^p(\mathbb{N})$ and $V : \ell^p(\mathbb{N}) \to \mathscr{X}$ are bounded linear operators.

Proof. Lemmas 4.2.5 and 4.2.6 give the characterization of dual frame as

$$g_n = \zeta_n\theta_f S_{f,\tau}^{-1} + \zeta_n U - \zeta_n\theta_f S_{f,\tau}^{-1}\theta_\tau U = f_n S_{f,\tau}^{-1} + \zeta_n U - f_n S_{f,\tau}^{-1}\theta_\tau U,$$

$$\omega_n = S_{f,\tau}^{-1}\theta_\tau e_n + Ve_n - V\theta_f S_{f,\tau}^{-1}\theta_\tau e_n = S_{f,\tau}^{-1}\tau_n + Ve_n - V\theta_f S_{f,\tau}^{-1}\tau_n, \quad \forall n \in \mathbb{N}$$

such that the operator

$$(S_{f,\tau}^{-1}\theta_\tau + V(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau))(\theta_f S_{f,\tau}^{-1} + (I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau)U)$$

is bounded invertible, where $U : \mathscr{X} \to \ell^p(\mathbb{N})$ and $V : \ell^p(\mathbb{N}) \to \mathscr{X}$ are bounded linear operators. By a direct expansion and simplification we get

$$\begin{aligned}
&(S_{f,\tau}^{-1}\theta_\tau + V(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau))(\theta_f S_{f,\tau}^{-1} + (I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau)U) \\
&\quad = S_{f,\tau}^{-1} + VU - V\theta_f S_{f,\tau}^{-1}\theta_\tau U.
\end{aligned}$$

□

We know that a bounded linear operator from $\ell^2(\mathbb{N})$ to $\mathscr{H}$ is given by a Bessel sequence (Theorem 1.2.58). Thus, for Hilbert spaces, Theorem 4.2.7 becomes Theorem 1.2.44.

4.3 SIMILARITY FOR p-APPROXIMATE SCHAUDER FRAMES

We define Definition 1.2.45 to Banach spaces as follows.

Definition 4.3.1. *Two p-ASFs $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathscr{X}$ are said to be **similar** or **equivalent** if there exist bounded invertible operators $T_{f,g}, T_{\tau,\omega} : \mathscr{X} \to \mathscr{X}$ such that*

$$g_n = f_n T_{f,g}, \quad \omega_n = T_{\tau,\omega}\tau_n, \quad \forall n \in \mathbb{N}.$$

Since the operators giving similarity are bounded invertible, the notion of similarity is symmetric. Further, a routine calculation shows that it is an equivalence relation (hence the name equivalent) on the set

$$\{(\{f_n\}_n, \{\tau_n\}_n) : (\{f_n\}_n, \{\tau_n\}_n) \text{ is a p-ASF for } \mathscr{X}\}.$$

We now characterize similarity using just operators. In the sequel, given a p-ASF $(\{f_n\}_n, \{\tau_n\}_n)$, we set $P_{f,\tau} := \theta_f S_{f,\tau}^{-1}\theta_\tau$.

Theorem 4.3.2. *For two p-ASFs $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathscr{X}$, the following are equivalent.*

(i) $g_n = f_nT_{f,g}, \omega_n = T_{\tau,\omega}\tau_n, \forall n \in \mathbb{N}$, *for some bounded invertible operators* $T_{f,g}, T_{\tau,\omega} : \mathscr{X} \to \mathscr{X}$.

(ii) $\theta_g = \theta_f T_{f,g}, \theta_\omega = T_{\tau,\omega}\theta_\tau$, *for some bounded invertible operators* $T_{f,g}, T_{\tau,\omega} : \mathscr{X} \to \mathscr{X}$.

(iii) $P_{g,\omega} = P_{f,\tau}$.

If one of the above conditions is satisfied, then invertible operators in (i) *and* (ii) *are unique and are given by* $T_{f,g} = S_{f,\tau}^{-1}\theta_\tau\theta_g, T_{\tau,\omega} = \theta_\omega\theta_f S_{f,\tau}^{-1}$. *In the case that* $(\{f_n\}_n, \{\tau_n\}_n)$ *is a simple p-ASF, then* $(\{g_n\}_n, \{\omega_n\}_n)$ *is a simple p-ASF if and only if* $T_{\tau,\omega}T_{f,g} = I_{\mathscr{X}}$ *if and only if* $T_{f,g}T_{\tau,\omega} = I_{\mathscr{X}}$.

Proof. (i) $\Rightarrow$ (ii) $\theta_g x = \{g_n(x)\}_n = \{f_n(T_{f,g}x)\}_n = \theta_f(T_{f,g}x), \forall x \in \mathscr{X}$, $\theta_\omega(\{a_n\}_n) = \sum_{n=1}^\infty a_n\omega_n = \sum_{n=1}^\infty a_nT_{\tau,\omega}\tau_n = T_{\tau,\omega}(\theta_\tau(\{a_n\}_n)), \forall \{a_n\}_n \in \ell^p(\mathbb{N})$.
(ii) $\Rightarrow$ (iii) $S_{g,\omega} = \theta_\omega\theta_g = T_{\tau,\omega}\theta_\tau\theta_f T_{f,g} = T_{\tau,\omega}S_{f,\tau}T_{f,g}$ and

$$P_{g,\omega} = \theta_g S_{g,\omega}^{-1}\theta_\omega = (\theta_f T_{f,g})(T_{\tau,\omega}S_{f,\tau}T_{f,g})^{-1}(T_{\tau,\omega}\theta_\tau) = P_{f,\tau}.$$

(ii) $\Rightarrow$ (i) $\sum_{n=1}^\infty g_n(x)e_n = \theta_g(x) = \theta_f(T_{f,g}x) = \sum_{n=1}^\infty f_n(T_{f,g}x)e_n, \forall x \in \mathscr{X}$. This clearly gives (i).
(iii) $\Rightarrow$ (ii) $\theta_g = P_{g,\omega}\theta_g = P_{f,\tau}\theta_g = \theta_f(S_{f,\tau}^{-1}\theta_\tau\theta_g)$, and

$$\theta_\omega = \theta_\omega P_{g,\omega} = \theta_\omega P_{f,\tau} = (\theta_\omega\theta_f S_{f,\tau}^{-1})\theta_\tau.$$

We show that $S_{f,\tau}^{-1}\theta_\tau\theta_g$ and $\theta_\omega\theta_f S_{f,\tau}^{-1}$ are invertible. For,

$$(S_{f,\tau}^{-1}\theta_\tau\theta_g)(S_{g,\omega}^{-1}\theta_\omega\theta_f) = S_{f,\tau}^{-1}\theta_\tau P_{g,\omega}\theta_f = S_{f,\tau}^{-1}\theta_\tau P_{f,\tau}\theta_f = I_{\mathscr{X}},$$
$$(S_{g,\omega}^{-1}\theta_\omega\theta_f)(S_{f,\tau}^{-1}\theta_\tau\theta_g) = S_{g,\omega}^{-1}\theta_\omega P_{f,\tau}\theta_g = S_{g,\omega}^{-1}\theta_\omega P_{g,\omega}\theta_g = I_{\mathscr{X}}$$

and

$$(\theta_\omega\theta_f S_{f,\tau}^{-1})(\theta_\tau\theta_g S_{g,\omega}^{-1}) = \theta_\omega P_{f,\tau}\theta_g S_{g,\omega}^{-1} = \theta_\omega P_{g,\omega}\theta_g S_{g,\omega}^{-1} = I_{\mathscr{X}},$$
$$(\theta_\tau\theta_g S_{g,\omega}^{-1})(\theta_\omega\theta_f S_{f,\tau}^{-1}) = \theta_\tau P_{g,\omega}\theta_f S_{f,\tau}^{-1} = \theta_\tau P_{f,\tau}\theta_f S_{f,\tau}^{-1} = I_{\mathscr{X}}.$$

Let $T_{f,g}, T_{\tau,\omega} : \mathscr{X} \to \mathscr{X}$ be bounded invertible and $g_n = f_nT_{f,g}, \omega_n = T_{\tau,\omega}\tau_n, \forall n \in \mathbb{N}$. Then $\theta_g = \theta_f T_{f,g}$ says that $\theta_\tau\theta_g = \theta_\tau\theta_f T_{f,g} = S_{f,\tau}T_{f,g}$ which implies $T_{f,g} = S_{f,\tau}^{-1}\theta_\tau\theta_g$, and $\theta_\omega = T_{\tau,\omega}\theta_\tau$ says $\theta_\omega\theta_f = T_{\tau,\omega}\theta_\tau\theta_f = T_{\tau,\omega}S_{f,\tau}$. Hence $T_{\tau,\omega} = \theta_\omega\theta_f S_{f,\tau}^{-1}$. □

It is easy to see that for Hilbert spaces, Theorem 4.3.2 reduces to Theorem 1.2.46.

Definition 4.2.1 introduced the notion of dual frames. A twin notion associated is the notion of orthogonality.

Definition 4.3.3. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. A p-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathscr{X}$ is* ***orthogonal*** *for $(\{f_n\}_n, \{\tau_n\}_n)$ if*

$$0 = \sum_{n=1}^{\infty} g_n(x)\tau_n = \sum_{n=1}^{\infty} f_n(x)\omega_n, \quad \forall x \in \mathscr{X}.$$

Unlike duality, the notion orthogonality is symmetric but not reflexive. Further, dual p-ASFs cannot be orthogonal to each other and orthogonal p-ASFs cannot be dual to each other. Moreover, if $(\{g_n\}_n, \{\omega_n\}_n)$ is orthogonal for $(\{f_n\}_n, \{\tau_n\}_n)$, then both $(\{f_n\}_n, \{\omega_n\}_n)$ and $(\{g_n\}_n, \{\tau_n\}_n)$ are not p-ASFs. Similar to Proposition 4.2.4 we have the following proposition.

Proposition 4.3.4. *For two p-ASFs $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathscr{X}$, the following are equivalent.*

(i) *$(\{g_n\}_n, \{\omega_n\}_n)$ is orthogonal for $(\{f_n\}_n, \{\tau_n\}_n)$.*

(ii) $\theta_\tau \theta_g = \theta_\omega \theta_f = 0$.

Usefulness of orthogonal frames is that we have interpolation result, i.e., these frames can be stitched along certain curves (in particular, on the unit circle centered at the origin) to get new frames.

Theorem 4.3.5. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ be two Parseval p-ASFs for $\mathscr{X}$ which are orthogonal. If $A, B, C, D : \mathscr{X} \to \mathscr{X}$ are bounded linear operators and $CA + DB = I_{\mathscr{X}}$, then*

$$(\{f_nA + g_nB\}_n, \{C\tau_n + D\omega_n\}_n)$$

is a simple p-ASF for $\mathscr{X}$. In particular, if scalars a, b, c, d satisfy $ca + db = 1$, then $(\{af_n + bg_n\}_n, \{c\tau_n + d\omega_n\}_n)$ is a simple p-ASF for $\mathscr{X}$.

Proof. By a calculation we find

$$\theta_{fA+gB}x = \{(f_nA + g_nB)(x)\}_n = \{f_n(Ax)\}_n + \{g_n(Bx)\}_n = \theta_f(Ax) + \theta_g(Bx), \quad \forall x \in \mathscr{X}$$

and

$$\theta_{C\tau+D\omega}(\{a_n\}_n) = \sum_{n=1}^{\infty} a_n(C\tau_n + D\omega_n) = C\theta_\tau(\{a_n\}_n) + D\theta_\omega(\{a_n\}_n), \quad \forall \{a_n\}_n \in \ell^p(\mathbb{N}).$$

So

$$\begin{aligned} S_{fA+gB,C\tau+D\omega} &= \theta_{C\tau+D\omega}\theta_{fA+gB} = (C\theta_\tau + D\theta_\omega)(\theta_f A + \theta_g B) \\ &= C\theta_\tau\theta_f A + C\theta_\tau\theta_g B + D\theta_\omega\theta_f A + D\theta_\omega\theta_g B \\ &= CS_{f,\tau}A + 0 + 0 + DS_{g,\omega}B = CI_{\mathscr{X}}A + DI_{\mathscr{X}}B = I_{\mathscr{X}}. \end{aligned}$$

□

Using Theorem 4.3.2 we finally relate three notions duality, similarity and orthogonality.

Proposition 4.3.6. *For every p-ASF* $(\{f_n\}_n, \{\tau_n\}_n)$*, the canonical dual for* $(\{f_n\}_n, \{\tau_n\}_n)$ *is the only dual p-ASF that is similar to* $(\{f_n\}_n, \{\tau_n\}_n)$*.*

Proof. Let us suppose that two p-ASFs $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ are similar and dual to each other. Then there exist bounded invertible operators $T_{f,g}, T_{\tau,\omega} : \mathscr{X} \to \mathscr{X}$ such that $g_n = f_n T_{f,g}, \omega_n = T_{\tau,\omega}\tau_n, \forall n \in \mathbb{N}$. Theorem 4.3.2 then gives

$$T_{f,g} = S_{f,\tau}^{-1}\theta_\tau\theta_g = S_{f,\tau}^{-1}I_{\mathscr{X}} = S_{f,\tau}^{-1} \text{ and } T_{\tau,\omega} = \theta_\omega\theta_f S_{f,\tau}^{-1} = I_{\mathscr{X}}S_{f,\tau}^{-1} = S_{f,\tau}^{-1}.$$

Hence $(\{g_n\}_n, \{\omega_n\}_n)$ is the canonical dual for $(\{f_n\}_n, \{\tau_n\}_n)$. □

Proposition 4.3.7. *Two similar p-ASFs cannot be orthogonal.*

Proof. Let $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ be two p-ASFs which are similar. Then there exist bounded invertible operators $T_{f,g}, T_{\tau,\omega} : \mathscr{X} \to \mathscr{X}$ such that $g_n = f_n T_{f,g}, \omega_n = T_{\tau,\omega}\tau_n, \forall n \in \mathbb{N}$. Theorem 4.3.2 then says $\theta_g = \theta_f T_{f,g}, \theta_\omega = T_{\tau,\omega}\theta_\tau$. Therefore

$$\theta_\tau\theta_g = \theta_\tau\theta_f T_{f,g} = S_{f,\tau}T_{f,g} \neq 0.$$

□

Remark 4.3.8. *For every p-ASF* $(\{f_n\}_n, \{\tau_n\}_n)$*, both p-ASFs*

$$(\{f_n S_{f,\tau}^{-1}\}_n, \{\tau_n\}_n) \text{ and } (\{f_n\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$$

are simple p-ASFs and are similar to $(\{f_n\}_n, \{\tau_n\}_n)$*. Therefore each p-ASF is similar to simple p-ASFs.*

4.4 DILATION THEOREM FOR p-APPROXIMATE SCHAUDER FRAMES

Here we derive a generalization of Theorem 1.2.37 (Naimark-Han-Larson dilation theorem) for frames in Hilbert spaces to p-ASFs for Banach spaces. In order to derive the dilation result we must have a notion of Riesz basis for Banach space. Theorem 1.2.25 gives various characterizations for Riesz bases for Hilbert spaces but all uses (implicitly or explicitly) inner product structures and orthonormal bases. These characterizations lead to the notion of p-Riesz basis for Banach spaces using a single sequence in the Banach space (Definition 1.3.1) but we consider a different notion in this chapter.
To define the notion of Riesz basis, which is compatible with Hilbert space situation, we first derive an operator-theoretic characterization for Riesz basis in Hilbert spaces, which does not use the inner product of Hilbert space. To do so, we need a result from Hilbert space frame theory.

Theorem 4.4.1. *For sequence $\{\tau_n\}_n$ in $\mathscr{H}$, the following are equivalent.*

(i) $\{\tau_n\}_n$ is a Riesz basis for $\mathscr{H}$.

(ii) $\{\tau_n\}_n$ is a frame for $\mathscr{H}$ and

$$\theta_\tau S_\tau^{-1}\theta_\tau^* = I_{\ell^2(\mathbb{N})}. \tag{4.4.1}$$

Proof. (i) $\implies$ (ii) From Theorem 1.2.22 that a Riesz basis is a frame. Now there exist an orthonormal basis $\{\omega_n\}_n$ for $\mathscr{H}$ and a bounded invertible operator $T:\mathscr{H}\to\mathscr{H}$ such that $T\omega_n=\tau_n$, for all $n\in\mathbb{N}$. We then have

$$\begin{aligned} S_\tau h &= \sum_{n=1}^\infty \langle h,\tau_n\rangle\tau_n = \sum_{n=1}^\infty \langle h, T\omega_n\rangle T\omega_n \\ &= T\left(\sum_{n=1}^\infty \langle T^*h,\omega_n\rangle\omega_n\right) = TT^*h, \quad \forall h\in\mathscr{H}. \end{aligned}$$

Therefore

$$\begin{aligned} \theta_\tau S_\tau^{-1}\theta_\tau^*\{a_n\}_n &= \theta_\tau(TT^*)^{-1}\theta_\tau^*\{a_n\}_n = \theta_\tau(T^*)^{-1}T^{-1}\theta_\tau^*\{a_n\}_n \\ &= \theta_\tau(T^*)^{-1}T^{-1}\left(\sum_{n=1}^\infty a_n\tau_n\right) = \theta_\tau(T^*)^{-1}T^{-1}\left(\sum_{n=1}^\infty a_nT\omega_n\right) \\ &= \theta_\tau\left(\sum_{n=1}^\infty a_n(T^*)^{-1}\omega_n\right) = \sum_{k=1}^\infty\left\langle \sum_{n=1}^\infty a_n(T^*)^{-1}\omega_n,\tau_k\right\rangle e_k \end{aligned}$$

$$= \sum_{k=1}^{\infty} \left\langle \sum_{n=1}^{\infty} a_n (T^*)^{-1} \omega_n, T\omega_k \right\rangle e_k$$
$$= \sum_{k=1}^{\infty} \left\langle \sum_{n=1}^{\infty} a_n \omega_n, \omega_k \right\rangle e_k = \{a_k\}_k, \quad \forall \{a_n\}_n \in \ell^2(\mathbb{N}).$$

(ii) $\implies$ (i) From Holub's theorem (Theorem 1.2.39), there exists a surjective bounded linear operator $T : \ell^2(\mathbb{N}) \to \mathscr{H}$ such that $Te_n = \tau_n$, for all $n \in \mathbb{N}$. Since all separable Hilbert spaces are isometrically isomorphic to one another and orthonormal bases map into orthonormal bases, without loss of generality we may assume that $\{e_n\}_n$ is an orthonormal basis for $\mathscr{H}$ and the domain of T is $\mathscr{H}$. Our job now reduces in showing T is invertible. Since T is already surjective, to show it is invertible, it suffices to show it is injective. Let $\{a_n\}_n \in \ell^2(\mathbb{N})$. Then $\{a_n\}_n = \theta_\tau(S_\tau^{-1}\theta_\tau^*\{a_n\}_n)$. Hence θ_τ is surjective. We now find

$$\theta_\tau h = \sum_{n=1}^{\infty} \langle h, \tau_n \rangle e_n = \sum_{n=1}^{\infty} \langle h, Te_n \rangle e_n = T^* h, \quad \forall h \in \mathscr{H}.$$

Therefore

$$\text{Kernel}(T) = T^*(\mathscr{H})^\perp = \theta_\tau(\mathscr{H})^\perp = \mathscr{H}^\perp = \{0\}.$$

Hence T is injective. □

Theorem 4.4.1 leads to the following definition of p-approximate Riesz basis.

Definition 4.4.2. *A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be a **p-approximate Riesz basis** for $\mathscr{X}$ if it is a p-ASF for $\mathscr{X}$ and $\theta_f S_{f,\tau}^{-1} \theta_\tau = I_{\ell^p(\mathbb{N})}$.*

Example 4.4.3. *Let $p \in [1, \infty)$ and $U : \mathscr{X} \to \ell^p(\mathbb{N})$, $V : \ell^p(\mathbb{N}) \to \mathscr{X}$ be bounded invertible linear operators. Let $\{e_n\}_n$, $\{\zeta_n\}_n$, $\{f_n\}_n$, and $\{\tau_n\}_n$ be as in Example 4.1.9. Then $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-approximate Riesz basis for $\mathscr{X}$.*

We now derive the dilation theorem.

Theorem 4.4.4. ***(Dilation theorem for p-approximate Schauder frames)*** *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Then there exist a Banach space $\mathscr{X}_1$ which contains $\mathscr{X}$ isometrically and a p-approximate Riesz basis $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathscr{X}_1$ such that*

$$f_n = g_n P_{|\mathscr{X}}, \quad \tau_n = P\omega_n, \quad \forall n \in \mathbb{N},$$

where $P : \mathscr{X}_1 \to \mathscr{X}$ is onto projection.

Proof. Let $\{e_n\}_n$ denote the standard Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Define

$$\mathscr{X}_1 \coloneqq \mathscr{X} \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N})), \quad P : \mathscr{X}_1 \ni x \oplus y \mapsto x \oplus 0 \in \mathscr{X}_1$$

and

$$\omega_n \coloneqq \tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n \in \mathscr{X}_1, \qquad g_n \coloneqq f_n \oplus \zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}) \in \mathscr{X}_1^*, \quad \forall n \in \mathbb{N}.$$

Then clearly $\mathscr{X}_1$ contains $\mathscr{X}$ isometrically, $P : \mathscr{X}_1 \to \mathscr{X}$ is onto projection and

$$(g_n P_{|\mathscr{X}})(x) = g_n(P_{|\mathscr{X}}x) = g_n(x) = (f_n \oplus \zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(x \oplus 0) = f_n(x), \quad \forall x \in \mathscr{X},$$
$$P\omega_n = P(\tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n) = \tau_n, \quad \forall n \in \mathbb{N}.$$

Since the operator $I_{\ell^p(\mathbb{N})} - P_{f,\tau}$ is idempotent, it follows that $(I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N}))$ is a closed subspace of $\ell^p(\mathbb{N})$ and hence a Banach space. Therefore $\mathscr{X}_1$ is a Banach space. Let $x \oplus y \in \mathscr{X}_1$ and we shall write $y = \{a_n\}_n \in \ell^p(\mathbb{N})$. We then see that

$$\begin{aligned}
\sum_{n=1}^{\infty} (\zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(y)\tau_n &= \sum_{n=1}^{\infty} \zeta_n(y)\tau_n - \sum_{n=1}^{\infty} \zeta_n(P_{f,\tau}(y))\tau_n \\
&= \sum_{n=1}^{\infty} \zeta_n(\{a_k\}_k)\tau_n - \sum_{n=1}^{\infty} \zeta_n(\theta_f S_{f,\tau}^{-1}\theta_\tau(\{a_k\}_k))\tau_n \\
&= \sum_{n=1}^{\infty} a_n\tau_n - \sum_{n=1}^{\infty} \zeta_n\left(\theta_f S_{f,\tau}^{-1}\left(\sum_{k=1}^{\infty} a_k\tau_k\right)\right)\tau_n \\
&= \sum_{n=1}^{\infty} a_n\tau_n - \sum_{n=1}^{\infty} \zeta_n\left(\sum_{k=1}^{\infty} a_k\theta_f S_{f,\tau}^{-1}\tau_k\right)\tau_n \\
&= \sum_{n=1}^{\infty} a_n\tau_n - \sum_{n=1}^{\infty} \zeta_n\left(\sum_{k=1}^{\infty} a_k \sum_{r=1}^{\infty} f_r(S_{f,\tau}^{-1}\tau_k)e_r\right)\tau_n \\
&= \sum_{n=1}^{\infty} a_n\tau_n - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_k \sum_{r=1}^{\infty} f_r(S_{f,\tau}^{-1}\tau_k)\zeta_n(e_r)\tau_n \\
&= \sum_{n=1}^{\infty} a_n\tau_n - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_k f_n(S_{f,\tau}^{-1}\tau_k)\tau_n \\
&= \sum_{n=1}^{\infty} a_n\tau_n - \sum_{k=1}^{\infty} a_k \sum_{n=1}^{\infty} f_n(S_{f,\tau}^{-1}\tau_k)\tau_n \\
&= \sum_{n=1}^{\infty} a_n\tau_n - \sum_{k=1}^{\infty} a_k\tau_k = 0
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} f_n(x)(I_{\ell^p(\mathbb{N})}-P_{f,\tau})e_n=\sum_{n=1}^{\infty} f_n(x)e_n-\sum_{n=1}^{\infty} f_n(x)P_{f,\tau}e_n$$
$$=\sum_{n=1}^{\infty} f_n(x)e_n-\sum_{n=1}^{\infty} f_n(x)\theta_f S_{f,\tau}^{-1}\theta_\tau e_n$$
$$=\sum_{n=1}^{\infty} f_n(x)e_n-\sum_{n=1}^{\infty} f_n(x)\theta_f S_{f,\tau}^{-1}\tau_n$$
$$=\sum_{n=1}^{\infty} f_n(x)e_n-\sum_{n=1}^{\infty} f_n(x)\sum_{k=1}^{\infty} f_k(S_{f,\tau}^{-1}\tau_n)e_k$$
$$=\sum_{n=1}^{\infty} f_n(x)e_n-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_n(x)f_k(S_{f,\tau}^{-1}\tau_n)e_k$$
$$=\sum_{n=1}^{\infty} f_n(x)e_n-\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} f_n(x)f_k(S_{f,\tau}^{-1}\tau_n)e_k$$
$$=\sum_{n=1}^{\infty} f_n(x)e_n-\sum_{k=1}^{\infty} f_k\left(\sum_{n=1}^{n} f_n(x)S_{f,\tau}^{-1}\tau_n\right)e_k$$
$$=\sum_{n=1}^{\infty} f_n(x)e_n-\sum_{k=1}^{\infty} f_k(x)e_k=0.$$

By using previous two calculations, we get

$$S_{g,\omega}(x\oplus y)=\sum_{n=1}^{\infty} g_n(x\oplus y)\omega_n$$
$$=\sum_{n=1}^{\infty}(f_n\oplus\zeta_n(I_{\ell^p(\mathbb{N})}-P_{f,\tau}))(x\oplus y)(\tau_n\oplus(I_{\ell^p(\mathbb{N})}-P_{f,\tau})e_n)$$
$$=\sum_{n=1}^{\infty}(f_n(x)+(\zeta_n(I_{\ell^p(\mathbb{N})}-P_{f,\tau}))(y))(\tau_n\oplus(I_{\ell^p(\mathbb{N})}-P_{f,\tau})e_n)$$
$$=\left(\sum_{n=1}^{\infty} f_n(x)\tau_n+\sum_{n=1}^{\infty}(\zeta_n(I_{\ell^p(\mathbb{N})}-P_{f,\tau}))(y)\tau_n\right)\oplus$$
$$\left(\sum_{n=1}^{\infty} f_n(x)(I_{\ell^p(\mathbb{N})}-P_{f,\tau})e_n+\sum_{n=1}^{\infty}(\zeta_n(I_{\ell^p(\mathbb{N})}-P_{f,\tau}))(y)(I_{\ell^p(\mathbb{N})}-P_{f,\tau})e_n\right)$$
$$=(S_{f,\tau}x+0)\oplus\left(0+(I_{\ell^p(\mathbb{N})}-P_{f,\tau})\sum_{n=1}^{\infty}\zeta_n((I_{\ell^p(\mathbb{N})}-P_{f,\tau})y)e_n\right)$$
$$=S_{f,\tau}x\oplus(I_{\ell^p(\mathbb{N})}-P_{f,\tau})(I_{\ell^p(\mathbb{N})}-P_{f,\tau})y=S_{f,\tau}x\oplus(I_{\ell^p(\mathbb{N})}-P_{f,\tau})y$$
$$=(S_{f,\tau}\oplus(I_{\ell^p(\mathbb{N})}-P_{f,\tau}))(x\oplus y).$$

Since the operator $I_{\ell^p(\mathbb{N})}-P_{f,\tau}$ is idempotent, $I_{\ell^p(\mathbb{N})}-P_{f,\tau}$ becomes the identity operator on the space $(I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N}))$. Hence we get that the operator $S_{g,\omega} = S_{f,\tau} \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})$ is bounded invertible from $\mathscr{X}_1$ onto itself. We next show that $(\{g_n\}_n, \{\omega_n\}_n)$ is a p-approximate Riesz basis for $\mathscr{X}_1$. For this, first we find θ_g and θ_ω. Consider

$$\begin{aligned}\theta_g(x\oplus y) &= \{g_n(x\oplus y)\}_n = \{(f_n \oplus \zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(x\oplus y)\}_n\\ &= \{f_n(x) + \zeta_n((I_{\ell^p(\mathbb{N})} - P_{f,\tau})y)\}_n = \{f_n(x)\}_n + \{\zeta_n((I_{\ell^p(\mathbb{N})} - P_{f,\tau})y)\}_n\\ &= \theta_f x + \sum_{n=1}^{\infty} \zeta_n((I_{\ell^p(\mathbb{N})} - P_{f,\tau})y)e_n = \theta_f x + (I_{\ell^p(\mathbb{N})} - P_{f,\tau})y, \quad \forall x\oplus y \in \mathscr{X}_1\end{aligned}$$

and

$$\begin{aligned}\theta_\omega\{a_n\}_n &= \sum_{n=1}^{\infty} a_n\omega_n = \sum_{n=1}^{\infty} a_n(\tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n)\\ &= \left(\sum_{n=1}^{\infty} a_n\tau_n\right) \oplus \left(\sum_{n=1}^{\infty} a_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n\right)\\ &= \theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\left(\sum_{n=1}^{\infty} a_n e_n\right)\\ &= \theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n, \quad \forall \{a_n\}_n \in \ell^p(\mathbb{N}).\end{aligned}$$

Therefore

$$\begin{aligned}P_{g,\omega}\{a_n\}_n &= \theta_g S_{g,\omega}^{-1}\theta_\omega\{a_n\}_n = \theta_g S_{g,\omega}^{-1}(\theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n)\\ &= \theta_g(S_{f,\tau}^{-1} \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(\theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n)\\ &= \theta_g(S_{f,\tau}^{-1}\theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})^2\{a_n\}_n)\\ &= \theta_g(S_{f,\tau}^{-1}\theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n)\\ &= \theta_f(S_{f,\tau}^{-1}\theta_\tau\{a_n\}_n) + (I_{\ell^p(\mathbb{N})} - P_{f,\tau})(I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n\\ &= P_{f,\tau}\{a_n\}_n + (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n = \{a_n\}_n, \quad \forall \{a_n\}_n \in \ell^p(\mathbb{N}).\end{aligned}$$

□

Corollary 4.4.5. *(Han and Larson (2000); Kashin and Kulikova (2002)) Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. Then there exist a Hilbert space $\mathscr{H}_1$ which contains $\mathscr{H}$ isometrically and a Riesz basis $\{\omega_n\}_n$ for $\mathscr{H}_1$ such that*

$$\tau_n = P\omega_n, \quad \forall n \in \mathbb{N},$$

where P is the orthogonal projection from $\mathscr{H}_1$ onto $\mathscr{H}$.

Proof. Let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. Define

$$f_n : \mathscr{H} \ni h \mapsto f_n(h) := \langle h, \tau_n \rangle \in \mathbb{K}, \quad \forall n \in \mathbb{N}.$$

Then $\theta_f = \theta_\tau$. Note that now $(\{f_n\}_n, \{\tau_n\}_n)$ is a 2-approximate frame for $\mathscr{H}$. Theorem 4.4.4 now says that there exist a Banach space $\mathscr{X}_1$ which contains $\mathscr{H}$ isometrically and a 2-approximate Riesz basis $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathscr{X}_1 = \mathscr{H} \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)(\ell^2(\mathbb{N}))$ such that

$$f_n = g_n P_{|\mathscr{H}}, \quad \tau_n = P\omega_n, \quad \forall n \in \mathbb{N},$$

where $P : \mathscr{X}_1 \to \mathscr{H}$ is onto projection. Since $(I_{\ell^2(\mathbb{N})} - P_\tau)(\ell^2(\mathbb{N}))$ is a closed subspace of the Hilbert space $\ell^2(\mathbb{N})$, $\mathscr{X}_1$ now becomes a Hilbert space. From the definition of P we get that it is an orthogonal projection. To prove $\{\omega_n\}_n$ is a Riesz basis for $\mathscr{X}_1$, we use Theorem 4.4.1. Since $\{\tau_n\}_n$ is a frame for $\mathscr{H}$, there exist $a, b > 0$ such that

$$a\|h\|^2 \leq \sum_{n=1}^\infty |\langle h, \tau_n \rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathscr{H}.$$

Let $h \oplus (I_{\ell^2(\mathbb{N})} - P_{f,\tau})\{a_k\}_k \in \mathscr{X}_1$. Then by noting $b \geq 1$, we get

$$\begin{aligned}
&\sum_{n=1}^\infty |\langle h \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, \omega_n \rangle|^2 \\
&= \sum_{n=1}^\infty |\langle h \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, \tau_n \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)e_n \rangle|^2 \\
&= \sum_{n=1}^\infty |\langle h, \tau_n \rangle|^2 + \sum_{n=1}^\infty |\langle (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, (I_{\ell^2(\mathbb{N})} - P_\tau)e_n \rangle|^2 \\
&= \sum_{n=1}^\infty |\langle h, \tau_n \rangle|^2 + \sum_{n=1}^\infty |\langle (I_{\ell^2(\mathbb{N})} - P_\tau)(I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, e_n \rangle|^2 \\
&= \sum_{n=1}^\infty |\langle h, \tau_n \rangle|^2 + \sum_{n=1}^\infty |\langle (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, e_n \rangle|^2 \\
&= \sum_{n=1}^\infty |\langle h, \tau_n \rangle|^2 + \|(I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k\|^2 \\
&\leq b\|h\|^2 + \|(I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k\|^2 \\
&\leq b(\|h\|^2 + \|(I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k\|^2) \\
&= b\|h \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k\|^2.
\end{aligned}$$

Previous calculation tells that $\{\omega_n\}_n$ is a Bessel sequence for $\mathscr{X}_1$. Hence $S_\omega : \mathscr{X}_1 \ni$

$x \oplus \{a_k\}_k \mapsto \sum_{n=1}^\infty \langle x \oplus \{a_k\}_k, \omega_n\rangle \omega_n \in \mathscr{X}_1$ is a well-defined bounded linear operator. Next we claim that

$$g_n(x\oplus\{a_k\}_k) = \langle x\oplus\{a_k\}_k, \omega_n\rangle, \quad \forall x\oplus\{a_k\}_k \in \mathscr{X}_1, \forall n\in\mathbb{N}. \tag{4.4.2}$$

Consider

$$\begin{aligned}
&g_n(x\oplus\{a_k\}_k) = (f_n\oplus\zeta_n(I_{\ell^2(\mathbb{N})}-P_\tau))(x\oplus\{a_k\}_k)\\
&= f_n(x)+\zeta_n((I_{\ell^2(\mathbb{N})}-P_\tau)\{a_k\}_k) = f_n(x)+\zeta_n(\{a_k\}_k)-\zeta_n(P_\tau\{a_k\}_k)\\
&= f_n(x)+\zeta_n(\{a_k\}_k)-\zeta_n(\theta_\tau S_\tau^{-1}\theta_\tau^*\{a_k\}_k) = f_n(x)+a_n-\zeta_n\left(\theta_\tau S_\tau^{-1}\left(\sum_{k=1}^\infty a_k\tau_k\right)\right)\\
&= f_n(x)+a_n-\zeta_n\left(\sum_{k=1}^\infty a_k\theta_\tau S_\tau^{-1}\tau_k\right) = f_n(x)+a_n-\zeta_n\left(\sum_{k=1}^\infty a_k\sum_{r=1}^\infty \langle S_\tau^{-1}\tau_k,\tau_r\rangle e_r\right)\\
&= f_n(x)+a_n-\sum_{k=1}^\infty a_k\langle S_\tau^{-1}\tau_k,\tau_n\rangle = \langle x,\tau_n\rangle+a_n-\sum_{k=1}^\infty a_k\langle S_\tau^{-1}\tau_k,\tau_n\rangle \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
&\langle x\oplus\{a_k\}_k,\omega_n\rangle = \langle x\oplus\{a_k\}_k, \tau_n\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)e_n\rangle\\
&= \langle x,\tau_n\rangle+\langle\{a_k\}_k,(I_{\ell^2(\mathbb{N})}-P_\tau)e_n\rangle = \langle x,\tau_n\rangle+\langle\{a_k\}_k,e_n\rangle+\langle\{a_k\}_k,P_\tau e_n\rangle\\
&= \langle x,\tau_n\rangle+a_n-\left\langle\{a_k\}_k,\theta_\tau S_\tau^{-1}\theta_\tau^* e_n\right\rangle = \langle x,\tau_n\rangle+a_n-\left\langle\{a_k\}_k,\theta_\tau S_\tau^{-1}\tau_n\right\rangle\\
&= \langle x,\tau_n\rangle+a_n-\langle\{a_k\}_k,\{\langle S_\tau^{-1}\tau_n,\tau_k\rangle\}_k\rangle = \langle x,\tau_n\rangle+a_n-\sum_{k=1}^\infty a_k\overline{\langle S_\tau^{-1}\tau_n,\tau_k\rangle}\\
&= \langle x,\tau_n\rangle+a_n-\sum_{k=1}^\infty a_k\langle \tau_k,S_\tau^{-1}\tau_n\rangle = \langle x,\tau_n\rangle+a_n-\sum_{k=1}^\infty a_k\langle S_\tau^{-1}\tau_k,\tau_n\rangle.
\end{aligned}$$

Thus Equation (4.4.2) holds. Therefore for all $x\oplus\{a_k\}_k\in\mathscr{X}_1$,

$$S_{g,\omega}(x\oplus\{a_k\}_k) = \sum_{n=1}^\infty g_n(x\oplus\{a_k\}_k)\omega_n = \sum_{n=1}^\infty \langle x\oplus\{a_k\}_k,\omega_n\rangle\omega_n = S_\omega(x\oplus\{a_k\}_k).$$

Since $S_{g,\omega}$ is invertible, S_ω becomes invertible. Clearly S_ω is positive. Therefore

$$\frac{1}{\|S_\omega\|^{-1}}\|g\|^2 \le \langle S_\omega g, g\rangle \le \|S_\omega\|\|g\|^2, \quad \forall g\in\mathscr{X}_1.$$

Hence

$$\frac{1}{\|S_\omega\|^{-1}}\|g\|^2 \le \sum_{n=1}^\infty |\langle g,\omega_n\rangle|^2 \le \|S_\omega\|\|g\|^2, \quad \forall g\in\mathscr{X}_1.$$

Hence $\{\omega_n\}_n$ is a frame for $\mathscr{X}_1$.

Finally we show Equation (4.4.1) in Theorem 4.4.1 for the frame $\{\omega_n\}_n$. Consider

$$\begin{aligned}
&\theta_\omega S_\omega^{-1}\theta_\omega^*\{a_n\}_n = \theta_\omega S_\omega^{-1}\left(\sum_{n=1}^{\infty} a_n\omega_n\right) = \theta_\omega\left(\sum_{n=1}^{\infty} a_n S_\omega^{-1}\omega_n\right)\\
&= \sum_{k=1}^{\infty}\left\langle \sum_{n=1}^{\infty} a_n S_\omega^{-1}\omega_n, \omega_k\right\rangle = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle S_\omega^{-1}\omega_n, \omega_k\rangle\\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle (S_\tau^{-1}\oplus(I_{\ell^2(\mathbb{N})}-P_\tau))(\tau_n\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)e_n), \tau_k\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)e_k\rangle\\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle S_\tau^{-1}\tau_n\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)^2 e_n, \tau_k\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)e_k\rangle\\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n(\langle S_\tau^{-1}\tau_n, \tau_k\rangle + \langle (I_{\ell^2(\mathbb{N})}-P_\tau)e_n, (I_{\ell^2(\mathbb{N})}-P_\tau)e_k\rangle)\\
&= \sum_{k=1}^{\infty}\left\langle \sum_{n=1}^{\infty} a_n S_\tau^{-1}\tau_n, \tau_k\right\rangle + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle (I_{\ell^2(\mathbb{N})}-P_{f,\tau})e_n, (I_{\ell^2(\mathbb{N})}-P_\tau)e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle (I_{\ell^2(\mathbb{N})}-P_\tau)e_n, e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle e_n, e_k\rangle - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle P_\tau e_n, e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^{\infty} a_k e_k - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle \theta_\tau S_\tau^{-1}\theta_\tau^* e_n, e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^{\infty} a_k e_k - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle S_\tau^{-1}\tau_n, \theta_\tau^* e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^{\infty} a_k e_k - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n\langle S_\tau^{-1}\tau_n, \tau_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^{\infty} a_k e_k - P_\tau\{a_n\}_n = \{a_n\}_n, \quad \forall \{a_n\}_n \in \ell^2(\mathbb{N}).
\end{aligned}$$

Thus $\{\omega_n\}_n$ is a Riesz basis for $\mathscr{X}_1$ which completes the proof. □

We now illustrate Theorem 4.4.4 with an example.

Example 4.4.6. *Let $p \in [1,\infty)$. Let $\{e_n\}_n$ denote the canonical Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$ respectively. Define*

$$R : \ell^p(\mathbb{N}) \ni (x_n)_{n=1}^{\infty} \mapsto (0, x_1, x_2, \dots) \in \ell^p(\mathbb{N}),$$
$$L : \ell^p(\mathbb{N}) \ni (x_n)_{n=1}^{\infty} \mapsto (x_2, x_3, x_4, \dots) \in \ell^p(\mathbb{N}).$$

Then $LR = I_{\ell^p(\mathbb{N})}$. *Example 4.4.3 says that* $(\{f_n := \zeta_n R\}_n, \{\tau_n := Le_n\}_n)$ *is a p-ASF for* $\ell^p(\mathbb{N})$. *Note that* $\theta_f = R$ *and* $\theta_\tau = L$. *Therefore* $S_{f,\tau} = LR = I_{\ell^p(\mathbb{N})}$ *and* $P_{f,\tau} = RL$. *Then*

$$\begin{aligned}(I_{\ell^p(\mathbb{N})} - P_{f,\tau})(x_n)_{n=1}^\infty &= (x_n)_{n=1}^\infty - RL(x_n)_{n=1}^\infty \\ &= (x_n)_{n=1}^\infty - (0, x_2, x_3, \dots) = (x_1, 0, 0, \dots), \quad \forall (x_n)_{n=1}^\infty \in \ell^p(\mathbb{N})\end{aligned}$$

which says that $(I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N})) \cong \mathbb{K}$. *Using Theorem 4.4.4,*

$$\begin{aligned}&\mathscr{X}_1 = \ell^p(\mathbb{N}) \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N})) \cong \ell^p(\mathbb{N}) \oplus \mathbb{K} \cong \ell^p(\mathbb{N} \cup \{0\}) \\ &P : \ell^p(\mathbb{N} \cup \{0\}) \ni (x_n)_{n=0}^\infty \mapsto (x_n)_{n=1}^\infty \in \ell^p(\mathbb{N}),\end{aligned}$$

$$\begin{aligned}\omega_1 &= \tau_1 \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\tau_1 = Le_1 \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})Le_1 = 0 \oplus 0, \\ \omega_2 &= \tau_2 \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\tau_2 = Le_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})Le_2 \\ &= e_1 \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_1 = e_1 \oplus RLe_1 = e_1 \oplus 0, \\ \omega_n &= \tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\tau_n = Le_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})Le_n \\ &= e_{n-1} \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_{n-1} = e_{n-1} \oplus RLe_{n-1} = e_{n-1} \oplus e_{n-1}, \quad \forall n \geq 3, \\ g_n &= f_n \oplus \zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}) = \zeta_n R \oplus \zeta_n RL = \zeta_n R(I_{\ell^p(\mathbb{N})} \oplus L), \quad \forall n \in \mathbb{N}\end{aligned}$$

and $(\{g_n\}_n, \{\omega_n\}_n)$ *is a p-approximate Riesz basis for* $\ell^p(\mathbb{N})$.

4.5 NEW IDENTITY FOR PARSEVAL p-APPROXIMATE SCHAUDER FRAMES

Certain classes of Banach spaces known as homogeneous semi-inner product spaces admit a kind of inner product and can be studied with certain similarities with Hilbert spaces. These spaces are introduced by Lumer (1961) and studied extensively by Giles (1967). We now recall the fundamentals of semi-inner products. Let $\mathscr{X}$ be a vector space over $\mathbb{K}$. A map $[\cdot,\cdot] : \mathscr{X} \times \mathscr{X} \to \mathbb{K}$ is said to be a **homogeneous semi-inner product** if it satisfies the following.

(i) $[x,x] > 0$, for all $x \in \mathscr{X}, x \neq 0$.

(ii) $[\lambda x, y] = \lambda[x,y]$, for all $x, y \in \mathscr{X}$, for all $\lambda \in \mathbb{K}$.

(iii) $[x, \lambda y] = \overline{\lambda}[x,y]$, for all $x, y \in \mathscr{X}$, for all $\lambda \in \mathbb{K}$.

(iv) $[x+y, z] = [x,z] + [y,z]$, for all $x, y, z \in \mathscr{X}$.

(v) $|[x,y]|^2 \leq [x,x][y,y]$, for all $x,y \in \mathscr{X}$.

A homogeneous semi-inner product $[\cdot,\cdot]$ induces a **norm** which is defined as $\|x\| := \sqrt{[x,x]}$. A prototypical example of homogeneous semi-inner product spaces is the standard $\ell^p(\mathbb{N})$ space, $1 < p < \infty$, equipped with semi-inner product defined as follows. For $x = \{x_n\}_n$, $y = \{y_n\}_n \in \ell^p(\mathbb{N})$, define

$$[x,y] := \begin{cases} \dfrac{\sum\limits_{n=1}^{\infty} x_n \overline{y_n} |y_n|^{p-2}}{\|y\|_p^{p-2}} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

For certain classes of Banach spaces we have Riesz representation theorem.

Theorem 4.5.1. *(Giles (1967)) (**Riesz representation theorem for Banach spaces**) Let $\mathscr{X}$ be a complete homogeneous semi-inner product space. If $\mathscr{X}$ is continuous and uniformly convex, then for every bounded linear functional $f : \mathscr{X} \to \mathbb{K}$, there exists a unique $y \in \mathscr{X}$ such that $f(x) = [x,y]$, for all $x \in \mathscr{X}$.*

Theorem 4.5.1 leads to the notion of generalized adjoint whose existence is assured by the following theorem. Two state the result we need two definitions.

Definition 4.5.2. *(cf. Giles (1967)) Let $\mathscr{X}$ be a complete homogeneous semi-inner product space. Space $\mathscr{X}$ is said to be **continuous** if*

$$Re([x,y+\lambda x]) \to Re([x,y]), \text{ for all real } \lambda \to 0, \forall x,y \in \mathscr{X} \text{ such that } \|x\| = \|y\| = 1.$$

Definition 4.5.3. *(Giles (1967)) A Banach space is said to be **uniformly convex** if given $\varepsilon > 0$, there exists an $\delta > 0$ such that if $x,y \in \mathscr{X}$ are such that $\|x\| = \|y\| = 1$ and $\|x-y\| > \varepsilon$, then $\|x+y\| \leq 2(1-\delta)$.*

Theorem 4.5.4. *(Koehler (1971)) Let $\mathscr{X}$ be a complete homogeneous semi-inner product space. If $\mathscr{X}$ is continuous and uniformly convex, then for every bounded linear operator $A : \mathscr{X} \to \mathscr{X}$, there exists a unique map $A^\dagger : \mathscr{X} \to \mathscr{X}$, which may not be linear or continuous (called as **generalized adjoint** of A) such that*

$$[Ax,y] = [x,A^\dagger y], \quad \forall x,y \in \mathscr{X}.$$

Moreover, the following statements hold.

(i) $(\lambda A)^\dagger = \overline{\lambda} A^\dagger$, for all $\lambda \in \mathbb{K}$.

(ii) $A^\dagger$ is injective if and only if $\overline{A(\mathscr{X})} = \mathscr{X}$.

(iii) If the norm of $\mathscr{X}$ is strongly (Frechet) differentiable, then $A^\dagger$ is continuous.

Throughout this section we assume that $\mathscr{X}$ is a continuous, uniformly convex, homogeneous semi-inner product space. Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Theorem 4.5.1 then says that each f_n can be identified with unique $\omega_n \in \mathscr{X}$ which satisfies $f_n(x) = [x, \omega_n]$, for all $x \in \mathscr{X}$. Note that

$$\sum_{n=1}^{\infty}[x,(S_{\omega,\tau}^{-1})^\dagger\omega_n]S_{\omega,\tau}^{-1}\tau_n = S_{\omega,\tau}^{-1}\left(\sum_{n=1}^{\infty}[S_{\omega,\tau}^{-1}x,\omega_n]\tau_n\right) = S_{\omega,\tau}^{-1}x, \quad \forall x \in \mathscr{X}.$$

Hence $(\{\tilde{\omega}_n := (S_{\omega,\tau}^{-1})^\dagger\omega_n\}_n, \{\tilde{\tau}_n := S_{\omega,\tau}^{-1}\tau_n\}_n)$ is a p-ASF for $\mathscr{X}$ which is called as canonical dual frame for $(\{\omega_n\}_n, \{\tau_n\}_n)$. Given $\mathbb{M} \subseteq \mathbb{N}$, we define $S_{\mathbb{M}} : \mathscr{X} \ni x \mapsto \sum_{n\in\mathbb{M}}\langle x, \omega_n\rangle\tau_n \in \mathscr{X}$. Because of Inequalities (4.1.5) and (4.1.6), the map $S_{\mathbb{M}}$ is a well-defined bounded linear operator. Note that the operator $S_{\mathbb{M}}$ may not be invertible. In Proposition 2.2 in (Balan et al. (2007)) it is derived that if operators $U,V : \mathscr{H} \to \mathscr{H}$ satisfy $U+V = I_{\mathscr{H}}$, then $U-V = U^2 - V^2$. This remains valid for Banach spaces.

Lemma 4.5.5. *If operators $U,V : \mathscr{X} \to \mathscr{X}$ satisfy $U+V = I_{\mathscr{X}}$, then $U-V = U^2-V^2$.*

Proof. We follow the ideas in the proof of Proposition 2.2 in (Balan et al. (2007)):

$$\begin{aligned} U-V &= U-(I_{\mathscr{X}}-U) = 2U - I_{\mathscr{X}} = U^2-(I_{\mathscr{X}}-2U+U^2)\\ &= U^2-(I_{\mathscr{X}}-U)^2 = U^2-V^2. \end{aligned}$$

□

We now have Banach space version of Theorem 1.2.47.

Theorem 4.5.6. *Let $(\{\omega_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Then for every $\mathbb{M} \subseteq \mathbb{N}$, and for all $x \in \mathscr{X}$,*

$$\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x] - \sum_{n=1}^{\infty}[S_{\mathbb{M}}x,\tilde{\omega}_n][\tilde{\tau}_n, S_{\mathbb{M}}^\dagger x] = \sum_{n\in\mathbb{M}^c}[x,\omega_n][\tau_n,x] - \sum_{n=1}^{\infty}[S_{\mathbb{M}^c}x,\tilde{\omega}_n][\tilde{\tau}_n, S_{\mathbb{M}^c}^\dagger x].$$

Proof. For notational convenience, we denote $S_{f,\tau}$ by S. We clearly have $S_{\mathbb{M}} + S_{\mathbb{M}^c} = S$. Using $S^{-1}S_{\mathbb{M}} + S^{-1}S_{\mathbb{M}^c} = I_{\mathscr{X}}$ and Lemma 4.5.5, we get $S^{-1}S_{\mathbb{M}} - S^{-1}S_{\mathbb{M}^c} = (S^{-1}S_{\mathbb{M}})^2 - (S^{-1}S_{\mathbb{M}^c})^2 = S^{-1}S_{\mathbb{M}}S^{-1}S_{\mathbb{M}} - S^{-1}S_{\mathbb{M}^c}S^{-1}S_{\mathbb{M}^c}$ which gives

$$S^{-1}S_{\mathbb{M}} - S^{-1}S_{\mathbb{M}}S^{-1}S_{\mathbb{M}} = S^{-1}S_{\mathbb{M}^c} - S^{-1}S_{\mathbb{M}^c}S^{-1}S_{\mathbb{M}^c}.$$

Therefore for all $x,y \in \mathscr{X}$,

$$[S^{-1}S_{\mathbb{M}}x,y] - [S^{-1}S_{\mathbb{M}}S^{-1}S_{\mathbb{M}}x,y] = [S^{-1}S_{\mathbb{M}^c}x,y] - [S^{-1}S_{\mathbb{M}^c}S^{-1}S_{\mathbb{M}^c}x,y].$$

In particular, for all $x \in \mathscr{X}$,

$$[S^{-1}S_{\mathbb{M}}x, S^{\dagger}x] - [S^{-1}S_{\mathbb{M}}S^{-1}S_{\mathbb{M}}x, S^{\dagger}x] = [S^{-1}S_{\mathbb{M}^c}x, S^{\dagger}x] - [S^{-1}S_{\mathbb{M}^c}S^{-1}S_{\mathbb{M}^c}x, S^{\dagger}x]$$

which gives

$$[S_{\mathbb{M}}x, x] - [S^{-1}S_{\mathbb{M}}x, S_{\mathbb{M}}^{\dagger}x] = [S_{\mathbb{M}^c}x, x] - [S^{-1}S_{\mathbb{M}^c}x, S_{\mathbb{M}^c}^{\dagger}x], \quad \forall x \in \mathscr{X}. \tag{4.5.1}$$

Now note that

$$\begin{aligned}\sum_{n=1}^{\infty}[x, \tilde{\omega}_n][\tilde{\tau}_n, y] &= \sum_{n=1}^{\infty}[x, (S^{-1})^{\dagger}\omega_n][S^{-1}\tau_n, y] = \sum_{n=1}^{\infty}[S^{-1}x, \omega_n][S^{-1}\tau_n, y]\\ &= \left[\sum_{n=1}^{\infty}[S^{-1}x, \omega_n]S^{-1}\tau_n, y\right] = \left[S^{-1}\left(\sum_{n=1}^{\infty}[S^{-1}x, \omega_n]\tau_n\right), y\right]\\ &= [S^{-1}x, y], \quad \forall x, y \in \mathscr{X}.\end{aligned}$$

Equation (4.5.1) now gives

$$\begin{aligned}\sum_{n\in\mathbb{M}}[x, \omega_n][\tau_n, x] - \sum_{n=1}^{\infty}[S_{\mathbb{M}}x, \tilde{\omega}_n][\tilde{\tau}_n, S_{\mathbb{M}}^{\dagger}x] &= \sum_{n\in\mathbb{M}^c}[x, \omega_n][\tau_n, x]\\ &\quad - \sum_{n=1}^{\infty}[S_{\mathbb{M}^c}x, \tilde{\omega}_n][\tilde{\tau}_n, S_{\mathbb{M}^c}^{\dagger}x], \qquad \forall x \in \mathscr{X}.\end{aligned}$$

□

A look at Theorem 1.2.48 makes a guess of the following statement for Banach spaces. Let $(\{\omega_n\}_n, \{\tau_n\}_n)$ be a Parseval p-ASF for $\mathscr{X}$. Then for every $\mathbb{M} \subseteq \mathbb{N}$,

$$\sum_{n\in\mathbb{M}}[x, \omega_n][\tau_n, x] - \left\|\sum_{n\in\mathbb{M}}[x, \omega_n]\tau_n\right\|^2 = \sum_{n\in\mathbb{M}^c}[x, \omega_n][\tau_n, x] - \left\|\sum_{n\in\mathbb{M}^c}[x, \omega_n]\tau_n\right\|^2, \quad \forall x \in \mathscr{X}. \tag{4.5.2}$$

However, the correct Banach space version of Theorem 1.2.48 is not Equation (4.5.2) but it is stated in the next theorem.

Theorem 4.5.7. ***(Parseval p-ASF identity)*** *Let $(\{\omega_n\}_n, \{\tau_n\}_n)$ be a Parseval p-ASF for $\mathscr{X}$. Then for every $\mathbb{M} \subseteq \mathbb{N}$,*

$$\begin{aligned}&\sum_{n\in\mathbb{M}}[x, \omega_n][\tau_n, x] - \sum_{n\in\mathbb{M}}\sum_{k\in\mathbb{M}}[x, \omega_n][\tau_n, \omega_k][\tau_k, x]\\ &\quad = \sum_{n\in\mathbb{M}^c}[x, \omega_n][\tau_n, x] - \sum_{n\in\mathbb{M}^c}\sum_{k\in\mathbb{M}^c}[x, \omega_n][\tau_n, \omega_k][\tau_k, x], \quad \forall x \in \mathscr{X}.\end{aligned}$$

Proof. Using Theorem 4.5.6, for all $x \in \mathscr{X}$,

$$\begin{aligned}
&\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]-\sum_{n\in\mathbb{M}}\sum_{k\in\mathbb{M}}[x,\omega_n][\tau_n,\omega_k][\tau_k,x]\\
&=\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]-\left[\sum_{n\in\mathbb{M}}[x,\omega_n]\sum_{k\in\mathbb{M}}[\tau_n,\omega_k]\tau_k,x\right]\\
&=\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]-\left[\sum_{n\in\mathbb{M}}[x,\omega_n]S_{\mathbb{M}}\tau_n,x\right]\\
&=\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]-\left[S_{\mathbb{M}}\left(\sum_{n\in\mathbb{M}}[x,\omega_n]\tau_n\right),x\right]\\
&=\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]-[S_{\mathbb{M}}S_{\mathbb{M}}x,x]\\
&=\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]-\left[S_{\mathbb{M}}x,S_{\mathbb{M}}^{\dagger}x\right]\\
&=\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]-\left[\sum_{n=1}^{\infty}[S_{\mathbb{M}}x,\omega_n]\tau_n,S_{\mathbb{M}}^{\dagger}x\right]\\
&=\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]-\sum_{n=1}^{\infty}[S_{\mathbb{M}}x,\omega_n][\tau_n,S_{\mathbb{M}}^{\dagger}x]\\
&=\sum_{n\in\mathbb{M}^{c}}[x,\omega_n][\tau_n,x]-\sum_{n=1}^{\infty}[S_{\mathbb{M}^{c}}x,\omega_n][\tau_n,S_{\mathbb{M}^{c}}^{\dagger}x]\\
&=\sum_{n\in\mathbb{M}^{c}}[x,\omega_n][\tau_n,x]-\sum_{n\in\mathbb{M}^{c}}\sum_{k\in\mathbb{M}^{c}}[x,\omega_n][\tau_n,\omega_k][\tau_k,x].
\end{aligned}$$

□

In terms of $S_{\mathbb{M}}$ and $S_{\mathbb{M}}^{c}$, Theorem 4.5.7 can be written as

$$S_{\mathbb{M}}-S_{\mathbb{M}}^{2}=S_{\mathbb{M}^{c}}-S_{\mathbb{M}^{c}}^{2}\quad\text{or}\quad S_{\mathbb{M}}+S_{\mathbb{M}^{c}}^{2}=S_{\mathbb{M}^{c}}+S_{\mathbb{M}}^{2}. \tag{4.5.3}$$

We now give an application of Theorem 4.5.7. This is Banach space version of Theorem 1.2.49.

Theorem 4.5.8. *Let $(\{\omega_n\}_n,\{\tau_n\}_n)$ be a Parseval p-ASF for $\mathscr{X}$. Let $\mathbb{M}\subseteq\mathbb{N}$. If $x\in\mathscr{X}$ is such that $[(S_{\mathbb{M}}-\frac{1}{2}I_{\mathscr{X}})^2x,x]\geq 0$, then*

$$\begin{aligned}
&\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]+\sum_{n\in\mathbb{M}^c}\sum_{k\in\mathbb{M}^c}[x,\omega_n][\tau_n,\omega_k][\tau_k,x]\\
&=\sum_{n\in\mathbb{M}^c}[x,\omega_n][\tau_n,x]+\sum_{n\in\mathbb{M}}\sum_{k\in\mathbb{M}}[x,\omega_n][\tau_n,\omega_k][\tau_k,x]\geq\frac{3}{4}\|x\|^2,\quad \textit{for that } x.
\end{aligned}$$

Proof. We first compute

$$S_{\mathbb{M}}^2+S_{\mathbb{M}^c}^2=S_{\mathbb{M}}^2+(I_{\mathscr{X}}-S_{\mathbb{M}})^2=2S_{\mathbb{M}}^2-2S_{\mathbb{M}}+I_{\mathscr{X}}$$
$$=2\left(S_{\mathbb{M}}-\frac{1}{2}I_{\mathscr{X}}\right)^2+\frac{1}{2}I_{\mathscr{X}}.$$

Hence if $x\in\mathscr{X}$ satisfies $[(S_{\mathbb{M}}-\frac{1}{2}I_{\mathscr{X}})^2x,x]\geq 0$, then

$$[(S_{\mathbb{M}}^2+S_{\mathbb{M}^c}^2)x,x]\geq\frac{1}{2}\|x\|^2.$$

Now using Equation (4.5.3) we get

$$2\sum_{n\in\mathbb{M}}[x,\omega_n][\tau_n,x]+2\sum_{n\in\mathbb{M}^c}\sum_{k\in\mathbb{M}^c}[x,\omega_n][\tau_n,\omega_k][\tau_k,x]=2[S_{\mathbb{M}}x,x]+2[S_{\mathbb{M}^c}^2x,x]$$
$$=[2(S_{\mathbb{M}}+S_{\mathbb{M}^c}^2)x,x]=[((S_{\mathbb{M}}+S_{\mathbb{M}^c}^2)+(S_{\mathbb{M}}+S_{\mathbb{M}^c}^2))x,x]$$
$$=[((S_{\mathbb{M}}+S_{\mathbb{M}^c}^2)+(S_{\mathbb{M}^c}+S_{\mathbb{M}}^2))x,x]=[(I_{\mathscr{X}}+S_{\mathbb{M}^c}^2+S_{\mathbb{M}}^2)x,x]$$
$$=\|x\|^2+[(S_{\mathbb{M}}^2+S_{\mathbb{M}^c}^2)x,x]\geq\frac{3}{2}\|x\|^2,\quad\forall x\in\mathscr{X}.$$

□

4.6 PALEY-WIENER THEOREM FOR p-APPROXIMATE SCHAUDER FRAMES

In order to derive Paley-Wiener theorem for p-ASFs, we need a generalization of result of Hilding (1948).

Theorem 4.6.1. *(Casazza and Christensen (1997); Casazza and Kalton (1999); van Eijndhoven (1996))* ***(Casazza-Christensen-Kalton-van Eijndhoven perturbation)*** *Let $\mathscr{X},\mathscr{Y}$ be Banach spaces and $A:\mathscr{X}\to\mathscr{Y}$ be a bounded invertible operator. If a bounded operator $B:\mathscr{X}\to\mathscr{Y}$ is such that there exist $\alpha,\beta\in[0,1)$ with*

$$\|Ax-Bx\|\leq\alpha\|Ax\|+\beta\|Bx\|,\quad\forall x\in\mathscr{X},$$

then B is invertible and

$$\frac{1-\alpha}{1+\beta}\|Ax\|\leq\|Bx\|\leq\frac{1+\alpha}{1-\beta}\|Ax\|,\quad\forall x\in\mathscr{X};$$

$$\frac{1-\beta}{1+\alpha}\frac{1}{\|A\|}\|y\|\leq\|B^{-1}y\|\leq\frac{1+\beta}{1-\alpha}\|A^{-1}\|\|y\|,\quad\forall y\in\mathscr{Y}.$$

In the sequel, the standard Schauder basis for $\ell^p(\mathbb{N})$ is denoted by $\{e_n\}_n$.

Theorem 4.6.2. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Assume that a collection $\{\tau_n\}_n$ in $\mathscr{X}$ is such that there exist $\alpha, \beta, \gamma \geq 0$ with $\max\{\alpha+\gamma\|\theta_f S_{f,\tau}^{-1}\|, \beta\} < 1$ and*

$$\left\|\sum_{n=1}^{m} c_n(\tau_n - \omega_n)\right\| \leq \alpha \left\|\sum_{n=1}^{m} c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^{m}|c_n|^p\right)^{\frac{1}{p}} + \beta\left\|\sum_{n=1}^{m} c_n\omega_n\right\|, \\ \forall c_1,\dots,c_m \in \mathbb{K},\ m=1,2,\dots. \tag{4.6.1}$$

Then $(\{f_n\}_n, \{\omega_n\}_n)$ is a p-ASF for $\mathscr{X}$ with bounds

$$\frac{1-(\alpha+\gamma\|\theta_f S_{f,\tau}^{-1}\|)}{(1+\beta)\|S_{f,\tau}^{-1}\|} \quad \text{and} \quad \left(\frac{1+\alpha}{1-\beta}\|\theta_\tau\| + \frac{\gamma}{1-\beta}\right)\|\theta_f\|.$$

Proof. For $m = 1,2,\dots$ and for every $c_1,\dots,c_m \in \mathbb{K}$,

$$\begin{aligned}\left\|\sum_{n=1}^{m} c_n\omega_n\right\| &\leq \left\|\sum_{n=1}^{m} c_n(\tau_n-\omega_n)\right\| + \left\|\sum_{n=1}^{m} c_n\tau_n\right\| \\ &\leq (1+\alpha)\left\|\sum_{n=1}^{m} c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^{m}|c_n|^p\right)^{\frac{1}{p}} + \beta\left\|\sum_{n=1}^{m} c_n\omega_n\right\|.\end{aligned}$$

Hence

$$\left\|\sum_{n=1}^{m} c_n\omega_n\right\| \leq \frac{1+\alpha}{1-\beta}\left\|\sum_{n=1}^{m} c_n\tau_n\right\| + \frac{\gamma}{1-\beta}\left(\sum_{n=1}^{m}|c_n|^p\right)^{\frac{1}{p}}, \quad \forall c_1,\dots,c_m \in \mathbb{K}, m=1,2,\dots.$$

Therefore θ_ω is well-defined bounded linear operator with

$$\|\theta_\omega\| \leq \frac{1+\alpha}{1-\beta}\|\theta_\tau\| + \frac{\gamma}{1-\beta}.$$

Now Equation (4.6.1) gives

$$\left\|\sum_{n=1}^{\infty} c_n(\tau_n-\omega_n)\right\| \leq \alpha\left\|\sum_{n=1}^{\infty} c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^{\infty}|c_n|^p\right)^{\frac{1}{p}} + \beta\left\|\sum_{n=1}^{\infty} c_n\omega_n\right\|, \quad \forall \{c_n\}_n \in \ell^p(\mathbb{N}).$$

That is,

$$\|\theta_\tau\{c_n\}_n - \theta_\omega\{c_n\}_n\| \leq \alpha\|\theta_\tau\{c_n\}_n\| + \gamma\left(\sum_{n=1}^{\infty}|c_n|^p\right)^{\frac{1}{p}} + \beta\|\theta_\omega\{c_n\}_n\|,$$

$$\forall \{c_n\}_n \in \ell^p(\mathbb{N}). \tag{4.6.2}$$

By taking $\{c_n\}_n = \{f_n(S_{f,\tau}^{-1}x)\}_n = \theta_f S_{f,\tau}^{-1}x$ in Equation (4.6.2), we get

$$\|\theta_\tau\theta_f S_{f,\tau}^{-1}x - \theta_\omega\theta_f S_{f,\tau}^{-1}x\| \leq \alpha\|\theta_\tau\theta_f S_{f,\tau}^{-1}x\| + \gamma\left(\sum_{n=1}^{\infty}|f_n(S_{f,\tau}^{-1}x)|^p\right)^{\frac{1}{p}} + \beta\|\theta_\omega\theta_f S_{f,\tau}^{-1}x\|,$$

for all $x \in \mathscr{X}$. That is,

$$\begin{aligned}\|x - S_{f,\omega}S_{f,\tau}^{-1}x\| &\leq \alpha\|x\| + \gamma\|\theta_f S_{f,\tau}^{-1}x\| + \beta\|S_{f,\omega}S_{f,\tau}^{-1}x\| \\ &\leq (\alpha + \gamma\|\theta_f S_{f,\tau}^{-1}\|)\|x\| + \beta\|S_{f,\omega}S_{f,\tau}^{-1}x\|, \quad \forall x \in \mathscr{X}.\end{aligned}$$

Since $\max\{\alpha+\gamma\|\theta_f S_{f,\tau}^{-1}\|, \beta\} < 1$, we can use Theorem 4.6.1 to get the operator $S_{f,\omega}S_{f,\tau}^{-1}$ to be invertible and

$$\|(S_{g,\omega}S_{f,\tau}^{-1})^{-1}\| \leq \frac{1+\beta}{1-(\alpha+\gamma\|\theta_f S_{f,\tau}^{-1}\|)}.$$

Hence the operator $S_{f,\omega} = (S_{f,\omega}S_{f,\tau}^{-1})S_{f,\tau}$ is invertible. Therefore $(\{f_n\}_n, \{\omega_n\}_n)$ is a p-ASF for $\mathscr{X}$. We get the frame bounds from the following calculations:

$$\begin{aligned}&\|S_{f,\omega}^{-1}\| \leq \|S_{f,\tau}^{-1}\|\|S_{f,\tau}S_{f,\omega}^{-1}\| \leq \frac{\|S_{f,\tau}^{-1}\|(1+\beta)}{1-(\alpha+\gamma\|\theta_f S_{f,\tau}^{-1}\|)} \quad \text{and} \\ &\|S_{f,\omega}\| \leq \|\theta_\omega\|\|\theta_f\| \leq \left(\frac{1+\alpha}{1-\beta}\|\theta_\tau\| + \frac{\gamma}{1-\beta}\right)\|\theta_f\|.\end{aligned}$$

□

Remark 4.6.3. *Theorem 1.2.55 is a corollary of Theorem 4.6.2. In particular, Theorems 1.2.53 and 1.2.54 are corollaries of Theorem 4.6.2. Indeed, let $\{\tau_n\}_n$ be a frame for $\mathscr{H}$. We define*

$$f_n : \mathscr{H} \ni h \mapsto f_n(h) := \langle h, \tau_n\rangle \in \mathbb{K}, \quad \forall n \in \mathbb{N}.$$

Then $\theta_f = \theta_\tau$ and $(\{f_n\}_n, \{\tau_n\}_n)$ is a 2-approximate frame for $\mathscr{H}$. Theorem 4.6.2 now says that $(\{f_n\}_n, \{\omega_n\}_n)$ is a 2-ASF for $\mathscr{H}$. To prove Theorem 1.2.55, it now suffices to prove that $\{\omega_n\}_n$ is a frame for $\mathscr{H}$. Since $(\{f_n\}_n, \{\omega_n\}_n)$ is a 2-ASF for $\mathscr{H}$, it follows that θ_ω is surjective. We now use the following result to conclude that $\{\omega_n\}_n$ is a frame for $\mathscr{H}$.

Theorem 4.6.4. *(Christensen (1995b)) A collection $\{\tau_n\}_n$ is a frame for $\mathscr{H}$ if and only* *if the map*

$$T : \ell^2(\mathbb{N}) \ni \{c_n\}_n \mapsto \sum_{n=1}^{\infty} c_n\tau_n \in \mathscr{H}$$

is a well-defined bounded linear surjective operator.

Corollary 4.6.5. *Let q be the conjugate index of p. Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Assume that a collection $\{\tau_n\}_n$ in $\mathscr{X}$ is such that and*

$$\lambda := \sum_{n=1}^{\infty} \|\tau_n - \omega_n\|^q < \frac{1}{\|\theta_f S_{f,\tau}^{-1}\|^q}.$$

Then $(\{f_n\}_n, \{\omega_n\}_n)$ is a p-ASF for $\mathscr{X}$ with bounds

$$\frac{1 - \lambda^{1/p}\|\theta_f S_{f,\tau}^{-1}\|}{\|S_{f,\tau}^{-1}\|} \quad \textit{and} \quad (\|\theta_\tau\| + \lambda^{1/p}).$$

Proof. Take $\alpha = 0, \beta = 0, \gamma = \lambda^{1/p}$. Then $\max\{\alpha + \gamma\|\theta_f S_{f,\tau}^{-1}\|, \beta\} < 1$ and

$$\left\|\sum_{n=1}^{m} c_n(\tau_n - \omega_n)\right\| \leq \left(\sum_{n=1}^{m} \|\tau_n - \omega_n\|^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{m} |c_n|^p\right)^{\frac{1}{p}} \leq \gamma \left(\sum_{n=1}^{m} |c_n|^p\right)^{\frac{1}{p}}, \quad \forall c_1, \dots, c_m \in \mathbb{K},\ m = 1, 2, \dots.$$

By using Theorem 4.6.2 we now get the result. □

We next derive stability result which does not demand maximum condition on parameters α and γ.

Theorem 4.6.6. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathscr{X}$. Assume that a collection $\{\tau_n\}_n$ in $\mathscr{X}$ and a collection $\{g_n\}_n$ in $\mathscr{X}^*$ are such that there exist $r, s, t, \alpha, \beta, \gamma \geq 0$ with $\max\{\beta, s\} < 1$ and*

$$\left\|\sum_{n=1}^{m} (f_n - g_n)(x)e_n\right\| \leq r\left\|\sum_{n=1}^{m} f_n(x)e_n\right\| + t\|x\| + s\left\|\sum_{n=1}^{m} g_n(x)e_n\right\|, \quad \forall x \in \mathscr{X}, m = 1, 2, \dots,$$

$$\left\|\sum_{n=1}^{m} c_n(\tau_n - \omega_n)\right\| \leq \alpha\left\|\sum_{n=1}^{m} c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^{m} |c_n|^p\right)^{\frac{1}{p}} + \beta\left\|\sum_{n=1}^{m} c_n\omega_n\right\|, \quad \forall c_1, \dots, c_m \in \mathbb{K}, m = 1, 2, \dots.$$

Assume that one of the following holds.

(i) $\sum_{n=1}^{\infty}(\|f_n-g_n\|\|S_{f,\tau}^{-1}\tau_n\|+\|g_n\|\|S_{f,\tau}^{-1}(\tau_n-\omega_n)\|)<1.$

(ii) $\sum_{n=1}^{\infty}(\|f_n-g_n\|\|S_{f,\tau}^{-1}\omega_n\|+\|f_n\|\|S_{f,\tau}^{-1}(\tau_n-\omega_n)\|)<1.$

(iii) $\sum_{n=1}^{\infty}(\|(f_n-g_n)S_{f,\tau}^{-1}\|\|\tau_n\|+\|g_nS_{f,\tau}^{-1}\|\|\tau_n-\omega_n\|)<1.$

(iv) $\sum_{n=1}^{\infty}(\|(f_n-g_n)S_{f,\tau}^{-1}\|\|\omega_n\|+\|f_nS_{f,\tau}^{-1}\|\|\tau_n-\omega_n\|)<1.$

Then $(\{g_n\}_n,\{\omega_n\}_n)$ *is a p-ASF for* $\mathscr{X}$. *Moreover, an upper bound is*

$$\left(\frac{1+\alpha}{1-\beta}\|\theta_\tau\|+\frac{\gamma}{1-\beta}\right)\left(\frac{1+r}{1-s}\|\theta_f\|+\frac{t}{1-s}\right).$$

Proof. Following the initial lines in the proof of Theorem 4.6.2, we see that θ_g and θ_ω are well-defined bounded linear operators. We now consider four cases.
Assume (i). Then

$$\begin{aligned}\left\|x-\sum_{n=1}^{\infty}g_n(x)S_{f,\tau}^{-1}\omega_n\right\| &= \left\|\sum_{n=1}^{\infty}f_n(x)S_{f,\tau}^{-1}\tau_n-\sum_{n=1}^{\infty}g_n(x)S_{f,\tau}^{-1}\omega_n\right\|\\
&\leq \sum_{n=1}^{\infty}\|f_n(x)S_{f,\tau}^{-1}\tau_n-g_n(x)S_{f,\tau}^{-1}\omega_n\|\\
&\leq \sum_{n=1}^{\infty}\left\{\|f_n(x)S_{f,\tau}^{-1}\tau_n-g_n(x)S_{f,\tau}^{-1}\tau_n\|+\|g_n(x)S_{f,\tau}^{-1}\tau_n-g_n(x)S_{f,\tau}^{-1}\omega_n\|\right\}\\
&= \sum_{n=1}^{\infty}\left\{\|(f_n-g_n)(x)S_{f,\tau}^{-1}\tau_n\|+\|g_n(x)S_{f,\tau}^{-1}(\tau_n-\omega_n)\|\right\}\\
&\leq \left(\sum_{n=1}^{\infty}\left\{\|f_n-g_n\|\|S_{f,\tau}^{-1}\tau_n\|+\|g_n\|\|S_{f,\tau}^{-1}(\tau_n-\omega_n)\|\right\}\right)\|x\|.\end{aligned}$$

Therefore the operator $S_{f,\tau}^{-1}S_{g,\omega}$ is invertible.
Assume (ii). Then

$$\begin{aligned}\left\|x-\sum_{n=1}^{\infty}g_n(x)S_{f,\tau}^{-1}\omega_n\right\| &= \left\|\sum_{n=1}^{\infty}f_n(x)S_{f,\tau}^{-1}\tau_n-\sum_{n=1}^{\infty}g_n(x)S_{f,\tau}^{-1}\omega_n\right\|\\
&\leq \sum_{n=1}^{\infty}\|f_n(x)S_{f,\tau}^{-1}\tau_n-g_n(x)S_{f,\tau}^{-1}\omega_n\|\\
&\leq \sum_{n=1}^{\infty}\left\{\|f_n(x)S_{f,\tau}^{-1}\tau_n-f_n(x)S_{f,\tau}^{-1}\omega_n\|+\|f_n(x)S_{f,\tau}^{-1}\omega_n-g_n(x)S_{f,\tau}^{-1}\omega_n\|\right\}\\
&= \sum_{n=1}^{\infty}\left\{\|f_n(x)S_{f,\tau}^{-1}(\tau_n-\omega_n)\|+\|(f_n-g_n)(x)S_{f,\tau}^{-1}\omega_n\|\right\}\end{aligned}$$

$$\leq \left(\sum_{n=1}^{\infty} \left\{ \|f_n\| \|S_{f,\tau}^{-1}(\tau_n - \omega_n)\| + \|f_n - g_n\| \|S_{f,\tau}^{-1}\omega_n\| \right\} \right) \|x\|.$$

Therefore the operator $S_{f,\tau}^{-1}S_{g,\omega}$ is invertible.
Assume (iii). Then

$$\begin{aligned}
&\left\| x - \sum_{n=1}^{\infty} g_n(S_{f,\tau}^{-1}x)\omega_n \right\| = \left\| \sum_{n=1}^{\infty} f_n(S_{f,\tau}^{-1}x)\tau_n - \sum_{n=1}^{\infty} g_n(S_{f,\tau}^{-1}x)\omega_n \right\| \\
&\leq \sum_{n=1}^{\infty} \|f_n(S_{f,\tau}^{-1}x)\tau_n - g_n(S_{f,\tau}^{-1}x)\omega_n\| \\
&\leq \sum_{n=1}^{\infty} \left\{ \|f_n(S_{f,\tau}^{-1}x)\tau_n - g_n(S_{f,\tau}^{-1}x)\tau_n\| + \|g_n(S_{f,\tau}^{-1}x)\tau_n - g_n(S_{f,\tau}^{-1}x)\omega_n\| \right\} \\
&= \sum_{n=1}^{\infty} \left\{ \|(f_n - g_n)(S_{f,\tau}^{-1}x)\tau_n\| + \|g_n(S_{f,\tau}^{-1}x)(\tau_n - \omega_n)\| \right\} \\
&\leq \left(\sum_{n=1}^{\infty} \left\{ \|(f_n - g_n)S_{f,\tau}^{-1}\| \|\tau_n\| + \|g_nS_{f,\tau}^{-1}\| \|\tau_n - \omega_n\| \right\} \right) \|x\|.
\end{aligned}$$

Therefore the operator $S_{g,\omega}S_{f,\tau}^{-1}$ is invertible.
Assume (iv). Then

$$\begin{aligned}
&\left\| x - \sum_{n=1}^{\infty} g_n(S_{f,\tau}^{-1}x)\omega_n \right\| = \left\| \sum_{n=1}^{\infty} f_n(S_{f,\tau}^{-1}x)\tau_n - \sum_{n=1}^{\infty} g_n(S_{f,\tau}^{-1}x)\omega_n \right\| \\
&\leq \sum_{n=1}^{\infty} \|f_n(S_{f,\tau}^{-1}x)\tau_n - g_n(S_{f,\tau}^{-1}x)\omega_n\| \\
&\leq \sum_{n=1}^{\infty} \left\{ \|f_n(S_{f,\tau}^{-1}x)\tau_n - f_n(S_{f,\tau}^{-1}x)\omega_n\| + \|f_n(S_{f,\tau}^{-1}x)\omega_n - g_n(S_{f,\tau}^{-1}x)\omega_n\| \right\} \\
&= \sum_{n=1}^{\infty} \left\{ \|f_n(S_{f,\tau}^{-1}x)(\tau_n - \omega_n)\| + \|(f_n - g_n)(S_{f,\tau}^{-1}x)\omega_n\| \right\} \\
&\leq \left(\sum_{n=1}^{\infty} \left\{ \|f_nS_{f,\tau}^{-1}\| \|\tau_n - \omega_n\| + \|(f_n - g_n)S_{f,\tau}^{-1}\| \|\omega_n\| \right\} \right) \|x\|.
\end{aligned}$$

Therefore the operator $S_{g,\omega}S_{f,\tau}^{-1}$ is invertible.

Hence in each of the assumptions we get that $(\{g_n\}_n, \{\omega_n\}_n)$ is a p-ASF for $\mathscr{X}$. $\square$

We end this chapter by deriving results on the expansion of sequences to approximate Schauder frames.

A routine Hilbert space argument shows that a sequence $\{\tau_n\}_n$ is a Bessel sequence for Hilbert space $\mathscr{H}$ if and only if the map $S_\tau : \mathscr{H} \ni h \mapsto \sum_{n=1}^{\infty} \langle h, \tau_n \rangle \tau_n \in \mathscr{H}$ is a well-defined bounded linear operator. In fact, if $\{\tau_n\}_n$ is a Bessel sequence, then both maps $\theta_\tau : \mathscr{H} \ni h \mapsto \theta_\tau h := \{\langle h, \tau_n\rangle\}_n \in \ell^2(\mathbb{N})$ and $\theta_\tau^* : \ell^2(\mathbb{N}) \ni \{a_n\}_n \mapsto \theta_\tau^*\{a_n\}_n := \sum_{n=1}^\infty a_n\tau_n \in \mathscr{H}$ are well-defined bounded linear operators (Chapter 3 in Christensen (2016)). Now $\theta_\tau^*\theta_\tau = S_\tau$ and hence S_τ is a well-defined bounded linear operator. Conversely, let S_τ be a well-defined bounded linear operator. Definition of S_τ says that it is a positive operator. Thus there exists $b > 0$ such that $\langle S_\tau h, h\rangle \leq b\|h\|^2, \forall h \in \mathscr{H}$. Again using the definition of S_τ gives that $\{\tau_n\}_n$ is a Bessel sequence. This observation and Definition 4.1.2 make us to define the following.

Definition 4.6.7. *Let $\{\tau_n\}_n$ be a sequence in a Banach space $\mathscr{X}$ and $\{f_n\}_n$ be a sequence in $\mathscr{X}^*$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be a **weak reconstruction sequence** or **approximate Bessel sequence** (ABS) for $\mathscr{X}$ if $S_{f,\tau} : \mathscr{X} \ni x \mapsto S_{f,\tau}x := \sum_{n=1}^\infty f_n(x)\tau_n \in \mathscr{X}$ is a well-defined bounded linear operator.*

We next recall the reconstruction property of Banach spaces.

Definition 4.6.8. *(Casazza and Christensen (2008)) A Banach space $\mathscr{X}$ is said to have the **reconstruction property** if there exists a sequence $\{\tau_n\}_n$ in $\mathscr{X}$ and a sequence $\{f_n\}_n$ in $\mathscr{X}^*$ such that $x = \sum_{n=1}^\infty f_n(x)\tau_n, \forall x \in \mathscr{X}$.*

Using **approximation property of Banach spaces** (cf. Casazza (2001)), Casazza and Christensen proved the following result.

Theorem 4.6.9. *(Casazza and Christensen (2008)) There exists a Banach space $\mathscr{X}$ such that $\mathscr{X}$ does not have the reconstruction property.*

Now we have the following characterization. This is a result which is in contrast with Theorem 1.2.59.

Theorem 4.6.10. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a weak reconstruction sequence for $\mathscr{X}$. Then the following are equivalent.*

(i) $(\{f_n\}_n, \{\tau_n\}_n)$ can be expanded to an ASF for $\mathscr{X}$.

(ii) $\mathscr{X}$ has the reconstruction property.

Proof. (i) $\Rightarrow$ (ii) Let $\{\omega_n\}_n$ be a sequence in $\mathscr{X}$ and $\{g_n\}_n$ be a sequence in $\mathscr{X}^*$ such that $(\{f_n\}_n \cup \{g_n\}_n, \{\tau_n\}_n \cup \{\omega_n\}_n)$ is an ASF for $\mathscr{X}$. Let $S_{(f,g),(\tau,\omega)}$ be the frame operator for $(\{f_n\}_n \cup \{g_n\}_n, \{\tau_n\}_n \cup \{\omega_n\}_n)$. Then

$$x = S_{(f,g),(\tau,\omega)}^{-1}S_{(f,g),(\tau,\omega)}x = S_{(f,g),(\tau,\omega)}^{-1}\left(\sum_{n=1}^\infty f_n(x)\tau_n + \sum_{n=1}^\infty g_n(x)\omega_n\right)$$

$$= \sum_{n=1}^{\infty} f_n(x) S^{-1}_{(f,g),(\tau,\omega)} \tau_n + \sum_{n=1}^{\infty} g_n(x) S^{-1}_{(f,g),(\tau,\omega)} \omega_n, \quad \forall x \in \mathscr{X}$$

which shows that $\mathscr{X}$ has the reconstruction property.

(ii) $\Rightarrow$ (i) $\{\omega_n\}_n$ be a sequence in $\mathscr{X}$ and $\{g_n\}_n$ be a sequence in $\mathscr{X}^*$ such that $x = \sum_{n=1}^{\infty} g_n(x)\omega_n$, $\forall x \in \mathscr{X}$. Define $h_n := g_n$, $\rho_n := (I_{\mathscr{X}} - S_{f,\tau})\omega_n$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} f_n(x)\tau_n + \sum_{n=1}^{\infty} h_n(x)\rho_n &= \sum_{n=1}^{\infty} f_n(x)\tau_n + \sum_{n=1}^{\infty} g_n(x)(I_{\mathscr{X}} - S_{f,\tau})\omega_n \\
&= S_{f,\tau}x + (I_{\mathscr{X}} - S_{f,\tau})\left(\sum_{n=1}^{\infty} g_n(x)\omega_n\right) \\
&= S_{f,\tau}x + (I_{\mathscr{X}} - S_{f,\tau})x = x, \quad \forall x \in \mathscr{X}.
\end{aligned}$$

Therefore $(\{f_n\}_n \cup \{h_n\}_n, \{\tau_n\}_n \cup \{\rho_n\}_n)$ is an ASF for $\mathscr{X}$.

□

We now show that there are infinitely many ways to expand a weak reconstruction sequence into an ASF.

Corollary 4.6.11. *There exists a Banach space $\mathscr{X}$ such that given any weak reconstruction sequence $(\{f_n\}_n, \{\tau_n\}_n)$ for $\mathscr{X}$, $(\{f_n\}_n, \{\tau_n\}_n)$ can not be expanded to an ASF for $\mathscr{X}$.*

Proof. From Theorem 4.6.9, there exists a Banach space $\mathscr{X}$ which does not have the reconstruction property. Let $(\{f_n\}_n, \{\tau_n\}_n)$ be any weak reconstruction sequence for $\mathscr{X}$. Theorem 4.6.10 now says that $(\{f_n\}_n, \{\tau_n\}_n)$ can not be expanded to an ASF for $\mathscr{X}$. □

Following corollary is an easy consequence of Theorem 4.6.10.

Corollary 4.6.12. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a weak reconstruction sequence for $\mathscr{X}$. If $\mathscr{X}$ admits a Schauder basis, then $(\{f_n\}_n, \{\tau_n\}_n)$ can be expanded to an ASF for $\mathscr{X}$.*

Note that Theorem 4.6.10 may not add countably many elements to a weak reconstruction sequence to get an ASF. In the following example we show that it adds just one element to a weak reconstruction sequence and yields an ASF.

Example 4.6.13. *Let $p \in [1, \infty)$ and let $\{e_n\}_n$ denote the standard Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Define*

$$R : \ell^p(\mathbb{N}) \ni (x_n)_{n=1}^{\infty} \mapsto (0, x_1, x_2, \dots) \in \ell^p(\mathbb{N}),$$

$$L : \ell^p(\mathbb{N}) \ni (x_n)_{n=1}^\infty \mapsto (x_2, x_3, x_4, \dots) \in \ell^p(\mathbb{N}).$$

Clearly $(\{f_n := \zeta_n L\}_n, \{\tau_n := Re_n\}_n)$ *is a weak reconstruction sequence for* $\ell^p(\mathbb{N})$*. Note that* $S_{f,\tau} = RL$ *and*

$$(I_{\ell^p(\mathbb{N})} - S_{f,\tau})e_1 = e_1 - RLe_1 = e_1 - 0 = e_1,$$
$$(I_{\ell^p(\mathbb{N})} - S_{f,\tau})e_n = e_n - RLe_n = e_n - Re_{n-1} = e_n - e_n = 0, \quad \forall n \geq 2.$$

Let $g_n := \zeta_n$ *and* $\omega_n := e_n$*,* $\forall n \in \mathbb{N}$*. Theorem 4.6.10 now says that* $(\{f_n\}_n \cup \{h_1\}, \{\tau_n\}_n \cup \{\rho_1\})$ *is an ASF for* $\ell^p(\mathbb{N})$*.*

It may be possible to expand a weak reconstruction sequence to a tight ASF by adding finitely many elements Hilbert space. In this case, we can estimate the number of elements added to a tight ASF. This is given in the following theorem which can be compared with Theorem 1.2.60.

Theorem 4.6.14. *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a weak reconstruction sequence for* $\mathscr{X}$*. If* $(\{f_n\}_n \cup \{g_k\}_{k=1}^N, \{\tau_n\}_n \cup \{\omega_k\}_{k=1}^N)$ *is a* λ*-tight ASF for* $\mathscr{X}$*, then*

$$N \geq \dim(\lambda I_{\mathscr{X}} - S_{f,\tau})(\mathscr{X}). \tag{4.6.3}$$

Further, the Inequality (4.6.3) can not be improved.

Proof. Let $S_{(f,g),(\tau,\omega)}$ be the frame operator for $(\{f_n\}_n \cup \{g_k\}_{k=1}^N, \{\tau_n\}_n \cup \{\omega_k\}_{k=1}^N)$. Set $S_{g,\omega}(x) := \sum_{k=1}^N g_k(x)\omega_k, \forall x \in \mathscr{X}$. Then

$$\lambda x = S_{(f,g),(\tau,\omega)}x = \sum_{n=1}^\infty f_n(x)\tau_n + \sum_{k=1}^N g_k(x)\omega_k = S_{f,\tau}x + S_{g,\omega}x, \quad \forall x \in \mathscr{X}.$$

Therefore

$$N \geq \dim S_{g,\omega}(\mathscr{X}) = \dim(\lambda I_{\mathscr{X}} - S_{f,\tau})(\mathscr{X}).$$

Example 4.6.13 says that inequality in Theorem 4.6.14 can not be improved. □

We now state the definition of a p-weak reconstruction sequence and give a extension theorem for p-weak reconstruction sequences.

Definition 4.6.15. *Let* $p \in [1, \infty)$*. A weak reconstruction sequence* $(\{f_n\}_n, \{\tau_n\}_n)$ *for* $\mathscr{X}$ *is said to be a* ***p-weak reconstruction sequence*** *or* ***p-approximate Bessel sequence*** *(p-ABS) for* $\mathscr{X}$ *if both the maps* $\theta_f : \mathscr{X} \ni x \mapsto \theta_f x := \{f_n(x)\}_n \in \ell^p(\mathbb{N})$ *and* $\theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \theta_\tau\{a_n\}_n := \sum_{n=1}^\infty a_n\tau_n \in \mathscr{X}$ *are well-defined bounded linear operators.*

Theorem 4.6.16. *Let $p \in [1,\infty)$. If $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-weak reconstruction sequence for $\ell^p(\mathbb{N})$, then $(\{f_n\}_n, \{\tau_n\}_n)$ can be expanded to a p-ASF.*

Proof. Let $\{e_n\}_n$ and $\{\zeta_n\}_n$ be as in Example 4.6.13. Define $h_n := \zeta_n$, $\rho_n := (I_{\ell^p(\mathbb{N})} - S_{f,\tau})e_n$, for all $n \in \mathbb{N}$. Then it follows that $(\{f_n\}_n \cup \{h_n\}_n, \{\tau_n\}_n \cup \{\rho_n\}_n)$ is a p-ASF for $\ell^p(\mathbb{N})$. □

CHAPTER 5

WEAK OPERATOR-VALUED FRAMES

5.1 BASIC PROPERTIES

Let $\mathscr{H}, \mathscr{H}_0$ be Hilbert spaces and $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ be the collection of all bounded linear operators from $\mathscr{H}$ to $\mathscr{H}_0$. In this chapter, we study a generalization of the notion of operator-valued frame by studying the convergence of the series $\sum_{n=1}^{\infty} \Psi_n^* A_n$ to a bounded invertible operator in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$.

Definition 5.1.1. *Let $\{A_n\}_n$ and $\{\Psi_n\}_n$ be collections in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. The pair $(\{A_n\}_n, \{\Psi_n\}_n)$ is said to be a* ***weak operator-valued frame*** *(weak OVF) in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ if the series*

$$\textit{(Operator-valued frame operator)} \quad S_{A,\Psi} := \sum_{n=1}^{\infty} \Psi_n^* A_n$$

converges in the strong-operator topology on $\mathscr{B}(\mathscr{H})$ to a bounded invertible operator. If $S_{A,\Psi} = I_{\mathscr{H}}$, then it is called as a Parseval weak OVF.

We now give various examples of weak OVFs.

Example 5.1.2. *(i) By taking $\Psi_n := A_n$, for all $n \in \mathbb{N}$, it follows that an* ***operator-valued frame*** *is a weak OVF. In particular, a* ***G-frame*** *is a weak OVF.*

(ii) Let $(\{\tau_n\}_n, \{f_n\}_n)$ be a ***pseudo-frame*** *for $\mathscr{H}$. If we define $\Psi_n := f_n$ and $A_n h := \langle h, \tau_n \rangle$, for all $n \in \mathbb{N}$, for all $h \in \mathscr{H}$, then $(\{A_n\}_n, \{\Psi_n\}_n)$ is a weak OVF in $\mathscr{B}(\mathscr{H}, \mathbb{K})$. Similarly it follows that* ***frames for subspaces***, ***fusion frames***, ***outer frames***, ***oblique frames*** *and* ***quasi-projectors*** *are all weak OVFs.*

(iii) Let $C \in \mathscr{B}(\mathscr{H})$ be invertible and $\{\tau_n\}_n$ be a ***C-controlled frame*** *for $\mathscr{H}$ (Balazs et al. (2010)). If we define $\Psi_n h := \langle h, C\tau_n \rangle$ and $A_n h := \langle h, \tau_n \rangle$, for all $n \in \mathbb{N}$, for*

all $h \in \mathscr{H}$, then $(\{A_n\}_n, \{\Psi_n\}_n)$ is a weak OVF in $\mathscr{B}(\mathscr{H}, \mathbb{K})$. In particular, every ***weighted frame*** *is a weak OVF.*

(iv) Let $(\{\tau_n\}_n, \{f_n\}_n)$ be an ***approximate Schauder frame*** *for $\mathscr{H}$ (Freeman et al. (2014)). Note that it is possible for Hilbert spaces to have approximate Schauder frames which are not frames. If we define $\Psi_n := f_n$ and $A_n h := \langle h, \tau_n \rangle$, for all $n \in \mathbb{N}$, for all $h \in \mathscr{H}$, then $(\{A_n\}_n, \{\Psi_n\}_n)$ is a weak OVF in $\mathscr{B}(\mathscr{H}, \mathbb{K})$. In particular,* ***atomic decompositions*** *(Casazza et al. (1999)),* ***framings*** *(Casazza et al. (1999)),* ***cb-frames*** *(Liu and Ruan (2016)) and* ***Schauder frames*** *(Casazza et al. (2008a)) for Hilbert spaces are all weak OVFs.*

(v) Let $\{\tau_n\}_n$ be a ***signed frame*** *for $\mathscr{H}$ with signs $\{\sigma_n\}_n$ (Peng and Waldron (2002)). If we define $\Psi_n := h := \langle h, \sigma_n \tau_n \rangle$ and $A_n h := \langle h, \tau_n \rangle$, for all $n \in \mathbb{N}$, for all $h \in \mathscr{H}$, then $(\{A_n\}_n, \{\Psi_n\}_n)$ is a weak OVF in $\mathscr{B}(\mathscr{H}, \mathbb{K})$.*

Unlike in the case of OVFs, the frame operator $S_{A,\Psi}$ need not be positive. Since $S_{A,\Psi}$ is invertible, there are $a, b > 0$ such that

$$a\|h\| \leq \|S_{A,\Psi} h\| \leq b\|h\|, \quad \forall h \in \mathscr{H}.$$

We call such a, b as lower and upper frame bounds, respectively. Supremum of the set of all lower frame bounds is called as optimal lower frame bound and infimum of the set of all upper frame bounds is called as optimal upper frame bound. We easily get that

$$\begin{aligned} \text{optimal lower frame bound } &= \|S_{A,\Psi}^{-1}\|^{-1}, \\ \text{optimal upper frame bound } &= \|S_{A,\Psi}\|. \end{aligned}$$

We now define the notion of dual weak OVFs.

Definition 5.1.3. *A weak OVF $(\{B_n\}_n, \{\Phi_n\}_n)$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is said to be a* ***dual*** *for a weak OVF $(\{A_n\}_n, \{\Psi_n\}_n)$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ if*

$$\sum_{n=1}^{\infty} \Psi_n^* B_n = \sum_{n=1}^{\infty} \Phi_n^* A_n = I_{\mathscr{H}}.$$

Note that dual always exists for a given weak OVF. In fact, a direct calculation shows that

$$(\{\widetilde{A}_n := A_n S_{A,\Psi}^{-1}\}_n, \{\widetilde{\Psi}_n := \Psi_n (S_{A,\Psi}^{-1})^*\}_n)$$

is a weak OVF and is a dual for $(\{A_n\}_n, \{\Psi_n\}_n)$. This weak OVF is called as the **canonical dual** for $(\{A_n\}_n, \{\Psi_n\}_n)$. Canonical dual has two nice properties. Following two results establish them.

Proposition 5.1.4. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. If $h \in \mathscr{H}$ has representation $h = \sum_{n=1}^\infty A_n^* y_n = \sum_{n=1}^\infty \Psi_n^* z_n$, for some sequences $\{y_n\}_n, \{z_n\}_n$ in $\mathscr{H}_0$, then*

$$\sum_{n=1}^\infty \langle y_n, z_n\rangle = \sum_{n=1}^\infty \langle \widetilde{\Psi}_n h, \widetilde{A}_n h\rangle + \sum_{n=1}^\infty \langle y_n - \widetilde{\Psi}_n h, z_n - \widetilde{A}_n h\rangle.$$

Proof. We start from the right side and see

$$\begin{aligned}
&\sum_{n=1}^\infty \langle \widetilde{\Psi}_n h, \widetilde{A}_n h\rangle + \sum_{n=1}^\infty \langle y_n, z_n\rangle - \sum_{n=1}^\infty \langle y_n, \widetilde{A}_n h\rangle - \sum_{n=1}^\infty \langle \widetilde{\Psi}_n h, z_n\rangle + \sum_{n=1}^\infty \langle \widetilde{\Psi}_n h, \widetilde{A}_n h\rangle \\
&= 2\sum_{n=1}^\infty \langle \widetilde{\Psi}_n h, \widetilde{A}_n h\rangle + \sum_{n=1}^\infty \langle y_n, z_n\rangle - \sum_{n=1}^\infty \langle y_n, A_n S_{A,\Psi}^{-1} h\rangle - \sum_{n=1}^\infty \langle \Psi_n (S_{A,\Psi}^{-1})^* h, z_n\rangle \\
&= 2\left\langle \sum_{n=1}^\infty (S_{A,\Psi}^{-1})^* A_n^* \Psi_n (S_{A,\Psi}^{-1})^* h, h\right\rangle + \sum_{n=1}^\infty \langle y_n, z_n\rangle - \left\langle \sum_{n=1}^\infty A_n^* y_n, S_{A,\Psi}^{-1} h\right\rangle \\
&\quad - \left\langle (S_{A,\Psi}^{-1})^* h, \sum_{n=1}^\infty \Psi_n^* z_n\right\rangle \\
&= 2\langle (S_{A,\Psi}^{-1})^* h, h\rangle + \sum_{n=1}^\infty \langle y_n, z_n\rangle - \langle h, S_{A,\Psi}^{-1} h\rangle - \langle (S_{A,\Psi}^{-1})^* h, h\rangle
\end{aligned}$$

which gives the left side. □

Theorem 5.1.5. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a weak OVF with frame bounds a and b.*

(i) The canonical dual weak OVF for the canonical dual weak OVF for $(\{A_n\}_n, \{\Psi_n\}_n)$ is itself.

(ii) $\frac{1}{b}, \frac{1}{a}$ are frame bounds for the canonical dual of $(\{A_n\}_n, \{\Psi_n\}_n)$.

(iii) If a, b are optimal frame bounds for $(\{A_n\}_n, \{\Psi_n\}_n)$, then $\frac{1}{b}, \frac{1}{a}$ are optimal frame bounds for its canonical dual.

Proof. Since (ii) and (iii) follow from the property of invertible operators on Banach spaces, we have to argue for (i): frame operator for $(\{A_n S_{A,\Psi}^{-1}\}_n, \{\Psi_n (S_{A,\Psi}^{-1})^*\}_n)$ is

$$\sum_{n=1}^\infty (\Psi_n (S_{A,\Psi}^{-1})^*)^* (A_n S_{A,\Psi}^{-1}) = S_{A,\Psi}^{-1}\left(\sum_{n=1}^\infty \Psi_n^* A_n\right) S_{A,\Psi}^{-1} = S_{A,\Psi}^{-1} S_{A,\Psi} S_{A,\Psi}^{-1} = S_{A,\Psi}^{-1}.$$

Therefore, its canonical dual is $(\{(A_n S_{A,\Psi}^{-1}) S_{A,\Psi}\}_n, \{(\Psi_n (S_{A,\Psi}^{-1})^*) S_{A,\Psi}^*\}_n)$ which is the original frame. □

For the further study of weak OVFs, we impose some conditions so that the frame operator splits.

Definition 5.1.6. *A weak OVF $(\{A_n\}_n, \{\Psi_n\}_n)$ is said to be* ***factorable*** *if both the maps (called* ***analysis operator****)*

$$\theta_A : \mathscr{H} \ni h \mapsto \theta_A h := \sum_{n=1}^{\infty} L_n A_n h \in \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$$
$$\theta_\Psi : \mathscr{H} \ni h \mapsto \theta_\Psi h := \sum_{n=1}^{\infty} L_n \Psi_n h \in \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$$

are well-defined bounded linear operators.

We next give an example which shows that a weak OVF need not be factorable.

Example 5.1.7. *On $\mathbb{C}$, define $A_n x := \frac{x}{\sqrt{n}}, \forall x \in \mathbb{C}, \forall n \in \mathbb{N}$, and $\Psi_1 x := x, \Psi_n x := 0, \forall x \in \mathbb{C}, \forall n \in \mathbb{N} \setminus \{1\}$. Then $\sum_{n=1}^{\infty} \Psi_n^* A_n x$ converges to an invertible operator but $\sum_{n=1}^{\infty} L_n A_n x$ does not converge. In fact, using Equation (1.6.1),*

$$\left\| \sum_{n=1}^{m} L_n A_n 1 \right\|^2 = \sum_{n=1}^{m} \|A_n 1\|^2 = \sum_{n=1}^{m} \frac{1}{n} \to \infty \quad \text{as} \quad m \to \infty.$$

Equation (1.6.1) gives the following theorem easily.

Theorem 5.1.8. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a factorable weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$.*

(i) Analysis operator

$$\theta_A : \mathscr{H} \ni h \mapsto \theta_A h := \sum_{n=1}^{\infty} L_n A_n h \in \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$$

is a well-defined bounded linear injective operator.

(ii) Synthesis operator

$$\theta_\Psi^* : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \ni z \mapsto \sum_{n=1}^{\infty} \Psi_n^* L_n^* z \in \mathscr{H}$$

is a well-defined bounded linear surjective operator.

(iii) Frame operator factors as $S_{A,\Psi} = \theta_\Psi^* \theta_A$.

(iv) $P_{A,\Psi} := \theta_A S_{A,\Psi}^{-1} \theta_\Psi^* : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ is an idempotent onto $\theta_A(\mathscr{H})$.

We next define the notions of Riesz and orthonormal factorable weak OVFs.

Definition 5.1.9. *A factorable weak OVF* $(\{A_n\}_n, \{\Psi_n\}_n)$ *in* $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ *is said to be a* ***Riesz OVF*** *if* $P_{A,\Psi} = I_{\ell^2(\mathbb{N})} \otimes I_{\mathscr{H}_0}$*. A Parseval and Riesz OVF, i.e.,* $\theta_\Psi^*\theta_A = I_{\mathscr{H}}$ *and* $\theta_A\theta_\Psi^* = I_{\ell^2(\mathbb{N})} \otimes I_{\mathscr{H}_0}$ *is called as an* ***orthonormal OVF****.*

Proposition 5.1.10. *A factorable weak OVF* $(\{A_n\}_n, \{\Psi_n\}_n)$ *in* $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ *is an orthonormal OVF if and only if it is a Parseval OVF and* $A_n\Psi_m^* = \delta_{n,m}I_{\mathscr{H}_0}, \forall n,m \in \mathbb{N}$*.*

Proof. $(\Rightarrow)$ We have $\theta_A\theta_\Psi^* = I_{\ell^2(\mathbb{N})} \otimes I_{\mathscr{H}_0}$. Hence

$$\begin{aligned} e_m \otimes y = \theta_A\theta_\Psi^*(e_m \otimes y) &= \sum_{n=1}^\infty L_nA_n\left(\sum_{k=1}^\infty \Psi_k^*L_k^*(e_m \otimes y)\right) \\ &= \sum_{n=1}^\infty L_nA_n\Psi_m^*y = \sum_{n=1}^\infty (e_n \otimes A_n\Psi_m^*y) \\ &= e_m \otimes (A_m\Psi_m^*y) + \sum_{n=1, n\neq m}^\infty (e_n \otimes A_n\Psi_m^*y), \forall m \in \mathbb{N}, \quad y \in \mathscr{H}_0. \end{aligned}$$

We then have $A_n\Psi_m^*y = \delta_{n,m}y, \forall y \in \mathscr{H}_0$.

$(\Leftarrow)$ $\theta_A\theta_\Psi^* = \sum_{n=1}^\infty L_nA_n(\sum_{k=1}^\infty \Psi_k^*L_k^*) = \sum_{n=1}^\infty L_nL_n^* = I_{\ell^2(\mathbb{N})} \otimes I_{\mathscr{H}_0}$. □

We now derive a dilation result for factorable weak OVFs. First we need a lemma for this.

Lemma 5.1.11. *Let* $(\{A_n\}_n, \{\Psi_n\}_n)$ *be a factorable weak OVF in* $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$*. Then the range of* θ_A *is closed.*

Proof. Let $\{h_n\}_{n=1}^\infty$ in $\mathscr{H}$ be such that $\{\theta_Ah_n\}_{n=1}^\infty$ converges to $y \in \mathscr{H}_0$. This gives $S_{A,\Psi}h_n \to \theta_\Psi^*y$ as $n \to \infty$ and this in turn gives $h_n \to S_{A,\Psi}^{-1}\theta_\Psi^*y$ as $n \to \infty$. An application of θ_A gives $\theta_Ah_n \to \theta_AS_{A,\Psi}^{-1}\theta_\Psi^*y$ as $n \to \infty$. Therefore $y = \theta_A(S_{A,\Psi}^{-1}\theta_\Psi^*y)$. □

Theorem 5.1.12. *Let* $(\{A_n\}_n, \{\Psi_n\}_n)$ *be a Parseval factorable weak OVF in* $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ *such that* $\theta_A(\mathscr{H}) = \theta_\Psi(\mathscr{H})$ *and* $P_{A,\Psi}$ *is projection. Then there exist a Hilbert space* $\mathscr{H}_1$ *which contains* $\mathscr{H}$ *isometrically and bounded linear operators* $B_n, \Phi_n : \mathscr{H}_1 \to \mathscr{H}_0, \forall n$ *such that* $(\{B_n\}_n, \{\Phi_n\}_n)$ *is an orthonormal OVF in* $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_0)$ *and* $B_n|_{\mathscr{H}} = A_n, \Phi_n|_{\mathscr{H}} = \Psi_n, \forall n \in \mathbb{N}$*.*

Proof. We first see that $P_{A,\Psi}$ is the orthogonal projection from $\ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ onto $\theta_A(\mathscr{H}) = \theta_\Psi(\mathscr{H})$. Define $\mathscr{H}_1 := \mathscr{H} \oplus \theta_A(\mathscr{H})^\perp$. From Lemma 5.1.11, $\mathscr{H}_1$ becomes a Hilbert space. Then $\mathscr{H} \ni h \mapsto h \oplus 0 \in \mathscr{H}_1$ is an isometry. Set $P_{A,\Psi}^\perp := I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - P_{A,\Psi}$ and define

$$B_n : \mathscr{H}_1 \ni h \oplus g \mapsto A_nh + L_n^*P_{A,\Psi}^\perp g \in \mathscr{H}_0,$$

$$\Phi_n : \mathcal{H}_1 \ni h \oplus g \mapsto \Psi_n h + L_n^* P_{A,\Psi}^\perp g \in \mathcal{H}_0, \quad \forall n \in \mathbb{N}.$$

Then clearly $B_n|_{\mathcal{H}} = A_n, \Phi_n|_{\mathcal{H}} = \Psi_n, \forall n \in \mathbb{N}$. Now

$$\theta_B(h \oplus g) = \sum_{n=1}^\infty L_n A_n h + \sum_{n=1}^\infty L_n L_n^* P_{A,\Psi}^\perp g = \theta_A h + P_{A,\Psi}^\perp g, \quad \forall h \oplus g \in \mathcal{H}_1.$$

Similarly $\theta_\Phi(h \oplus g) = \theta_\Psi h + P_{A,\Psi}^\perp g, \forall h \oplus g \in \mathcal{H}_1$. Also

$$\begin{aligned}\langle \theta_B^* z, h \oplus g\rangle = \langle z, \theta_B(h \oplus g)\rangle &= \langle \theta_A^* z, h\rangle + \langle P_{A,\Psi}^\perp z, g\rangle \\ &= \langle \theta_A^* z \oplus P_{A,\Psi}^\perp z, h \oplus g\rangle, \quad \forall z \in \ell^2(\mathbb{N}) \otimes \mathcal{H}_0, \forall h \oplus g \in \mathcal{H}_1.\end{aligned}$$

Hence $\theta_B^* z = \theta_A^* z \oplus P_{A,\Psi}^\perp z, \forall z \in \ell^2(\mathbb{N}) \otimes \mathcal{H}_0$ and similarly $\theta_\Phi^* z = \theta_\Psi^* z \oplus P_{A,\Psi}^\perp z$, $\forall z \in \ell^2(\mathbb{N}) \otimes \mathcal{H}_0$. By using $\theta_A(\mathcal{H}) = \theta_\Psi(\mathcal{H})$ and $\theta_\Psi^* P_{A,\Psi}^\perp = 0 = P_{A,\Psi}^\perp \theta_A$, we get

$$\begin{aligned}S_{B,\Phi}(h \oplus g) = \theta_\Phi^*(\theta_A h + P_{A,\Psi}^\perp g) &= \theta_\Psi^*(\theta_A h + P_{A,\Psi}^\perp g) \oplus P_{A,\Psi}^\perp(\theta_A h + P_{A,\Psi}^\perp g) \\ &= (S_{A,\Psi} h + 0) \oplus (0 + P_{A,\Psi}^\perp g) = S_{A,\Psi} h \oplus P_{A,\Psi}^\perp g \\ &= I_{\mathcal{H}} h \oplus I_{\theta_A(\mathcal{H})^\perp} g, \quad \forall h \oplus g \in \mathcal{H}_1.\end{aligned}$$

Hence $(\{B_n\}_n, \{\Phi_n\}_n)$ is a Parseval weak OVF in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$. We further find

$$\begin{aligned}P_{B,\Phi} z = \theta_B S_{B,\Phi}^{-1} \theta_\Phi^* z = \theta_B \theta_\Phi^* z &= \theta_B(\theta_\Psi^* z \oplus P_{A,\Psi}^\perp z) \\ &= \theta_A(\theta_\Psi^* z) + P_{A,\Psi}^\perp(P_{A,\Psi}^\perp z) = P_{A,\Psi} z + P_{A,\Psi}^\perp z \\ &= P_{A,\Psi} z + ((I_{\ell^2(\mathbb{N})} \otimes I_{\mathcal{H}_0}) - P_{A,\Psi}) z = (I_{\ell^2(\mathbb{N})} \otimes I_{\mathcal{H}_0}) z, \quad \forall z \in \ell^2(\mathbb{N}) \otimes \mathcal{H}_0.\end{aligned}$$

Therefore $(\{B_n\}_n, \{\Phi_n\}_n)$ is a Riesz weak OVF in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$. Thus $(\{B_n\}_n, \{\Phi_n\}_n)$ is an orthonormal weak OVF in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$. □

Theorem 5.1.13. *A pair $(\{A_n\}_n, \{\Psi_n\}_n)$ is a factorable weak OVF in $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$ if and only if*

$$A_n = L_n^* U, \quad \Psi_n = L_n^* V, \quad \forall n \in \mathbb{N},$$

where $U, V : \mathcal{H} \to \ell^2(\mathbb{N}) \otimes \mathcal{H}_0$ are bounded linear operators such that V^*U is bounded invertible.

Proof. $(\Leftarrow)$ Clearly θ_A and θ_Ψ are well-defined bounded linear operators. Let $h \in \mathcal{H}$.

Then using Equation (1.6.1), we have

$$S_{A,\Psi}h = \sum_{n=1}^{\infty}(L_n^*V)^*L_n^*Uh = V^*\left(\sum_{n=1}^{\infty}L_nL_n^*\right)Uh = V^*Uh. \tag{5.1.1}$$

Hence $S_{A,\Psi}$ is bounded invertible.
($\Rightarrow$) Define $U := \sum_{n=1}^{\infty}L_nA_n$, $V := \sum_{n=1}^{\infty}L_n\Psi_n$. Then

$$L_n^*U = L_n^*\left(\sum_{k=1}^{\infty}L_kA_k\right) = \sum_{k=1}^{\infty}L_n^*L_kA_k = A_n,$$
$$L_n^*V = L_n^*\left(\sum_{k=1}^{\infty}L_k\Psi_k\right) = \sum_{k=1}^{\infty}L_n^*L_k\Psi_k = \Psi_n, \quad \forall n \in \mathbb{N}$$

and

$$V^*U = \left(\sum_{n=1}^{\infty}\Psi_n^*L_n^*\right)\left(\sum_{k=1}^{\infty}L_kA_k\right) = \sum_{n=1}^{\infty}\Psi_n^*A_n = S_{A,\Psi}$$

which is bounded invertible. ∎

Using Theorem 5.1.13 we can characterize Riesz and orthonormal factorable weak OVFs.

Corollary 5.1.14. *A pair* $(\{A_n\}_n, \{\Psi_n\}_n)$ *is a Riesz factorable weak OVF in* $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ *if and only if*

$$A_n = L_n^*U, \quad \Psi_n = L_n^*V, \quad \forall n \in \mathbb{N},$$

where $U, V : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ *are bounded linear operators such that* V^*U *is bounded invertible and* $U(V^*U)^{-1}V^* = I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0}$.

Proof. ($\Leftarrow$) $P_{A,\Psi} = U(V^*U)^{-1}V^* = I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0}$.
($\Rightarrow$) Let U and V be as in Theorem 5.1.13. Then $U(V^*U)^{-1}V^* = P_{A,\Psi} = I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0}$. ∎

Corollary 5.1.15. *A pair* $(\{A_n\}_n, \{\Psi_n\}_n)$ *is an orthonormal factorable weak OVF in* $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ *if and only if*

$$A_n = L_n^*U, \quad \Psi_n = L_n^*V, \quad \forall n \in \mathbb{N},$$

where $U, V : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ *are bounded linear operators such that* V^*U *is bounded invertible and* $V^*U = I_{\mathscr{H}}$, $I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} = UV^*$.

Proof. We use Corollary 5.1.14.

($\Leftarrow$) $S_{A,\Psi} = V^*U = I_{\mathscr{H}}, P_{A,\Psi} = \theta_A S_{A,\Psi}^{-1}\theta_\Psi^* = \theta_A\theta_\Psi^* = \theta_F UV^*\theta_F^* = I_{\ell^2(\mathbb{N})} \otimes I_{\mathscr{H}_0}$.

($\Rightarrow$) $V^*U = S_{A,\Psi} = I_{\mathscr{H}}$, and by using Proposition 5.1.10,

$$UV^* = \left(\sum_{n=1}^{\infty} L_n^* A_n\right)\left(\sum_{k=1}^{\infty} \Psi_k^* L_k\right) = \sum_{n=1}^{\infty} L_n L_n^* = I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0}.$$

□

Theorem 5.1.16. *Let $\{F_n\}_n$ be an orthonormal basis in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$. Then a pair $(\{A_n\}_n, \{\Psi_n\}_n)$ is a factorable weak OVF in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ if and only if*

$$A_n = F_n U, \quad \Psi_n = F_n V, \quad \forall n \in \mathbb{N},$$

where $U, V : \mathscr{H} \to \mathscr{H}$ are bounded linear operators such that V^*U is bounded invertible.

Proof. ($\Leftarrow$) $\sum_{n=1}^\infty L_n(F_nU) = (\sum_{n=1}^\infty L_nF_n)U, \sum_{n=1}^\infty L_n(F_nV) = (\sum_{n=1}^\infty L_nF_n)V$. These show analysis operators for $(\{F_nU\}_n, \{F_nV\}_n)$ are well-defined bounded linear operators and the equality

$$\sum_{n=1}^{\infty} (F_nV)^*(F_nU) = V^*U$$

shows that it is a factorable weak OVF.

($\Rightarrow$) Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a factorable weak OVF. Note that the series $\sum_{n=1}^\infty F_n^*A_n$ and $\sum_{n=1}^\infty F_n^*\Psi_n$ converge. In fact, for each $h \in \mathscr{H}$,

$$\begin{aligned}\left\|\sum_{n=1}^{m} F_n^*A_nh\right\|^2 &= \left\langle \sum_{n=1}^{m} F_n^*A_nh, \sum_{k=1}^{m} F_k^*A_kh\right\rangle \\ &= \sum_{n=1}^{m} \left\langle A_nh, F_n\left(\sum_{k=1}^{m} F_k^*A_kh\right)\right\rangle = \sum_{n=1}^{m} \|A_nh\|^2.\end{aligned}$$

which converges to $\|\theta_Ah\|^2 = \|\sum_{n=1}^\infty L_nA_nh\|^2 = \sum_{n=1}^\infty \|A_nh\|^2$. Define $U := \sum_{n=1}^\infty F_n^*A_n$ and $V := \sum_{n=1}^\infty F_n^*\Psi_n$. Then $F_nU = A_n, F_nV = \Psi_n, \forall n \in \mathbb{N}$ and

$$V^*U = \left(\sum_{n=1}^{\infty} \Psi_n^*F_n\right)\left(\sum_{k=1}^{\infty} F_k^*A_k\right) = \sum_{n=1}^{\infty} \Psi_n^*A_n = S_{A,\Psi}$$

which is bounded invertible. □

Corollary 5.1.17. *Let $\{F_n\}_n$ be an orthonormal basis in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$. Then a pair $(\{A_n\}_n, \{\Psi_n\}_n)$ is* *(i) a Riesz factorable weak OVF in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ if and only if*

$$A_n = F_nU, \quad \Psi_n = F_nV, \quad \forall n \in \mathbb{N},$$

where $U,V : \mathscr{H} \to \mathscr{H}$ are bounded linear operators such that V^*U is bounded invertible and $U(V^*U)^{-1}V^* = I_{\mathscr{H}}$.

(ii) an orthonormal factorable weak OVF in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ if and only if

$$A_n = F_nU, \quad \Psi_n = F_nV, \quad \forall n \in \mathbb{N},$$

where $U,V : \mathscr{H} \to \mathscr{H}$ are bounded linear operators such that V^*U is bounded invertible and $V^*U = I_{\mathscr{H}} = UV^*$.

Proof. (i) $(\Leftarrow)$

$$\begin{aligned}
P_{A,\Psi} = \theta_A S_{A,\Psi}^{-1}\theta_\Psi^* &= \left(\sum_{n=1}^\infty L_nF_nU\right)(V^*U)^{-1}\left(\sum_{k=1}^\infty V^*F_k^*L_k^*\right)\\
&= \theta_F U(V^*U)^{-1}V^*\theta_F^* = \theta_F I_{\mathscr{H}}\theta_F^* = \sum_{n=1}^\infty L_nF_n\left(\sum_{k=1}^\infty F_k^*L_k^*\right)\\
&= \sum_{n=1}^\infty L_nL_n^* = I_{\ell^2(\mathbb{N})}\otimes I_{\mathscr{H}_0}.
\end{aligned}$$

$(\Rightarrow)$ Let U and V be as in Theorem 5.1.16. Then

$$\begin{aligned}
U(V^*U)^{-1}V^* &= \left(\sum_{k=1}^\infty F_k^*A_k\right)S_{A,\Psi}^{-1}\left(\sum_{n=1}^\infty \Psi_n^*F_n\right)\\
&= \left(\sum_{r=1}^\infty F_r^*L_r^*\right)\left(\sum_{k=1}^\infty L_kA_k\right)S_{A,\Psi}^{-1}\left(\sum_{n=1}^\infty \Psi_n^*L_n^*\right)\left(\sum_{m=1}^\infty L_mF_m\right)\\
&= \left(\sum_{r=1}^\infty F_r^*L_r^*\right)\theta_A S_{A,\Psi}^{-1}\theta_\Psi^*\left(\sum_{m=1}^\infty L_mF_m\right)\\
&= \left(\sum_{r=1}^\infty F_r^*L_r^*\right)P_{A,\Psi}\left(\sum_{m=1}^\infty L_mF_m\right)\\
&= \left(\sum_{r=1}^\infty F_r^*L_r^*\right)(I_{\ell^2(\mathbb{N})}\otimes I_{\mathscr{H}_0})\left(\sum_{m=1}^\infty L_mF_m\right)\\
&= \left(\sum_{r=1}^\infty F_r^*L_r^*\right)\left(\sum_{m=1}^\infty L_mF_m\right) = \sum_{r=1}^\infty F_r^*F_r = I_{\mathscr{H}}.
\end{aligned}$$

(ii) We use (i).

($\Leftarrow$) $S_{A,\Psi} = V^*U = I_{\mathscr{H}}, P_{A,\Psi} = \theta_A S_{A,\Psi}^{-1}\theta_\Psi^* = \theta_A\theta_\Psi^* = \theta_F UV^*\theta_F^* = \theta_F I_{\mathscr{H}}\theta_F^* = I_{\ell^2(\mathbb{N})}\otimes I_{\mathscr{H}_0}$.

($\Rightarrow$) $V^*U = S_{A,\Psi} = I_{\mathscr{H}}$ and using Proposition 5.1.10,

$$UV^* = \left(\sum_{n=1}^\infty F_n^*A_n\right)\left(\sum_{k=1}^\infty \Psi_k^*F_k\right) = \sum_{n=1}^\infty F_n^*F_n = I_{\mathscr{H}}.$$

□

We next derive another characterization which is free from natural numbers.

Theorem 5.1.18. *Let $\{A_n\}_n, \{\Psi_n\}_n$ be in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. Then $(\{A_n\}_n, \{\Psi_n\}_n)$ is a factorable weak OVF*

(i) if and only if

$$U : \ell^2(\mathbb{N})\otimes\mathscr{H}_0 \ni y \mapsto \sum_{n=1}^\infty A_n^*L_n^*y \in \mathscr{H}, \text{ and } V : \ell^2(\mathbb{N})\otimes\mathscr{H}_0 \ni z \mapsto \sum_{n=1}^\infty \Psi_n^*L_n^*z \in \mathscr{H}$$

are well-defined bounded linear operators such that VU^* is bounded invertible.

(ii) if and only if

$$U : \ell^2(\mathbb{N})\otimes\mathscr{H}_0 \ni y \mapsto \sum_{n=1}^\infty A_n^*L_n^*y \in \mathscr{H}, \text{ and } S : \mathscr{H} \ni g \mapsto \sum_{n=1}^\infty L_n\Psi_n g \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0$$

are well-defined bounded linear operators such that S^*U^* is bounded invertible.

(iii) if and only if

$$R : \mathscr{H} \ni h \mapsto \sum_{n=1}^\infty L_nA_n h \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0, \text{ and } V : \ell^2(\mathbb{N})\otimes\mathscr{H}_0 \ni z \mapsto \sum_{n=1}^\infty \Psi_n^*L_n^*z \in \mathscr{H}$$

are well-defined bounded linear operators such that VR is bounded invertible.

(iv) if and only if

$$R : \mathscr{H} \ni h \mapsto \sum_{n=1}^\infty L_nA_n h \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0, \text{ and } S : \mathscr{H} \ni g \mapsto \sum_{n=1}^\infty L_n\Psi_n g \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0$$

are well-defined bounded linear operators such that S^*R is bounded invertible.

Proof. We prove (i) and others are similar.

($\Rightarrow$) Now $U = \theta_A^*$, $V = \theta_\Psi^*$ and hence $VU^* = \theta_\Psi^*\theta_A = S_{A,\Psi}$.

($\Leftarrow$) Now $\theta_A = U^*$, $\theta_\Psi = V^*$ and hence $S_{A,\Psi} = \theta_\Psi^*\theta_A = VU^*$. $\square$

Now we try to characterize all dual OVFs.

Lemma 5.1.19. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a factorable weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. Then a factorable weak OVF $(\{B_n\}_n, \{\Phi_n\}_n)$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is a dual for $(\{A_n\}_n, \{\Psi_n\}_n)$ if and only if*

$$B_n = L_n^*U, \quad \Phi_n = L_n^*V^*, \quad \forall n \in \mathbb{N}$$

where $U : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ is a bounded right-inverse of θ_Ψ^* and $V : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \mathscr{H}$ is a bounded left-inverse of θ_A such that VU is bounded invertible.

Proof. ($\Leftarrow$) “If" part of proof of Theorem 5.1.13, says that $(\{B_n\}_n, \{\Phi_n\}_n)$ is a factorable weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. We now check for the duality of $(\{B_n\}_n, \{\Phi_n\}_n)$. Consider $\theta_\Phi^*\theta_A = V^*\theta_A = I_\mathscr{H}$, $\theta_\Psi^*\theta_B = \theta_\Psi^*U = I_\mathscr{H}$.

($\Rightarrow$) Let $(\{B_n\}_n, \{\Phi_n\}_n)$ be a dual factorable weak OVF for $(\{A_n\}_n, \{\Psi_n\}_n)$. Then $\theta_\Psi^*\theta_B = I_\mathscr{H} = \theta_\Phi^*\theta_A$. Define $U := \theta_B, V := \theta_\Phi^*$. Then $U : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ is a bounded right-inverse of θ_Ψ^* and $V : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \mathscr{H}$ is a left inverse of θ_A such that $VU = \theta_\Phi^*\theta_B = S_{B,\Phi}$ is bounded invertible. We now see

$$L_n^*U = L_n^*\left(\sum_{k=1}^\infty L_kB_k\right) = B_n, \quad L_n^*V^* = L_n^*\left(\sum_{k=1}^\infty L_k\Phi_k\right) = \Phi_n, \quad \forall n \in \mathbb{N}.$$

$\square$

Lemma 5.1.20. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a factorable weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. Then*

(i) $R : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ is a bounded right-inverse of θ_Ψ^* if and only if

$$R = \theta_A S_{A,\Psi}^{-1} + (I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*)U,$$

where $U : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ is a bounded linear operator.

(ii) $L : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \mathscr{H}$ is a bounded left-inverse of θ_A if and only if

$$L = S_{A,\Psi}^{-1}\theta_\Psi^* + V(I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*),$$

where $V : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \mathscr{H}$ is a bounded linear operator.

Proof. (i) $(\Leftarrow)$ Let $U : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ be a bounded linear operator. Then

$$\theta_\Psi^*(\theta_A S_{A,\Psi}^{-1} + (I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*)U) = I_{\mathscr{H}} + \theta_\Psi^* U - \theta_\Psi^* U = I_{\mathscr{H}}.$$

Therefore $\theta_A S_{A,\Psi}^{-1} + (I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*)U$ is a bounded right-inverse of θ_Ψ^*.

$(\Rightarrow)$ Let $R : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ be a bounded right-inverse of θ_Ψ^*. Define $U := R$. Then $\theta_A S_{A,\Psi}^{-1} + (I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*)U = \theta_A S_{A,\Psi}^{-1} + (I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*)R = \theta_A S_{A,\Psi}^{-1} + R - \theta_A S_{A,\Psi}^{-1} = R$.

(ii) $(\Leftarrow)$ Let $V : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \mathscr{H}$ be a bounded linear operator. Then

$$(S_{A,\Psi}^{-1}\theta_\Psi^* + V(I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*))\theta_A = I_{\mathscr{H}} + V\theta_A - V\theta_A I_{\mathscr{H}} = I_{\mathscr{H}}.$$

Therefore $S_{A,\Psi}^{-1}\theta_\Psi^* + V(I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*)$ is a bounded left-inverse of θ_A.

$(\Rightarrow)$ Let $L : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \mathscr{H}$ be a bounded left-inverse of θ_A. Define $V := L$. Then $S_{A,\Psi}^{-1}\theta_\Psi^* + V(I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*) = S_{A,\Psi}^{-1}\theta_\Psi^* + L(I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*) = S_{A,\Psi}^{-1}\theta_\Psi^* + L - I_{\mathscr{H}} S_{A,\Psi}^{-1}\theta_\Psi^* = L$.

□

Theorem 5.1.21. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a factorable weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. Then a factorable weak OVF $(\{B_n\}_n, \{\Phi_n\}_n)$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is a dual for $(\{A_n\}_n, \{\Psi_n\}_n)$ if and only if*

$$\begin{aligned}
B_n &= A_n S_{A,\Psi}^{-1} + L_n^* U - A_n S_{A,\Psi}^{-1}\theta_\Psi^* U,\\
\Phi_n &= \Psi_n (S_{A,\Psi}^{-1})^* + L_n^* V^* - \Psi_n (S_{A,\Psi}^{-1})^* \theta_A^* V^*, \quad \forall n \in \mathbb{N}
\end{aligned}$$

such that the operator

$$S_{A,\Psi}^{-1} + VU - V\theta_A S_{A,\Psi}^{-1}\theta_\Psi^* U$$

is bounded invertible, where $U : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ and $V : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \mathscr{H}$ are bounded linear operators.

Proof. Lemmas 5.1.19 and 5.1.20 give the characterization of dual weak OVF as

$$\begin{aligned}
B_n &= L_n^*(\theta_A S_{A,\Psi}^{-1} + (I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*)U)\\
&= A_n S_{A,\Psi}^{-1} + L_n^* U - A_n S_{A,\Psi}^{-1}\theta_\Psi^* U,\\
\Phi_n &= L_n^*(\theta_\Psi (S_{A,\Psi}^{-1})^* + (I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_\Psi (S_{A,\Psi}^{-1})^* \theta_A^*)V^*)\\
&= \Psi_n (S_{A,\Psi}^{-1})^* + L_n^* V^* - \Psi_n (S_{A,\Psi}^{-1})^* \theta_A^* V^*, \quad \forall n \in \mathbb{N}
\end{aligned}$$

such that the operator

$$(S_{A,\Psi}^{-1}\theta_\Psi^* + V(I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*))(\theta_A S_{A,\Psi}^{-1} + (I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*)U)$$

is bounded invertible, where $U : \mathscr{H} \to \ell^2(\mathbb{N}) \otimes \mathscr{H}_0$ and $V : \ell^2(\mathbb{N}) \otimes \mathscr{H}_0 \to \mathscr{H}$ are bounded linear operators. We expand and get

$$\begin{aligned}
&(S_{A,\Psi}^{-1}\theta_\Psi^* + V(I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*))(\theta_A S_{A,\Psi}^{-1} + (I_{\ell^2(\mathbb{N})\otimes\mathscr{H}_0} - \theta_A S_{A,\Psi}^{-1}\theta_\Psi^*)U) \\
&= S_{A,\Psi}^{-1} + VU - V\theta_A S_{A,\Psi}^{-1}\theta_\Psi^* U.
\end{aligned}$$

□

We now define the orthogonality for weak OVFs.

Definition 5.1.22. *A weak OVF $(\{B_n\}_n, \{\Phi_n\}_n)$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is said to be* ***orthogonal*** *to a weak OVF $(\{A_n\}_n, \{\Psi_n\}_n)$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ if*

$$\sum_{n=1}^{\infty} \Psi_n^* B_n = \sum_{n=1}^{\infty} \Phi_n^* A_n = 0.$$

Remarkable property of orthogonal frames is that we can interpolate as well as we can take direct sum of them to get new frames. These are illustrated in the following two results.

Proposition 5.1.23. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ and $(\{B_n\}_n, \{\Phi_n\}_n)$ be two Parseval OVFs in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ which are orthogonal. If $C, D, E, F \in \mathscr{B}(\mathscr{H})$ are such that $C^*E + D^*F = I_{\mathscr{H}}$, then*

$$(\{A_nC + B_nD\}_n, \{\Psi_nE + \Phi_nF\}_n)$$

is a Parseval weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. In particular, if scalars c, d, e, f satisfy $\bar{c}e + \bar{d}f = 1$, then $(\{cA_n + dB_n\}_n, \{e\Psi_n + f\Phi_n\}_n)$ is a Parseval weak OVF.

Proof. We use the definition of frame operator and get

$$\begin{aligned}
S_{AC+BD,\Psi E+\Phi F} &= \sum_{n=1}^{\infty} (\Psi_nE + \Phi_nF)^*(A_nC + B_nD) \\
&= E^*S_{A,\Psi}C + E^*\left(\sum_{n=1}^{\infty} \Psi_n^* B_n\right) D + F^*\left(\sum_{n=1}^{\infty} \Phi_n^* A_n\right) C + F^*S_{B,\Phi}D \\
&= E^*I_{\mathscr{H}}C + E^*0D + F^*0C + F^*I_{\mathscr{H}}D = I_{\mathscr{H}}.
\end{aligned}$$

□

Proposition 5.1.24. *If* $(\{A_n\}_n, \{\Psi_n\}_n)$ *and* $(\{B_n\}_n, \{\Phi_n\}_n)$ *are orthogonal weak OVFs in* $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$*, then* $(\{A_n \oplus B_n\}_n, \{\Psi_n \oplus \Phi_n\}_n)$ *is a weak OVF in* $\mathscr{B}(\mathscr{H} \oplus \mathscr{H}, \mathscr{H}_0)$*. Further, if both* $(\{A_n\}_n, \{\Psi_n\}_n)$ *and* $(\{B_n\}_n, \{\Phi_n\}_n)$ *are Parseval, then* $(\{A_n \oplus B_n\}_n, \{\Psi_n \oplus \Phi_n\}_n)$ *is Parseval.*

Proof. Let $h \oplus g \in \mathscr{H} \oplus \mathscr{H}$. Then

$$\begin{aligned}
S_{A\oplus B,\Psi\oplus\Phi}(h \oplus g) &= \sum_{n=1}^{\infty} (\Psi_n \oplus \Phi_n)^*(A_n \oplus B_n)(h \oplus g) = \sum_{n=1}^{\infty} (\Psi_n \oplus \Phi_n)^*(A_n h + B_n g) \\
&= \sum_{n=1}^{\infty} (\Psi_n^*(A_n h + B_n g) \oplus \Phi_n^*(A_n h + B_n g)) \\
&= \left(\sum_{n=1}^{\infty} \Psi_n^* A_n h + \sum_{n=1}^{\infty} \Psi_n^* B_n g \right) \oplus \left(\sum_{n=1}^{\infty} \Phi_n^* A_n h + \sum_{n=1}^{\infty} \Phi_n^* B_n g \right) \\
&= (S_{A,\Psi} h + 0) \oplus (0 + S_{B,\Phi} g) = (S_{A,\Psi} \oplus S_{B,\Phi})(h \oplus g).
\end{aligned}$$

□

5.2 EQUIVALENCE OF WEAK OPERATOR-VALUED FRAMES

Definition 1.6.19 introduced similarity for OVFs. Here is the similar notion for factorable weak OVFs.

Definition 5.2.1. *A factorable weak OVF* $(\{B_n\}_n, \{\Phi_n\}_n)$ *in* $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ *is said to be* ***similar*** *to a factorable weak OVF* $(\{A_n\}_n, \{\Psi_n\}_n)$ *in* $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ *if there exist bounded invertible* $R_{A,B}, R_{\Psi,\Phi} \in \mathscr{B}(\mathscr{H})$ *such that*

$$B_n = A_n R_{A,B}, \quad \Phi_n = \Psi_n R_{\Psi,\Phi}, \quad \forall n \in \mathbb{N}.$$

Since $R_{A,B}$ and $R_{\Psi,\Phi}$ are bounded invertible, it easily follows that the notion similarity is symmetric. We further have that the relation “similarity" is an equivalence relation on the set

$$\{(\{A_n\}_n, \{\Psi_n\}_n) : (\{A_n\}_n, \{\Psi_n\}_n) \text{ is a factorable weak OVF}\}.$$

Similar frames have nice property that knowing analysis, synthesis and frame operators of one give that of another.

Lemma 5.2.2. *Let* $(\{A_n\}_n, \{\Psi_n\}_n)$ *and* $(\{B_n\}_n, \{\Phi_n\}_n)$ *be similar factorable weak OVFs and* $B_n = A_n R_{A,B}, \Phi_n = \Psi_n R_{\Psi,\Phi}, \forall n \in \mathbb{N}$*, for some invertible* $R_{A,B}, R_{\Psi,\Phi} \in \mathscr{B}(\mathscr{H})$*. Then*

(i) $\theta_B = \theta_A R_{A,B}, \theta_\Phi = \theta_\Psi R_{\Psi,\Phi}$.

(ii) $S_{B,\Phi} = R^*_{\Psi,\Phi} S_{A,\Psi} R_{A,B}$.

(iii) $P_{B,\Phi} = P_{A,\Psi}$.

Proof. $\theta_B = \sum_{n=1}^\infty L_n B_n = \sum_{n=1}^\infty L_n A_n R_{A,B} = \theta_A R_{A,B}$. Similarly $\theta_\Phi = \theta_\Psi R_{\Psi,\Phi}$. Now using operators θ_B and θ_Φ we get $S_{B,\Phi} = \sum_{n=1}^\infty \Phi_n^* B_n = \sum_{n=1}^\infty (\Psi_n R_{\Psi,\Phi})^*(A_n R_{A,B}) = R^*_{\Psi,\Phi}\left(\sum_{n=1}^\infty \Psi_n^* A_n\right) R_{A,B} = R^*_{\Psi,\Phi} S_{A,\Psi} R_{A,B}$. We now use (i) and (ii) to get

$$P_{B,\Phi} = \theta_B S^{-1}_{B,\Phi}\theta^*_\Phi = (\theta_A R_{A,B})(R^*_{\Psi,\Phi} S_{A,\Psi} R_{A,B})^{-1}(\theta_\Psi R_{\Psi,\Phi})^* = P_{A,\Psi}.$$

□

We now classify similarity using operators.

Theorem 5.2.3. *For two factorable weak OVFs* $(\{A_n\}_n, \{\Psi_n\}_n)$ *and* $(\{B_n\}_n, \{\Phi_n\}_n)$, *the following are equivalent.*

(i) $B_n = A_n R_{A,B}, \Phi_n = \Psi_n R_{\Psi,\Phi}, \forall n \in \mathbb{N}$, *for some invertible* $R_{A,B}, R_{\Psi,\Phi} \in \mathscr{B}(\mathscr{H})$.

(ii) $\theta_B = \theta_A R_{A,B}, \theta_\Phi = \theta_\Psi R_{\Psi,\Phi}$ *for some invertible* $R_{A,B}, R_{\Psi,\Phi} \in \mathscr{B}(\mathscr{H})$.

(iii) $P_{B,\Phi} = P_{A,\Psi}$.

If one of the above conditions is satisfied, then invertible operators in (i) *and* (ii) *are unique and are given by* $R_{A,B} = S^{-1}_{A,\Psi}\theta^*_\Psi\theta_B$, $R_{\Psi,\Phi} = (S^{-1}_{A,\Psi})^*\theta^*_A\theta_\Phi$. *In the case that* $(\{A_n\}_n, \{\Psi_n\}_n)$ *is Parseval, then* $(\{B_n\}_n, \{\Phi_n\}_n)$ *is Parseval if and only if* $R^*_{\Psi,\Phi} R_{A,B} = I_\mathscr{H}$ *if and only if* $R_{A,B} R^*_{\Psi,\Phi} = I_\mathscr{H}$.

Proof. The implications (i) ⇒ (ii) ⇒ (iii) follow from Lemma 5.2.2. Assume (ii) holds. We show (i) holds. Using Equation (1.6.1), $B_n = L_n^*\theta_B = L_n^*\theta_A R'_{A,B} = A_n R'_{A,B}$; the same procedure gives Φ_n also. Assume (iii). We note the following $\theta_B = P_{B,\Phi}\theta_B$ and $\theta_\Phi = P^*_{B,\Phi}\theta_\Phi$. Using these, $\theta_B = P_{A,\Psi}\theta_B = \theta_A(S^{-1}_{A,\Psi}\theta^*_\Psi\theta_B)$ and $\theta_\Phi = P^*_{A,\Psi}\theta_\Phi = (\theta_A S^{-1}_{A,\Psi}\theta^*_\Psi)^*\theta_\Phi = \theta_\Psi((S^{-1}_{A,\Psi})^*\theta^*_A\theta_\Phi)$. We now try to show that both $S^{-1}_{A,\Psi}\theta^*_\Psi\theta_B$ and $(S^{-1}_{A,\Psi})^*\theta^*_A\theta_\Phi$ are invertible. This is achieved via,

$$(S^{-1}_{A,\Psi}\theta^*_\Psi\theta_B)(S^{-1}_{B,\Phi}\theta^*_\Phi\theta_A) = S^{-1}_{A,\Psi}\theta^*_\Psi P_{B,\Phi}\theta_A = S^{-1}_{A,\Psi}\theta^*_\Psi P_{A,\Psi}\theta_A = S^{-1}_{A,\Psi}\theta^*_\Psi\theta_A = I_\mathscr{H},$$
$$(S^{-1}_{B,\Phi}\theta^*_\Phi\theta_A)(S^{-1}_{A,\Psi}\theta^*_\Psi\theta_B) = S^{-1}_{B,\Phi}\theta^*_\Phi P_{A,\Psi}\theta_B = S^{-1}_{B,\Phi}\theta^*_\Phi P_{B,\Phi}\theta_B = S^{-1}_{B,\Phi}\theta^*_\Phi\theta_B = I_\mathscr{H}$$

and

$$((S^{-1}_{A,\Psi})^*\theta^*_A\theta_\Phi)((S^{-1}_{B,\Phi})^*\theta^*_B\theta_\Psi) = (S^{-1}_{A,\Psi})^*\theta^*_A P^*_{B,\Phi}\theta_\Psi = (S^{-1}_{A,\Psi})^*\theta^*_A P^*_{A,\Psi}\theta_\Psi$$

$$
\begin{aligned}
&= (S_{A,\Psi}^{-1})^*\theta_A^*\theta_\Psi = I_{\mathscr{H}},\\
((S_{B,\Phi}^{-1})^*\theta_B^*\theta_\Psi)((S_{A,\Psi}^{-1})^*\theta_A^*\theta_\Phi) &= (S_{B,\Phi}^{-1})^*\theta_B^*P_{A,\Psi}^*\theta_\Phi = (S_{B,\Phi}^{-1})^*\theta_B^*P_{B,\Phi}^*\theta_\Phi\\
&= (S_{B,\Phi}^{-1})^*\theta_B^*\theta_\Phi = I_{\mathscr{H}}.
\end{aligned}
$$

Let $R_{A,B}, R_{\Psi,\Phi} \in \mathscr{B}(\mathscr{H})$ be invertible. From the previous arguments, $R_{A,B}$ and $R_{\Psi,\Phi}$ satisfy (i) if and only if they satisfy (ii). Let $B_n = A_nR_{A,B}, \Phi_n = \Psi_nR_{\Psi,\Phi}, \forall n \in \mathbb{N}$. Using (ii), $\theta_B = \theta_AR_{A,B}, \theta_\Phi = \theta_\Psi R_{\Psi,\Phi} \implies \theta_\Psi^*\theta_B = \theta_\Psi^*\theta_AR_{A,B} = S_{A,\Psi}R_{A,B}, \theta_A^*\theta_\Phi = \theta_A^*\theta_\Psi R_{\Psi,\Phi} = S_{A,\Psi}^*R_{\Psi,\Phi}$. These imply the formula for $R_{A,B}$ and $R_{\Psi,\Phi}$. For the last, we recall $S_{B,\Phi} = R_{\Psi,\Phi}^*S_{A,\Psi}R_{A,B}$. □

Corollary 5.2.4. *For any given factorable weak OVF* $(\{A_n\}_n, \{\Psi_n\}_n)$*, the canonical dual of* $(\{A_n\}_n, \{\Psi_n\}_n)$ *is the only dual factorable weak OVF that is similar to* $(\{A_n\}_n, \{\Psi_n\}_n)$.

Proof. Let $(\{B_n\}_n, \{\Phi_n\}_n)$ be a factorable weak OVF which is both dual and similar for $(\{A_n\}_n, \{\Psi_n\}_n)$. Then we have $\theta_B^*\theta_\Psi = I_{\mathscr{H}} = \theta_\Phi^*\theta_A$ and there exist invertible $R_{A,B}, R_{\Psi,\Phi} \in \mathscr{B}(\mathscr{H})$ such that $B_n = A_nR_{A,B}, \Phi_n = \Psi_nR_{\Psi,\Phi}, \forall n \in \mathbb{N}$. Theorem 5.2.3 gives $R_{A,B} = S_{A,\Psi}^{-1}\theta_\Psi^*\theta_B, R_{\Psi,\Phi} = S_{A,\Psi}^{-1}\theta_A^*\theta_\Phi$. But then $R_{A,B} = S_{A,\Psi}^{-1}I_{\mathscr{H}} = S_{A,\Psi}^{-1}$, $R_{\Psi,\Phi} = (S_{A,\Psi}^{-1})^*I_{\mathscr{H}} = (S_{A,\Psi}^{-1})^*$. Therefore $(\{B_n\}_n, \{\Phi_n\}_n)$ is the canonical dual for $(\{A_n\}_n, \{\Psi_n\}_n)$. □

Corollary 5.2.5. *Two similar factorable weak OVF cannot be orthogonal.*

Proof. Let a factorable weak OVF $(\{B_n\}_n, \{\Phi_n\}_n)$ be similar to $(\{A_n\}_n, \{\Psi_n\}_n)$. Choose invertible $R_{A,B}, R_{\Psi,\Phi} \in \mathscr{B}(\mathscr{H})$ such that $B_n = A_nR_{A,B}, \Phi_n = \Psi_nR_{\Psi,\Phi}, \forall n \in \mathbb{N}$. Using Theorem 5.2.3 and the invertibility of $R_{A,B}^*$ and $S_{A,\Psi}^*$, we get

$$\theta_B^*\theta_\Psi = (\theta_AR_{A,B})^*\theta_\Psi = R_{A,B}^*\theta_A^*\theta_\Psi = R_{A,B}^*S_{A,\Psi}^* \neq 0.$$

□

For every factorable weak OVF $(\{A_n\}_n, \{\Psi_n\}_n)$, each of 'OVFs' $(\{A_nS_{A,\Psi}^{-1}\}_n, \{\Psi_n\}_n)$ and $(\{A_n\}_n, \{\Psi_n(S_{A,\Psi}^{-1})^*\}_n)$ is a Parseval OVF which is similar to $(\{A_n\}_n, \{\Psi_n\}_n)$. Thus every OVF is similar to Parseval OVFs.

5.3 WEAK OPERATOR-VALUED FRAMES GENERATED BY GROUPS AND GROUP LIKE UNITARY SYSTEMS

In this section G denotes discrete group and π denotes unitary representation of G. Identity element of G is denoted by e.

Definition 5.3.1. *Let π be a unitary representation of a discrete group G on a Hilbert space $\mathscr{H}$. An operator A in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ is called a* ***factorable operator frame generator*** *(resp. a Parseval frame generator) w.r.t. an operator Ψ in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ if $(\{A_g := A\pi_{g^{-1}}\}_{g\in G}, \{\Psi_g := \Psi\pi_{g^{-1}}\}_{g\in G})$ is a factorable weak OVF (resp. Parseval) in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$. In this case, we write (A,Ψ) is an operator frame generator for π.*

Proposition 5.3.2. *Let (A,Ψ) and (B,Φ) be operator frame generators in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ for a unitary representation π of G on $\mathscr{H}$. Then*

(i) $\theta_A\pi_g = (\lambda_g\otimes I_{\mathscr{H}_0})\theta_A, \theta_\Psi\pi_g = (\lambda_g\otimes I_{\mathscr{H}_0})\theta_\Psi, \forall g\in G.$

(ii) $\theta_A^*\theta_B, \theta_\Psi^*\theta_\Phi, \theta_A^*\theta_\Phi$ *are in the commutant* $\pi(G)'$ *of* $\pi(G)''$. *Further,* $S_{A,\Psi}\in\pi(G)'$.

(iii) $\theta_A T\theta_\Psi^*, \theta_A T\theta_B^*, \theta_\Psi T\theta_\Phi^* \in \mathcal{R}(G)\otimes\mathscr{B}(\mathscr{H}_0), \forall T\in\pi(G)'$. *In particular,* $P_{A,\Psi}\in\mathcal{R}(G)\otimes\mathscr{B}(\mathscr{H}_0)$.

Proof. Let $g,p,q\in G$ and $h\in\mathscr{H}_0$.

(i) From the definition of λ_g and χ_q, we get $\lambda_g\chi_q = \chi_{gq}$. Therefore $L_{gq}h = \chi_{gq}\otimes h = \lambda_g\chi_q\otimes h = (\lambda_g\otimes I_{\mathscr{H}_0})(\chi_q\otimes h) = (\lambda_g\otimes I_{\mathscr{H}_0})L_q h$. Using this,

$$\begin{aligned}\theta_A\pi_g &= \sum_{p\in G} L_pA_p\pi_g = \sum_{p\in G} L_pA\pi_{p^{-1}}\pi_g = \sum_{p\in G} L_pA\pi_{p^{-1}g}\\ &= \sum_{q\in G} L_{gq}A\pi_{q^{-1}} = \sum_{q\in G}(\lambda_g\otimes I_{\mathscr{H}_0})L_qA\pi_{q^{-1}} = (\lambda_g\otimes I_{\mathscr{H}_0})\theta_A.\end{aligned}$$

Similarly $\theta_\Psi\pi_g = (\lambda_g\otimes I_{\mathscr{H}_0})\theta_\Psi$.

(ii) $\theta_A^*\theta_B\pi_g = \theta_A^*(\lambda_g\otimes I_{\mathscr{H}_0})\theta_B = ((\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})\theta_A)^*\theta_B = (\theta_A\pi_{g^{-1}})^*\theta_B = \pi_g\theta_A^*\theta_B$. In the same way, $\theta_\Psi^*\theta_\Phi, \theta_A^*\theta_\Phi\in\pi(G)'$. By taking $B=A$ and $\Phi=\Psi$ we get $S_{A,\Psi}\in\pi(G)'$.

(iii) Let $T\in\pi(G)'$. Then

$$\begin{aligned}\theta_A T\theta_\Psi^*(\lambda_g\otimes I_{\mathscr{H}_0}) &= \theta_A T((\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})\theta_\Psi)^* = \theta_A T\pi_g\theta_\Psi^*\\ &= \theta_A\pi_g T\theta_\Psi^* = (\lambda_g\otimes I_{\mathscr{H}_0})\theta_A T\theta_\Psi^*.\end{aligned}$$

From the construction of $\mathcal{L}(G)$, we now get $\theta_A T\theta_\Psi^*\in(\mathcal{L}(G)\otimes\{I_{\mathscr{H}_0}\})' = \mathcal{L}(G)'\otimes\{I_{\mathscr{H}_0}\}' = \mathcal{R}(G)\otimes\mathscr{B}(\mathscr{H}_0)$. Similarly $\theta_A T\theta_B^*, \theta_\Psi S\theta_\Phi^*\in\mathcal{R}(G)\otimes\mathscr{B}(\mathscr{H}_0), \forall S\in\pi(G)'$. For the choice $T = S_{A,\Psi}^{-1}$ we get $P_{A,\Psi}\in\mathcal{R}(G)\otimes\mathscr{B}(\mathscr{H}_0)$.

□

Theorem 5.3.3. *Let G be a discrete group and $(\{A_g\}_{g\in G},\ \{\Psi_g\}_{g\in G})$ be a Parseval factorable weak OVF in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$. Then there is a unitary representation π of G on $\mathscr{H}$ for which*

$$A_g = A_e\pi_{g^{-1}}, \quad \Psi_g = \Psi_e\pi_{g^{-1}}, \quad \forall g \in G$$

if and only if

$$A_{gp}A_{gq}^* = A_pA_q^*, \quad A_{gp}\Psi_{gq}^* = A_p\Psi_q^*, \quad \Psi_{gp}\Psi_{gq}^* = \Psi_p\Psi_q^*, \quad \forall g,p,q \in G.$$

Proof. $(\Rightarrow)$

$$A_{gp}\Psi_{gq}^* = A_e\pi_{(gp)^{-1}}(\Psi_e\pi_{(gq)^{-1}})^* = A_e\pi_{p^{-1}}\pi_{g^{-1}}\pi_g\pi_q\Psi_e^* = A_p\Psi_q^*, \quad \forall g,p,q \in G.$$

Similarly we get other two equalities.

$(\Leftarrow)$ Using assumptions, we use the following three equalities in the proof, among them we derive the second, remainings are similar. For all $g \in G$,

$$(\lambda_g\otimes I_{\mathscr{H}_0})\theta_A\theta_A^* = \theta_A\theta_A^*(\lambda_g\otimes I_{\mathscr{H}_0}),\ (\lambda_g\otimes I_{\mathscr{H}_0})\theta_A\theta_\Psi^* = \theta_A\theta_\Psi^*(\lambda_g\otimes I_{\mathscr{H}_0}),$$
$$(\lambda_g\otimes I_{\mathscr{H}_0})\theta_\Psi\theta_\Psi^* = \theta_\Psi\theta_\Psi^*(\lambda_g\otimes I_{\mathscr{H}_0}).$$

Noticing λ_g is unitary, we get $(\lambda_g\otimes I_{\mathscr{H}_0})^{-1} = (\lambda_g\otimes I_{\mathscr{H}_0})^*$; also we observed in the proof of Proposition 5.3.2 that $(\lambda_g\otimes I_{\mathscr{H}_0})L_q = L_{gq}$. So

$$\begin{aligned}
(\lambda_g\otimes I_{\mathscr{H}_0})\theta_A\theta_\Psi^*(\lambda_g\otimes I_{\mathscr{H}_0})^* &= \left(\sum_{p\in G}(\lambda_g\otimes I_{\mathscr{H}_0})L_pA_p\right)\left(\sum_{q\in G}(\lambda_g\otimes I_{\mathscr{H}_0})L_q\Psi_q\right)^* \\
&= \sum_{p\in G}L_{gp}\left(\sum_{q\in G}A_p\Psi_q^*L_{gq}^*\right) = \sum_{r\in G}L_r\left(\sum_{s\in G}A_{g^{-1}r}\Psi_{g^{-1}s}^*L_s^*\right) \\
&= \sum_{r\in G}L_r\left(\sum_{s\in G}A_r\Psi_s^*L_s^*\right) = \theta_A\theta_\Psi^*.
\end{aligned}$$

Define $\pi : G \ni g \mapsto \pi_g := \theta_\Psi^*(\lambda_g\otimes I_{\mathscr{H}_0})\theta_A \in \mathscr{B}(\mathscr{H})$. By using the Parsevalness,

$$\begin{aligned}
\pi_g\pi_h &= \theta_\Psi^*(\lambda_g\otimes I_{\mathscr{H}_0})\theta_A\theta_\Psi^*(\lambda_h\otimes I_{\mathscr{H}_0})\theta_A = \theta_\Psi^*\theta_A\theta_\Psi^*(\lambda_g\otimes I_{\mathscr{H}_0})(\lambda_h\otimes I_{\mathscr{H}_0})\theta_A \\
&= \theta_\Psi^*(\lambda_{gh}\otimes I_{\mathscr{H}_0})\theta_A = \pi_{gh}, \quad \forall g,h \in G
\end{aligned}$$

and

$$\pi_g\pi_g^* = \theta_\Psi^*(\lambda_g\otimes I_{\mathscr{H}_0})\theta_A\theta_A^*(\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})\theta_\Psi$$

$$
\begin{aligned}
&= \theta_\Psi^*\theta_A\theta_A^*(\lambda_g\otimes I_{\mathscr{H}_0})(\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})\theta_\Psi = I_{\mathscr{H}},\\
\pi_g^*\pi_g &= \theta_A^*(\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})\theta_\Psi\theta_\Psi^*(\lambda_g\otimes I_{\mathscr{H}_0})\theta_A\\
&= \theta_A^*(\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})(\lambda_g\otimes I_{\mathscr{H}_0})\theta_\Psi\theta_\Psi^*\theta_A = I_{\mathscr{H}}, \quad \forall g\in G.
\end{aligned}
$$

Since G has the discrete topology, this proves π is a unitary representation. It remains to prove $A_g = A_e\pi_{g^{-1}}, \Psi_g = \Psi_e\pi_{g^{-1}}$ for all $g\in G$. Indeed,

$$
\begin{aligned}
A_e\pi_{g^{-1}} &= L_e^*\theta_A\theta_\Psi^*(\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})\theta_A = L_e^*(\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})\theta_A\theta_\Psi^*\theta_A\\
&= ((\lambda_g\otimes I_{\mathscr{H}_0})L_e)^*\theta_A = L_{ge}^*\theta_A = A_g,
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_e\pi_{g^{-1}} &= L_e^*\theta_\Psi\theta_\Psi^*(\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})\theta_A = L_e^*(\lambda_{g^{-1}}\otimes I_{\mathscr{H}_0})\theta_\Psi\theta_\Psi^*\theta_A\\
&= ((\lambda_g\otimes I_{\mathscr{H}_0})L_e)^*\theta_\Psi = L_{ge}^*\theta_\Psi = \Psi_g.
\end{aligned}
$$

□

In the direct part of Theorem 5.3.3, we can remove the word 'Parseval' since it has not been used in the proof; same is true in the following corollary.

Corollary 5.3.4. *Let G be a discrete group and $(\{A_g\}_{g\in G}, \{\Psi_g\}_{g\in G})$ be a factorable weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. Then there is a unitary representation π of G on $\mathscr{H}$ for which*

(i) $A_g = A_eS_{A,\Psi}^{-1}\pi_{g^{-1}}S_{A,\Psi}, \Psi_g = \Psi_e\pi_{g^{-1}}$ for all $g\in G$ if and only if

$$
\begin{aligned}
&A_{gp}S_{A,\Psi}^{-1}(S_{A,\Psi}^{-1})^*A_{gq}^* = A_pS_{A,\Psi}^{-1}(S_{A,\Psi}^{-1})^*A_q^*, \quad A_{gp}S_{A,\Psi}^{-1}\Psi_{gq}^* = A_pS_{A,\Psi}^{-1}\Psi_q^*,\\
&\Psi_{gp}\Psi_{gq}^* = \Psi_p\Psi_q^*, \quad \forall g,p,q\in G.
\end{aligned}
$$

(ii) $A_g = A_e\pi_{g^{-1}}, \Psi_g = \Psi_e(S_{A,\Psi}^{-1})^*\pi_{g^{-1}}S_{A,\Psi}$ for all $g\in G$ if and only if

$$
\begin{aligned}
&A_{gp}A_{gq}^* = A_pA_q^*, \quad A_{gp}S_{A,\Psi}^{-1}\Psi_{gq}^* = A_pS_{A,\Psi}^{-1}\Psi_q^*,\\
&\Psi_{gp}(S_{A,\Psi}^{-1})^*S_{A,\Psi}^{-1}\Psi_{gq}^* = \Psi_p(S_{A,\Psi}^{-1})^*S_{A,\Psi}^{-1}\Psi_q^*, \quad \forall g,p,q\in G.
\end{aligned}
$$

Proof. (i) We apply Theorem 5.3.3 to the factorable Parseval OVF $(\{A_gS_{A,\Psi}^{-1}\}_{g\in G}, \{\Psi_g\}_{g\in G})$ to get: there is a unitary representation π of G on $\mathscr{H}$ for which $A_gS_{A,\Psi}^{-1} = (A_eS_{A,\Psi}^{-1})\pi_{g^{-1}}, \Psi_g = \Psi_e\pi_{g^{-1}}$ for all $g\in G$ if and only if

$$
(A_{gp}S_{A,\Psi}^{-1})(A_{gq}S_{A,\Psi}^{-1})^* = (A_pS_{A,\Psi}^{-1})(A_qS_{A,\Psi}^{-1})^*, \quad (A_{gp}S_{A,\Psi}^{-1})\Psi_{gq}^* = (A_pS_{A,\Psi}^{-1})\Psi_q^*,
$$

$$\Psi_{gp}\Psi_{gq}^* = \Psi_p\Psi_q^*, \quad \forall g, p, q \in G.$$

(ii) We apply Theorem 5.3.3 to the factorable Parseval OVF $(\{A_g\}_{g\in G}, \{\Psi_g(S_{A,\Psi}^{-1})^*\}_{g\in G})$ to get: there is a unitary representation π of G on $\mathscr{H}$ for which $A_g = A_e\pi_{g^{-1}}$, $\Psi_g S_{A,\Psi}^{-1} = (\Psi_e(S_{A,\Psi}^{-1})^*)\pi_{g^{-1}}$ for all $g \in G$ if and only if

$$A_{gp}A_{gq}^* = A_pA_q^*, \quad A_{gp}(\Psi_{gq}(S_{A,\Psi}^{-1})^*)^* = A_p(\Psi_q(S_{A,\Psi}^{-1})^*)^*,$$
$$(\Psi_{gp}(S_{A,\Psi}^{-1})^*)(\Psi_{gq}(S_{A,\Psi}^{-1})^*)^* = (\Psi_p(S_{A,\Psi}^{-1})^*)(\Psi_q(S_{A,\Psi}^{-1})^*)^*, \quad \forall g, p, q \in G.$$

□

We next address the situation of factorable weak OVF whenever it is indexed by group-like unitary systems. Group-like unitary systems arose from the study of Weyl-Heisenberg frames. This was first formally defined by Gabardo and Han (2001). In the sequel, by $\mathbb{T}$, we mean the standard unit circle group centered at the origin equipped with usual multiplication.

Definition 5.3.5. *(Gabardo and Han (2001)) A collection $\mathscr{U} \subseteq \mathscr{B}(\mathscr{H})$ containing $I_{\mathscr{H}}$ is called as a* ***unitary system****. If the group generated by unitary system $\mathscr{U}$, denoted by* group$(\mathscr{U})$ *is such that*

(i) group$(\mathscr{U}) \subseteq \mathbb{T}\mathscr{U} := \{\alpha U : \alpha \in \mathbb{T}, U \in \mathscr{U}\}$, *and*

(ii) $\mathscr{U}$ is linearly independent, i.e., $\mathbb{T}U \neq \mathbb{T}V$ whenever $U, V \in \mathscr{U}$ are such that $U \neq V$,

then $\mathscr{U}$ is called as a ***group-like unitary system****.*

Let $\mathscr{U}$ be a group-like unitary system. As in (Gabardo and Han (2003)), we define mappings

$$f : \text{group}(\mathscr{U}) \to \mathbb{T} \quad \text{and} \quad \sigma : \text{group}(\mathscr{U}) \to \mathscr{U}.$$

in the following way. For each $U \in \text{group}(\mathscr{U})$ there are unique $\alpha \in \mathbb{T}, V \in \mathscr{U}$ such that $U = \alpha V$. Define $f(U) = \alpha$ and $\sigma(U) = V$. These f, σ are well-defined and satisfy

$$U = f(U)\sigma(U), \quad \forall U \in \text{group}(\mathscr{U}).$$

These mappings are called as **corresponding mappings** associated to $\mathscr{U}$. We can picturize these maps as follows.

$$\begin{array}{ccc} \text{group}(\mathscr{U}) \subseteq \mathbb{T}\mathscr{U} & & \\ \downarrow \sigma & \searrow^{f} & \\ \mathscr{U} & \longleftarrow & \mathbb{T} \end{array}$$

Next result gives certain fundamental properties of corresponding mappings associated with group-like unitary systems.

Proposition 5.3.6. *(Gabardo and Han (2003)) For a group-like unitary system $\mathscr{U}$ and f, σ as above,*

(i) $f(U\sigma(VW))f(VW) = f(\sigma(UV)W)f(UV), \forall U, V, W \in \text{group}(\mathscr{U})$.

(ii) $\sigma(U\sigma(VW)) = \sigma(\sigma(UV)W), \forall U, V, W \in \text{group}(\mathscr{U})$.

(iii) $\sigma(U) = U$ *and* $f(U) = 1$ *for all* $U \in \mathscr{U}$.

(iv) If $V, W \in \text{group}(\mathscr{U})$, *then*

$$\begin{aligned} \mathscr{U} &= \{\sigma(UV) : U \in \mathscr{U}\} = \{\sigma(VU^{-1}) : U \in \mathscr{U}\} \\ &= \{\sigma(VU^{-1}W) : U \in \mathscr{U}\} = \{\sigma(V^{-1}U) : U \in \mathscr{U}\}. \end{aligned}$$

(v) For fixed $V, W \in \mathscr{U}$, *the following mappings are injective from* $\mathscr{U}$ *to itself:*

$$\begin{aligned} U \mapsto \sigma(VU) \quad (\textit{resp. } &\sigma(UV), \sigma(UV^{-1}), \sigma(V^{-1}U), \\ &\sigma(VU^{-1}), \sigma(U^{-1}V), \sigma(VU^{-1}W)). \end{aligned}$$

Since $\text{group}(\mathscr{U})$ is a group, we note that, in (iv) of Proposition 5.3.6, we can replace V by V^{-1}. Hence, whenever $V \in \text{group}(\mathscr{U})$, we have $\sum_{U \in \mathscr{U}} x_U = \sum_{U \in \mathscr{U}} x_{\sigma(VU)}$.

Definition 5.3.7. *(Gabardo and Han (2003)) A* ***unitary representation*** π *of a group-like unitary system* $\mathscr{U}$ *on* $\mathscr{H}$ *is an injective mapping from* $\mathscr{U}$ *into the set of unitary operators on* $\mathscr{H}$ *such that*

$$\pi(U)\pi(V) = f(UV)\pi(\sigma(UV)), \quad \pi(U)^{-1} = f(U^{-1})\pi(\sigma(U^{-1})), \; \forall U, V \in \mathscr{U},$$

where f *and* σ *are the corresponding mappings associated with* $\mathscr{U}$.

Since π is injective, once we have a unitary representation of a group-like unitary system $\mathscr{U}$ on $\mathscr{H}$, then $\pi(\mathscr{U})$ is also a group-like unitary system.

Let $\mathscr{U}$ be a group-like unitary system and $\{\chi_U\}_{U\in\mathscr{U}}$ be the standard orthonormal basis for $\ell^2(\mathscr{U})$. We define λ on $\mathscr{U}$ by $\lambda_U\chi_V = f(UV)\chi_{\sigma(UV)}, \forall U,V \in \mathscr{U}$. Then λ is a unitary representation which we call as left regular representation of $\mathscr{U}$. Similarly, we define right regular representation of $\mathscr{U}$ by $\rho_U\chi_V = f(VU^{-1})\chi_{\sigma(VU^{-1})}, \forall U,V \in \mathscr{U}$ (Gabardo and Han (2003)). Like frame generators for groups, we now define the frame generator for group-like unitary systems.

Definition 5.3.8. *Let $\mathscr{U}$ be a group-like unitary system. An operator A in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ is called an **operator frame generator** (resp. a Parseval frame generator) w.r.t. Ψ in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ if $(\{A_U := A\pi(U)^{-1}\}_{U\in\mathscr{U}}, \{\Psi_U := \Psi\pi(U)^{-1}\}_{U\in\mathscr{U}})$ is a factorable weak OVF (resp. a Parseval) in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$. We write (A,Ψ) is an operator frame generator for π.*

Theorem 5.3.9. *Let $\mathscr{U}$ be a group-like unitary system, I be the identity of $\mathscr{U}$ and $(\{A_U\}_{U\in\mathscr{U}}, \{\Psi_U\}_{U\in\mathscr{U}})$ be a factorable Parseval weak OVF in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ with θ_A^* injective. Then there is a unitary representation π of $\mathscr{U}$ on $\mathscr{H}$ for which*

$$A_U = A_I\pi(U)^{-1}, \quad \Psi_U = \Psi_I\pi(U)^{-1}, \quad \forall U \in \mathscr{U}$$

if and only if

$$\begin{aligned}
A_{\sigma(UV)}A^*_{\sigma(UW)} &= f(UV)\overline{f(UW)}A_VA^*_W,\\
A_{\sigma(UV)}\Psi^*_{\sigma(UW)} &= f(UV)\overline{f(UW)}A_V\Psi^*_W,\\
\Psi_{\sigma(UV)}\Psi^*_{\sigma(UW)} &= f(UV)\overline{f(UW)}\Psi_V\Psi^*_W, \quad \forall U,V,W \in \mathscr{U}.
\end{aligned}$$

Proof. $(\Rightarrow)$ For all $U,V,W \in \mathscr{U}$, we have

$$\begin{aligned}
A_{\sigma(UV)}A^*_{\sigma(UW)} &= A_I\pi(\sigma(UV))^{-1}(A_I\pi(\sigma(UW))^{-1})^*\\
&= A_I(\overline{f(UV)}\pi(U)\pi(V))^{-1}\overline{f(UW)}\pi(U)\pi(W)A_I^*\\
&= f(UV)\overline{f(UW)}A_I\pi(V)^{-1}(A_I\pi(W)^{-1})^*\\
&= f(UV)\overline{f(UW)}A_VA^*_W.
\end{aligned}$$

Others can be shown similarly.

$(\Leftarrow)$ We have to construct unitary representation which satisfies the stated conditions. Following observation plays an important role in this part. Let $h \in \mathscr{H}$. Then

$$\begin{aligned} L_{\sigma(UV)}h = \chi_{\sigma(UV)} \otimes h = \overline{f(UV)}\lambda_U\chi_V \otimes h = \overline{f(UV)}(\lambda_U\chi_V \otimes h) \\ = \overline{f(UV)}(\lambda_U \otimes I_{\mathscr{H}_0})(\chi_V \otimes h) = \overline{f(UV)}(\lambda_U \otimes I_{\mathscr{H}_0})L_V h. \end{aligned}$$

As in the proof of Theorem 5.3.3, we argue the following, for which now we prove the first. For all $U \in \mathscr{U}$,

$$\begin{aligned} &(\lambda_U \otimes I_{\mathscr{H}_0})\theta_A\theta_A^* = \theta_A\theta_A^*(\lambda_U \otimes I_{\mathscr{H}_0}), \quad (\lambda_U \otimes I_{\mathscr{H}_0})\theta_A\theta_\Psi^* = \theta_A\theta_\Psi^*(\lambda_U \otimes I_{\mathscr{H}_0}), \\ &(\lambda_U \otimes I_{\mathscr{H}_0})\theta_\Psi\theta_\Psi^* = \theta_\Psi\theta_\Psi^*(\lambda_U \otimes I_{\mathscr{H}_0}). \end{aligned}$$

Consider

$$\begin{aligned} (\lambda_U \otimes I_{\mathscr{H}_0})\theta_A\theta_A^*(\lambda_U \otimes I_{\mathscr{H}_0})^* &= \left(\sum_{V\in\mathscr{U}} (\lambda_U \otimes I_{\mathscr{H}_0})L_V A_V\right)\left(\sum_{W\in\mathscr{U}} (\lambda_U \otimes I_{\mathscr{H}_0})L_W A_W\right)^* \\ &= \left(\sum_{V\in\mathscr{U}} f(UV)L_{\sigma(UV)}A_V\right)\left(\sum_{W\in\mathscr{U}} f(UW)L_{\sigma(UW)}A_W\right)^* \\ &= \sum_{V\in\mathscr{U}} L_{\sigma(UV)}\left(\sum_{W\in\mathscr{U}} f(UV)\overline{f(UW)}A_V A_W^* L_{\sigma(UW)}^*\right) \\ &= \sum_{V\in\mathscr{U}} L_{\sigma(UV)}\left(\sum_{W\in\mathscr{U}} A_{\sigma(UV)}A_{\sigma(UW)}^* L_{\sigma(UW)}^*\right) \\ &= \left(\sum_{V\in\mathscr{U}} L_{\sigma(UV)}A_{\sigma(UV)}\right)\left(\sum_{W\in\mathscr{U}} L_{\sigma(UW)}A_{\sigma(UW)}\right)^* \\ &= \theta_A\theta_A^* \end{aligned}$$

where last part of Proposition 5.3.6 is used in the last equality.

Define $\pi : \mathscr{U} \ni U \mapsto \pi(U) := \theta_\Psi^*(\lambda_U \otimes I_{\mathscr{H}_0})\theta_A \in \mathscr{B}(\mathscr{H})$. Then

$$\begin{aligned} \pi(U)\pi(V) &= \theta_\Psi^*(\lambda_U \otimes I_{\mathscr{H}_0})\theta_A\theta_\Psi^*(\lambda_V \otimes I_{\mathscr{H}_0})\theta_A \\ &= \theta_\Psi^*\theta_A\theta_\Psi^*(\lambda_U \otimes I_{\mathscr{H}_0})(\lambda_V \otimes I_{\mathscr{H}_0})\theta_A \\ &= \theta_\Psi^*(\lambda_U\lambda_V \otimes I_{\mathscr{H}_0})\theta_A \\ &= \theta_\Psi^*(f(UV)\lambda_{\sigma(UV)} \otimes I_{\mathscr{H}_0})\theta_A \\ &= f(UV)\theta_\Psi^*(\lambda_{\sigma(UV)} \otimes I_{\mathscr{H}_0})\theta_A \\ &= f(UV)\pi(\sigma(UV)), \quad \forall U, V \in \mathscr{U} \end{aligned}$$

and

$$\begin{aligned}\pi(U)\pi(U)^* &= \theta_\Psi^*(\lambda_U\otimes I_{\mathscr{H}_0})\theta_A\theta_A^*(\lambda_U^*\otimes I_{\mathscr{H}_0})\theta_\Psi\\ &= \theta_\Psi^*\theta_A\theta_A^*(\lambda_U\otimes I_{\mathscr{H}_0})(\lambda_U^*\otimes I_{\mathscr{H}_0})\theta_\Psi = I_{\mathscr{H}},\\ \pi(U)^*\pi(U) &= \theta_A^*(\lambda_U^*\otimes I_{\mathscr{H}_0})\theta_\Psi\theta_\Psi^*(\lambda_U\otimes I_{\mathscr{H}_0})\theta_A\\ &= \theta_A^*(\lambda_U^*\otimes I_{\mathscr{H}_0})(\lambda_U\otimes I_{\mathscr{H}_0})\theta_\Psi\theta_\Psi^*\theta_A = I_{\mathscr{H}}, \quad \forall U\in\mathscr{U}.\end{aligned}$$

Further,

$$\begin{aligned}\pi(U)f(U^{-1})\pi(\sigma(U^{-1})) &= \theta_\Psi^*(\lambda_U\otimes I_{\mathscr{H}_0})\theta_A f(U^{-1})\theta_\Psi^*(\lambda_{\sigma(U^{-1})}\otimes I_{\mathscr{H}_0})\theta_A\\ &= f(U^{-1})\theta_\Psi^*\theta_A\theta_\Psi^*(\lambda_U\otimes I_{\mathscr{H}_0})(\lambda_{\sigma(U^{-1})}\otimes I_{\mathscr{H}_0})\theta_A\\ &= f(U^{-1})\theta_\Psi^*(\lambda_U\otimes I_{\mathscr{H}_0})(\lambda_{\sigma(U^{-1})}\otimes I_{\mathscr{H}_0})\theta_A\\ &= f(U^{-1})\theta_\Psi^*(\lambda_U\lambda_{\sigma(U^{-1})}\otimes I_{\mathscr{H}_0})\theta_A\\ &= f(U^{-1})\theta_\Psi^*(f(U\sigma(U^{-1}))\lambda_{\sigma(U\sigma(U^{-1}))}\otimes I_{\mathscr{H}_0})\theta_A\\ &= \theta_\Psi^*(f(U\sigma(U^{-1}I))f(U^{-1}I)\lambda_{\sigma(U\sigma(U^{-1}I))}\otimes I_{\mathscr{H}_0})\theta_A\\ &= \theta_\Psi^*(f(\sigma(UU^{-1})I)f(UU^{-1})\lambda_{\sigma(\sigma(UU^{-1})I)}\otimes I_{\mathscr{H}_0})\theta_A\\ &= \theta_\Psi^*(\lambda_I\otimes I_{\mathscr{H}_0})\theta_A = I_{\mathscr{H}}\end{aligned}$$

$\Rightarrow \pi(U)^{-1} = f(U^{-1})\pi(\sigma(U^{-1}))$ for all $U\in\mathscr{U}$. We shall now use θ_A^* is injective to show π is injective and thereby to get π is a unitary representation. Let $\pi(U)=\pi(V)$. Then

$$\begin{aligned}&\theta_\Psi^*(\lambda_U\otimes I_{\mathscr{H}_0})\theta_A = \theta_\Psi^*(\lambda_V\otimes I_{\mathscr{H}_0})\theta_A \Rightarrow \theta_\Psi^*(\lambda_U\otimes I_{\mathscr{H}_0})\theta_A\theta_A^* = \theta_\Psi^*(\lambda_V\otimes I_{\mathscr{H}_0})\theta_A\theta_A^*\\ &\Rightarrow \theta_\Psi^*\theta_A\theta_A^*(\lambda_U\otimes I_{\mathscr{H}_0}) = \theta_\Psi^*\theta_A\theta_A^*(\lambda_V\otimes I_{\mathscr{H}_0}) \Rightarrow \lambda_U\otimes I_{\mathscr{H}_0} = \lambda_V\otimes I_{\mathscr{H}_0}.\end{aligned}$$

We show U and V are identical at elementary tensors. For $h\in\ell^2(\mathscr{U}), y\in\mathscr{H}_0$, we get, $(\lambda_U\otimes I_{\mathscr{H}_0})(h\otimes y) = (\lambda_V\otimes I_{\mathscr{H}_0})(h\otimes y) \Rightarrow \lambda_U h\otimes y = \lambda_V h\otimes y \Rightarrow (\lambda_U-\lambda_V)h\otimes y = 0 \Rightarrow 0 = \langle(\lambda_U-\lambda_V)h\otimes y, (\lambda_U-\lambda_V)h\otimes y\rangle = \|(\lambda_U-\lambda_V)h\|^2\|y\|^2$. We may assume $y\neq 0$ (if $y=0$, then $h\otimes y=0$). But then $(\lambda_U-\lambda_V)(h)=0$, and λ is a unitary representation (it is injective) gives $U=V$. We now show $A_U = A_I\pi(U)^{-1}$ and $\Psi_U = \Psi_I\pi(U)^{-1}$ for all $U\in\mathscr{U}$ in the following:

$$\begin{aligned}A_I\pi(U)^{-1} &= L_I^*\theta_A(\theta_\Psi^*(\lambda_U\otimes I_{\mathscr{H}_0})\theta_A)^* = L_I^*(\theta_\Psi^*(\lambda_U\otimes I_{\mathscr{H}_0})\theta_A\theta_A^*)^*\\ &= L_I^*(\theta_\Psi^*\theta_A\theta_A^*(\lambda_U\otimes I_{\mathscr{H}_0}))^* = L_I^*(\theta_A^*(\lambda_U\otimes I_{\mathscr{H}_0}))^*\\ &= (\theta_A^*(\lambda_U\otimes I_{\mathscr{H}_0})L_I)^* = (\theta_A^*\overline{f(UI)}(\lambda_U\otimes I_{\mathscr{H}_0})L_I)^*\end{aligned}$$

$$= (\theta_A^* L_{\sigma(UI)})^* = L_U^* \theta_A = A_U$$

and

$$\begin{aligned}\Psi_I \pi(U)^{-1} &= L_I^* \theta_\Psi (\theta_\Psi^* (\lambda_U \otimes I_{\mathscr{H}_0}) \theta_A)^* = L_I^* (\theta_\Psi^* (\lambda_U \otimes I_{\mathscr{H}_0}) \theta_A \theta_\Psi^*)^* \\ &= L_I^* (\theta_\Psi^* \theta_A \theta_\Psi^* (\lambda_U \otimes I_{\mathscr{H}_0}))^* = L_I^* (\theta_\Psi^* (\lambda_U \otimes I_{\mathscr{H}_0}))^* \\ &= (\theta_\Psi^* (\lambda_U \otimes I_{\mathscr{H}_0}) L_I)^* = (\theta_\Psi^* \overline{f(UI)} (\lambda_U \otimes I_{\mathscr{H}_0}) L_I)^* \\ &= (\theta_\Psi^* L_{\sigma(UI)})^* = L_U^* \theta_\Psi = \Psi_U.\end{aligned}$$

□

Note that neither Parsevalness of the frame nor θ_A^* is injective was used in the direct part of Theorem 5.3.9. Since θ_A acts between Hilbert spaces, we know that $\overline{\theta_A(\mathscr{H})} = \operatorname{Ker}(\theta_A^*)^\perp$ and $\operatorname{Ker}(\theta_A^*) = \theta_A(\mathscr{H})^\perp$. From Lemma 5.1.11, the range of θ_A is closed. Therefore $\theta_A(\mathscr{H}) = \operatorname{Ker}(\theta_A^*)^\perp$. Thus the condition θ_A^* is injective in the Theorem 5.3.9 can be replaced by θ_A is onto.

Corollary 5.3.10. *Let $\mathscr{U}$ be a group-like unitary system, I be the identity of $\mathscr{U}$ and $(\{A_U\}_{U\in\mathscr{U}}, \{\Psi_U\}_{U\in\mathscr{U}})$ be a factorable weak OVF in $\mathscr{B}(\mathscr{H},\mathscr{H}_0)$ with θ_A^* is injective. Then there is a unitary representation π of $\mathscr{U}$ on $\mathscr{H}$ for which*

(i) $A_U = A_I S_{A,\Psi}^{-1} \pi(U)^{-1} S_{A,\Psi}, \Psi_U = \Psi_I \pi(U)^{-1}$ *for all $U \in \mathscr{U}$ if and only if*

$$\begin{aligned}&A_{\sigma(UV)} S_{A,\Psi}^{-1} (S_{A,\Psi}^{-1})^* A_{\sigma(UW)}^* = f(UV)\overline{f(UW)} A_V S_{A,\Psi}^{-1} (S_{A,\Psi}^{-1})^* A_W^*, \\ &A_{\sigma(UV)} S_{A,\Psi}^{-1} \Psi_{\sigma(UW)}^* = f(UV)\overline{f(UW)} A_V S_{A,\Psi}^{-1} \Psi_W^*, \\ &\Psi_{\sigma(UV)} \Psi_{\sigma(UW)}^* = f(UV)\overline{f(UW)} \Psi_V \Psi_W^*, \quad \forall U,V,W \in \mathscr{U}.\end{aligned}$$

(ii) $A_U = A_I \pi(U)^{-1}, \Psi_U = \Psi_I (S_{A,\Psi}^{-1})^* \pi(U)^{-1} S_{A,\Psi}$ *for all $U \in \mathscr{U}$ if and only if*

$$\begin{aligned}&A_{\sigma(UV)} A_{\sigma(UW)}^* = f(UV)\overline{f(UW)} A_V A_W^*, \\ &A_{\sigma(UV)} S_{A,\Psi}^{-1} \Psi_{\sigma(UW)}^* = f(UV)\overline{f(UW)} A_V S_{A,\Psi}^{-1} \Psi_W^*, \\ &\Psi_{\sigma(UV)} (S_{A,\Psi}^{-1})^* S_{A,\Psi}^{-1} \Psi_{\sigma(UW)}^* = f(UV)\overline{f(UW)} \Psi_V (S_{A,\Psi}^{-1})^* S_{A,\Psi}^{-1} \Psi_W^*, \quad \forall U,V,W \in \mathscr{U}.\end{aligned}$$

Proof. (i) We apply Theorem 5.3.9 to the factorable Parseval OVF $(\{A_U S_{A,\Psi}^{-1}\}_{U\in\mathscr{U}}, \{\Psi_U\}_{U\in\mathscr{U}})$. There is a unitary representation π of $\mathscr{U}$ on $\mathscr{H}$ for which $A_U S_{A,\Psi}^{-1} = (A_I S_{A,\Psi}^{-1}) \pi(U)^{-1}, \Psi_U = \Psi_I \pi(U)^{-1}$ for all $U \in \mathscr{U}$ if and only if

$$(A_{\sigma(UV)} S_{A,\Psi}^{-1})(A_{\sigma(UW)} S_{A,\Psi}^{-1})^* = f(UV)\overline{f(UW)} (A_V S_{A,\Psi}^{-1})(A_W S_{A,\Psi}^{-1})^*,$$

$$(A_{\sigma(UV)}S_{A,\Psi}^{-1})\Psi_{\sigma(UW)}^* = f(UV)\overline{f(UW)}(A_V S_{A,\Psi}^{-1})\Psi_W^*,$$
$$\Psi_{\sigma(UV)}\Psi_{\sigma(UW)}^* = f(UV)\overline{f(UW)}\Psi_V\Psi_W^*, \quad \forall U,V,W \in \mathscr{U}.$$

(ii) We apply Theorem 5.3.9 to the factorable Parseval OVF $(\{A_U\}_{U\in\mathscr{U}}, \{\Psi_U(S_{A,\Psi}^{-1})^*\}_{U\in\mathscr{U}})$. There is a unitary representation π of $\mathscr{U}$ on $\mathscr{H}$ for which $A_U = A_I\pi(U)^{-1}, \Psi_U(S_{A,\Psi}^{-1})^* = (\Psi_I(S_{A,\Psi}^{-1})^*)\pi(U)^{-1}$ for all $U \in \mathscr{U}$ if and only if

$$A_{\sigma(UV)}A_{\sigma(UW)}^* = f(UV)\overline{f(UW)}A_V A_W^*,$$
$$A_{\sigma(UV)}(\Psi_{\sigma(UW)}(S_{A,\Psi}^{-1})^*)^* = f(UV)\overline{f(UW)}A_V(\Psi_W(S_{A,\Psi}^{-1})^*)^*,$$
$$(\Psi_{\sigma(UV)}(S_{A,\Psi}^{-1})^*)(\Psi_{\sigma(UW)}(S_{A,\Psi}^{-1})^*)^* = f(UV)\overline{f(UW)}(\Psi_V(S_{A,\Psi}^{-1})^*)(\Psi_W(S_{A,\Psi}^{-1})^*)^*,$$
$$\forall U,V,W \in \mathscr{U}.$$

□

5.4 PERTURBATIONS OF WEAK OPERATOR-VALUED FRAMES

In this section we derive stability results for factorable weak operator-valued frames.

Theorem 5.4.1. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a factorable weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. Suppose $\{B_n\}_n$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is such that there exist $\alpha, \beta, \gamma \geq 0$ with $\max\{\alpha + \gamma\|\theta_\Psi(S_{A,\Psi}^*)^{-1}\|, \beta\} < 1$ and for all $m = 1, 2, \ldots,$*

$$\left\|\sum_{n=1}^m (A_n^* - B_n^*)L_n^* y\right\| \leq \alpha \left\|\sum_{n=1}^m A_n^* L_n^* y\right\| + \beta \left\|\sum_{n=1}^m B_n^* L_n^* y\right\| + \gamma \left(\sum_{n=1}^m \|L_n^* y\|^2\right)^{\frac{1}{2}},$$
$$\forall y \in \ell^2(\mathbb{N}) \otimes \mathscr{H}_0. \tag{5.4.1}$$

Then $(\{B_n\}_n, \{\Psi_n\}_n)$ is a factorable weak OVF with bounds

$$\frac{1 - (\alpha + \gamma\|\theta_\Psi(S_{A,\Psi}^*)^{-1}\|)}{(1+\beta)\|(S_{A,\Psi}^*)^{-1}\|} \quad \textit{and} \quad \frac{\|\theta_\Psi\|((1+\alpha)\|\theta_A\| + \gamma)}{1-\beta}.$$

Proof. For $m = 1, 2, \ldots,$ and for every y in $\ell^2(\mathbb{N}) \otimes \mathscr{H}_0$,

$$\left\|\sum_{n=1}^m B_n^* L_n^* y\right\| \leq \left\|\sum_{n=1}^m (A_n^* - B_n^*)L_n^* y\right\| + \left\|\sum_{n=1}^m A_n^* L_n^* y\right\|$$
$$\leq (1+\alpha)\left\|\sum_{n=1}^m A_n^* L_n^* y\right\| + \beta \left\|\sum_{n=1}^m B_n^* L_n^* y\right\| + \gamma \left(\sum_{n=1}^m \|L_n^* y\|^2\right)^{\frac{1}{2}}$$

which implies

$$\left\|\sum_{n=1}^{m}B_n^*L_n^*y\right\| \leq \frac{1+\alpha}{1-\beta}\left\|\sum_{n=1}^{m}A_n^*L_n^*y\right\| + \frac{\gamma}{1-\beta}\left(\sum_{n=1}^{m}\|L_n^*y\|^2\right)^{\frac{1}{2}}, \quad \forall y \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0. \tag{5.4.2}$$

Since

$$\langle y,y\rangle = \langle (I_{\ell^2(\mathbb{N})}\otimes I_{\mathscr{H}_0})y,y\rangle = \left\langle \sum_{n=1}^{\infty}L_nL_n^*y,y\right\rangle = \sum_{n=1}^{\infty}\|L_n^*y\|^2, \quad \forall y \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0,$$

Inequality (5.4.2) shows that $\sum_{n=1}^{\infty}B_n^*L_n^*y$ exists for all $y \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0$. From the continuity of norm, Inequality (5.4.2) gives

$$\begin{aligned}\left\|\sum_{n=1}^{\infty}B_n^*L_n^*y\right\| &\leq \frac{1+\alpha}{1-\beta}\left\|\sum_{n=1}^{\infty}A_n^*L_n^*y\right\| + \frac{\gamma}{1-\beta}\left(\sum_{n=1}^{\infty}\|L_n^*y\|^2\right)^{\frac{1}{2}}\\ &= \frac{1+\alpha}{1-\beta}\|\theta_A^*y\| + \frac{\gamma}{1-\beta}\|y\|, \quad \forall y \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0\end{aligned} \tag{5.4.3}$$

and this gives $\sum_{n=1}^{\infty}B_n^*L_n^*$ is bounded; therefore its adjoint exists, which is θ_B; Inequality (5.4.3) now produces $\|\theta_B^*y\| \leq \frac{1+\alpha}{1-\beta}\left\|\theta_A^*y\right\| + \frac{\gamma}{1-\beta}\|y\|, \forall y \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0$ and from this $\|\theta_B\| = \|\theta_B^*\| \leq \frac{1+\alpha}{1-\beta}\left\|\theta_A^*\right\| + \frac{\gamma}{1-\beta} = \frac{1+\alpha}{1-\beta}\|\theta_A\| + \frac{\gamma}{1-\beta}$. Thus we derived $S_{B,\Psi}$ is a bounded linear operator. Continuity of the norm, existence of frame operators together with Inequality (5.4.1) give

$$\|\theta_A^*y - \theta_B^*y\| \leq \alpha\|\theta_A^*y\| + \beta\|\theta_B^*y\| + \gamma\|y\|, \quad \forall y \in \ell^2(\mathbb{N})\otimes\mathscr{H}_0$$

which implies

$$\begin{aligned}\|\theta_A^*(\theta_\Psi(S_{A,\Psi}^*)^{-1}h) - \theta_B^*(\theta_\Psi(S_{A,\Psi}^*)^{-1}h)\| &\leq \alpha\|\theta_A^*(\theta_\Psi(S_{A,\Psi}^*)^{-1}h)\|\\ &\quad + \beta\|\theta_B^*(\theta_\Psi(S_{A,\Psi}^*)^{-1}h)\| + \gamma\|\theta_\Psi(S_{A,\Psi}^*)^{-1}h\|,\\ &\quad \forall h \in \mathscr{H}.\end{aligned}$$

But $\theta_A^*\theta_\Psi(S_{A,\Psi}^*)^{-1} = I_{\mathscr{H}}$ and $\theta_B^*\theta_\Psi(S_{A,\Psi}^*)^{-1} = S_{B,\Psi}^*(S_{A,\Psi}^*)^{-1}$. Therefore

$$\begin{aligned}\|h - S_{B,\Psi}^*(S_{A,\Psi}^*)^{-1}h\| &\leq \alpha\|h\| + \beta\|S_{B,\Psi}^*(S_{A,\Psi}^*)^{-1}h\| + \gamma\|\theta_\Psi(S_{A,\Psi}^*)^{-1}h\|\\ &\leq (\alpha + \gamma\|\theta_\Psi(S_{A,\Psi}^*)^{-1}\|)\|h\| + \beta\|S_{B,\Psi}^*(S_{A,\Psi}^*)^{-1}h\|, \quad \forall h \in \mathscr{H}.\end{aligned}$$

Since $\max\{\alpha + \gamma\|\theta_\Psi(S_{A,\Psi}^*)^{-1}\|, \beta\} < 1$, Theorem 4.6.1 tells that $S_{B,\Psi}^*(S_{A,\Psi}^*)^{-1}$ is invertible and $\|(S^*_{B,\Psi}(S^*_{A,\Psi})^{-1})^{-1}\| \leq \frac{1+\beta}{1-(\alpha+\gamma\|\theta_\Psi(S^*_{A,\Psi})^{-1}\|)}$. From these, we get

$$(S^*_{B,\Psi}(S^*_{A,\Psi})^{-1})S^*_{A,\Psi} = S^*_{B,\Psi}$$

is invertible and

$$\|S^{-1}_{B,\Psi}\| \leq \|(S^*_{A,\Psi})^{-1}\|\|S^*_{A,\Psi}S^{-1}_{B,\Psi}\| \leq \frac{\|(S^*_{A,\Psi})^{-1}\|(1+\beta)}{1-(\alpha+\gamma\|\theta_\Psi(S^*_{A,\Psi})^{-1}\|)}.$$

Therefore $(\{B_n\}_n, \{\Psi_n\}_n)$ is a factorable weak OVF. Observing that

$$\|S_{B,\Psi}\| \leq \|\theta_\Psi\|\|\theta_B\| \leq \frac{\|\theta_\Psi\|((1+\alpha)\|\theta_A\|+\gamma)}{1-\beta}$$

and $\|S^{-1}_{B,\Psi}\|^{-1}$ and $\|S_{B,\Psi}\|$ are optimal lower and upper frame bounds for $(\{B_n\}_n, \{\Psi_n\}_n)$, we get the frame bounds stated in the theorem. □

Corollary 5.4.2. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a factorable weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. Suppose $\{B_n\}_n$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is such that*

$$r := \sum_{n=1}^{\infty} \|A_n - B_n\|^2 < \frac{1}{\|\theta_\Psi(S^*_{A,\Psi})^{-1}\|^2}.$$

Then $(\{B_n\}_n, \{\Psi_n\}_n)$ is a factorable weak OVF with bounds

$$\frac{1-\sqrt{r}\|\theta_\Psi(S^*_{A,\Psi})^{-1}\|}{\|(S^*_{A,\Psi})^{-1}\|} \quad \text{and} \quad \|\theta_\Psi\|(\|\theta_A\|+\sqrt{r}).$$

Proof. We apply Theorem 5.4.1 by taking $\alpha = 0, \beta = 0, \gamma = \sqrt{r}$. Then $\max\{\alpha + \gamma\|\theta_\Psi(S^*_{A,\Psi})^{-1}\|, \beta\} < 1$ and for all $m = 1, 2, \ldots,$

$$\begin{aligned}\left\|\sum_{n=1}^{m}(A^*_n - B^*_n)L^*_n y\right\| &\leq \left(\sum_{n=1}^{m}\|A^*_n - B^*_n\|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{m}\|L^*_n y\|^2\right)^{\frac{1}{2}} \\ &\leq \gamma\left(\sum_{n=1}^{m}\|L^*_n y\|^2\right)^{\frac{1}{2}}, \quad \forall y \in \ell^2(\mathbb{N}) \otimes \mathscr{H}_0.\end{aligned}$$

□

We next derive another stability result with different condition.

Theorem 5.4.3. *Let $(\{A_n\}_n, \{\Psi_n\}_n)$ be a factorable weak OVF in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$. Suppose $\{B_n\}_n$ in $\mathscr{B}(\mathscr{H}, \mathscr{H}_0)$ is such that $\sum_{n=1}^{\infty}\|A_n - B_n\|^2$ converges, and $\sum_{n=1}^{\infty}\|A_n -$*

$B_n\|\|\Psi_n(S_{A,\Psi}^*)^{-1}\| < 1$. *Then* $(\{B_n\}_n, \{\Psi_n\}_n)$ *is a factorable weak OVF with bounds*

$$\frac{1-\sum_{n=1}^{\infty}\|A_n-B_n\|\|\Psi_n(S_{A,\Psi}^*)^{-1}\|}{\|(S_{A,\Psi}^*)^{-1}\|} \quad \text{and} \quad \|\theta_\Psi\|\left(\left(\sum_{n=1}^{\infty}\|A_n-B_n\|^2\right)^{1/2}+\|\theta_A\|\right).$$

Proof. Let $\alpha = \sum_{n=1}^{\infty}\|A_n-B_n\|^2$ and $\beta = \sum_{n=1}^{\infty}\|A_n-B_n\|\|\Psi_n(S_{A,\Psi}^*)^{-1}\|$. For $m = 1,2,\ldots$ and for every y in $\ell^2(\mathbb{N})\otimes\mathscr{H}_0$,

$$\begin{aligned}
\left\|\sum_{n=1}^{m}B_n^*L_n^*y\right\| &\leq \left\|\sum_{n=1}^{m}(A_n^*-B_n^*)L_n^*y\right\| + \left\|\sum_{n=1}^{m}A_n^*L_n^*y\right\| \\
&\leq \sum_{n=1}^{m}\|A_n-B_n\|\|L_n^*y\| + \left\|\sum_{n=1}^{m}A_n^*L_n^*y\right\| \\
&\leq \left(\sum_{n=1}^{m}\|A_n-B_n\|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{m}\|L_n^*y\|^2\right)^{\frac{1}{2}} + \left\|\sum_{n=1}^{m}A_n^*L_n^*y\right\| \\
&\leq \alpha^{\frac{1}{2}}\left(\sum_{n=1}^{m}\|L_n^*y\|^2\right)^{\frac{1}{2}} + \left\|\sum_{n=1}^{m}A_n^*L_n^*y\right\| \\
&= \alpha^{\frac{1}{2}}\left\langle\sum_{n=1}^{m}L_nL_n^*y, y\right\rangle^{\frac{1}{2}} + \left\|\sum_{n=1}^{m}A_n^*L_n^*y\right\|,
\end{aligned}$$

which converges to $\sqrt{\alpha}\|y\| + \|\theta_A^*y\|$. Hence θ_B exists and $\|\theta_B\| \leq \sqrt{\alpha} + \|\theta_A\|$. Therefore $S_{B,\Psi} = \theta_\Psi^*\theta_B = \sum_{n=1}^{\infty}\Psi_n^*B_n$ exists. Now

$$\begin{aligned}
\|I_{\mathscr{H}} - S_{B,\Psi}(S_{A,\Psi}^*)^{-1}\| &= \left\|\sum_{n=1}^{\infty}A_n^*\Psi_n(S_{A,\Psi}^*)^{-1} - \sum_{n=1}^{\infty}B_n^*\Psi_n(S_{A,\Psi}^*)^{-1}\right\| \\
&= \left\|\sum_{n=1}^{\infty}(A_n^*-B_n^*)\Psi_n(S_{A,\Psi}^*)^{-1}\right\| \\
&\leq \sum_{n=1}^{\infty}\|A_n-B_n\|\|\Psi_n(S_{A,\Psi}^*)^{-1}\| = \beta < 1.
\end{aligned}$$

Therefore $S_{B,\Psi}(S_{A,\Psi}^*)^{-1}$ is invertible and $\|(S_{B,\Psi}(S_{A,\Psi}^*)^{-1})^{-1}\| \leq 1/(1-\beta)$. Calculation of frame bounds is similar to proof of Theorem 5.4.1. $\square$

CHAPTER 6

CONCLUSION AND FUTURE WORK

In Chapter 2 we initiated the study of frames for metric spaces. Since metric spaces are more general objects and have less structure than Banach spaces, study of frames for metric spaces goes in a different way than that of frames for Hilbert as well as for Banach spaces. Arens-Eells space is used as a tool which allows to use the functional analysis technique to Lipschitz functions. However, this works good only when the co domain of Lipschitz functions is a Banach space. Most of the results in Chapter 2 are concentrated whenever the codomain of Lipschitz function is a Banach space. In future we are interested to work on frames for arbitrary metric spaces.

In Chapter 3 we defined multipliers for metric spaces. We obtained some fundamental properties of multipliers. One of the future work is to explore further on multipliers in metric spaces.

In Chapter 4 we studied a special class of approximate Schauder frames. We characterized a class of approximate Schauder frames and its duals. It is planned to obtain the description of frames and its duals for Banach spaces.

In Chapter 5 we initiated the study of the series $\sum_{n=1}^{\infty}\Psi_n^*A_n$. We mainly obtained results whenever this series is factored as the product of two bounded linear operators. In the future we are planning to study the series without factorability condition. We are also interested in studying path-connectedness of weak OVFs and try to get a result similar to Theorem 1.6.25.

APPENDIX A: DILATIONS OF LINEAR MAPS ON VECTOR SPACES

6.1 DILATIONS OF FUNCTIONS ON SETS

One of the most useful results in the study of isometries on Hilbert spaces is the Wold decomposition. It describes the structure of an isometry. It uses the notion of a shift.

Definition 6.1.1. *(cf. Sz.-Nagy et al. (2010)) Let $\mathscr{H}$ be a Hilbert space. An operator $T : \mathscr{H} \to \mathscr{H}$ is called a **shift** if $\cap_{n=0}^{\infty} T^n(\mathscr{H}) = \{0\}$.*

Theorem 6.1.2. *(cf. Sz.-Nagy et al. (2010); Wold (1954)) (**Wold decomposition**) Let T be an isometry on a Hilbert space $\mathscr{H}$. Then $\mathscr{H}$ decomposes uniquely as $\mathscr{H} = \mathscr{H}_u \oplus \mathscr{H}_s$, where $\mathscr{H}_u$ and $\mathscr{H}_s$ are T-reducing subspaces of $\mathscr{H}$, $T_{|\mathscr{H}_u} : \mathscr{H}_u \to \mathscr{H}_u$ is a unitary and $T_{|\mathscr{H}_s} : \mathscr{H}_s \to \mathscr{H}_s$ is a shift.*

Using functional calculus and Weierstrass polynomial approximation theorem, Halmos in 1950 proved an important result that every contraction on a Hilbert space can be lifted to unitary.

Theorem 6.1.3. *(Halmos (1950)) (**Halmos dilation**) Let $\mathscr{H}$ be a Hilbert space and $T : \mathscr{H} \to \mathscr{H}$ be a contraction. Then the operator*

$$U := \begin{pmatrix} T & \sqrt{I - TT^*} \\ \sqrt{I - T^*T} & -T^* \end{pmatrix}$$

is unitary on $\mathscr{H} \oplus \mathscr{H}$. In other words,

$$T = P_{\mathscr{H}} U_{|\mathscr{H}},$$

where $P_{\mathscr{H}} : \mathscr{H} \oplus \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ is the orthogonal projection onto $\mathscr{H}$.

Three years later, Sz. Nagy extended the result of Halmos which reads as follows.

Theorem 6.1.4. *(Sz.-Nagy (1953)) (**Sz. Nagy dilation**) Let $\mathscr{H}$ be a Hilbert space and $T : \mathscr{H} \to \mathscr{H}$ be a contraction. Then there exists a Hilbert space $\mathscr{K}$ which contains $\mathscr{H}$ isometrically and a unitary $U : \mathscr{K} \to \mathscr{K}$ such that*

$$T^n = P_{\mathscr{H}} U_{|\mathscr{H}},^n \quad \forall n = 1, 2, \ldots,$$

where $P_{\mathscr{H}} : \mathscr{K} \to \mathscr{K}$ is the orthogonal projection onto $\mathscr{H}$.

Unitary operator U in Theorem 6.1.4 is known as dilation operator and the space $\mathscr{K}$ is called as dilation space. If

$$\mathscr{K} = \overline{\text{span}}\{U^n h :, n \in \mathbb{Z}_+, h \in \mathscr{H}\},$$

then $(\mathscr{K}, U)$ is said to be a minimal dilation. It is known that in Theorem 6.1.4, the space $\mathscr{K}$ can be taken as a minimal space. It was Schaffer (1955) who gave a proof of Sz. Nagy dilation theorem using infinite matrices. In the following theorem, $\oplus_{n=-\infty}^{\infty}\mathscr{H}$ is the Hilbert space defined by

$$\oplus_{n=-\infty}^{\infty}\mathscr{H} := \left\{\{h_n\}_{n=-\infty}^{\infty}, h_n \in \mathscr{H}, \forall n \in \mathbb{Z}, \sum_{n=-\infty}^{\infty} \|h_n\|^2 < \infty\right\}$$

with respect to the inner product

$$\langle \{h_n\}_{n=-\infty}^{\infty}, \{g_n\}_{n=-\infty}^{\infty}\rangle := \sum_{n=-\infty}^{\infty} \langle h_n, g_n\rangle, \quad \forall \{h_n\}_{n=-\infty}^{\infty}, \{g_n\}_{n=-\infty}^{\infty} \in \oplus_{n=-\infty}^{\infty}\mathscr{H}.$$

Theorem 6.1.5. *(Schaffer (1955)) Let $\mathscr{H}$ be a Hilbert space and $T : \mathscr{H} \to \mathscr{H}$ be a contraction. Let $U := [u_{n,m}]_{-\infty<n,m<\infty}$ be the* ***Schaffer operator*** *defined on $\oplus_{n=-\infty}^{\infty}\mathscr{H}$ given by the infinite matrix defined as follows:*

$$u_{0,0} := T, \quad u_{0,1} := \sqrt{I - TT^*}, \quad u_{-1,0} := \sqrt{I - T^*T},$$
$$u_{-1,1} := -T^*, \quad u_{n,n+1} := I, \ \forall n \in \mathbb{Z}, n \neq 0, 1, \quad u_{n,m} := 0, \quad \textit{otherwise},$$

i.e.,

$$U = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & I & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \sqrt{I-T^*T} & -T^* & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{T} & \sqrt{I-TT^*} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}_{\infty\times\infty}$$

where T is in the $(0,0)$ position (which is in the box), is invertible on $\oplus_{n=-\infty}^{\infty}\mathscr{H}$ and

$$T^n = P_{\mathscr{H}} U^n_{|\mathscr{H}}, \quad \forall n \in \mathbb{N},$$

where $P_{\mathscr{H}} : \oplus_{n=-\infty}^{\infty}\mathscr{H} \to \oplus_{n=-\infty}^{\infty}\mathscr{H}$ *is the orthogonal projection onto* $\mathscr{H}$.

After a year of work of Sz. Nagy, it was Egervary who observed that Halmos dilation of contraction can be extended finitely so that power of dilation will be dilation of power of contraction.

Theorem 6.1.6. *(Egervary (1954)) (**N-dilation**) Let* $\mathscr{H}$ *be a Hilbert space and* $T : \mathscr{H} \to \mathscr{H}$ *be a contraction. Let N be a natural number. Then the operator*

$$U := \begin{pmatrix} T & 0 & 0 & \cdots & 0 & \sqrt{I-TT^*} \\ \sqrt{I-T^*T} & 0 & 0 & \cdots & 0 & -T^* \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & I & 0 \end{pmatrix}_{(N+1)\times(N+1)}$$

is unitary on $\oplus_{k=1}^{N+1}\mathscr{H}$ *and*

$$T^k = P_{\mathscr{H}} U^k_{|\mathscr{H}}, \quad \forall k = 1, \ldots, N,$$

where $P_{\mathscr{H}} : \oplus_{k=1}^{N+1}\mathscr{H} \to \oplus_{k=1}^{N+1}\mathscr{H}$ *is the orthogonal projection onto* $\mathscr{H}$.

A very useful result which can be derived using Theorem 6.1.6 is the von Neumann's inequality. It was derived by von Neumann (1951) using the theory of analytic functions.

Theorem 6.1.7. *(**von Neumann inequality**) (Rainone (2007); Shalit (2021); von Neumann (1951)) Let* $\mathscr{H}$ *be a Hilbert space and* $T : \mathscr{H} \to \mathscr{H}$ *be a contraction. Then for every polynomial* $p \in \mathbb{C}[z]$,

$$\|p(T)\| \leq \sup_{|z|=1} |p(z)|.$$

Sz. Nagy's dilation theorem leads to the study of dilating more than one operator which are commuting. After a decade of work of Sz. Nagy, Ando derived the following result.

Theorem 6.1.8. *(Ando (1963)) (**Ando dilation**) Let* $\mathscr{H}$ *be a Hilbert space and* $T_1, T_2 : \mathscr{H} \to \mathscr{H}$ *be commuting contractions. Then there exist a Hilbert space* $\mathscr{K}$ *which con-*

tains $\mathscr{H}$ isometrically and a pair of commuting unitaries $U_1, U_2 : \mathscr{K} \to \mathscr{K}$ such that

$$T_1^n T_2^m = P_{\mathscr{H}} U_1^n U_2{}^m_{\mathscr{H}}, \quad \forall n, m = 1, 2, \ldots,$$

where $P_{\mathscr{H}} : \mathscr{K} \to \mathscr{K}$ is the orthogonal projection onto $\mathscr{H}$.

An easy consequence of Ando dilation is generalization of von Neumann inequality.

Theorem 6.1.9. *(**Ando-von Neumann inequality**) (cf. Ando (1963); Bhattacharyya (2002)) Let $\mathscr{H}$ be a Hilbert space and $T_1, T_2 : \mathscr{H} \to \mathscr{H}$ be commuting contractions. Then for every polynomial $p \in \mathbb{C}[z, w]$,*

$$\|p(T_1, T_2)\| \leq \sup_{|z|=|w|=1} |p(z, w)|.$$

It is known that Ando dilation theorem can not be extended for more than two commuting contractions (cf. Bhattacharyya (2002); Crabb and Davie (1975); Drury (1983); Parrott (1970); Varopoulos (1974)). We next consider inter-twining lifting theorem. This says that any operator which intertwins contractions can be lifted so that the lifted operator intertwins dilation operator.

Theorem 6.1.10. *(Sz.-Nagy and Foias (1971)) (**Inter-twining lifting theorem**) Let $T_1 : \mathscr{H}_1 \to \mathscr{H}_1$, $T_2 : \mathscr{H}_2 \to \mathscr{H}_2$ be contractions, where $\mathscr{H}_1$, $\mathscr{H}_2$ are Hilbert spaces. Let $V_1 : \mathscr{K}_1 \to \mathscr{K}_1$, $V_2 : \mathscr{K}_2 \to \mathscr{K}_2$ be minimal isometric dilations of T_1, T_2, respectively. Assume that $S : \mathscr{H}_2 \to \mathscr{H}_1$ is a bounded linear operator such that $T_1 S = S T_2$. Then there exists a bounded linear operator $R : \mathscr{K}_2 \to \mathscr{K}_1$ such that*

$$V_1 R = R V_2, \quad P_{\mathscr{H}_1} R_{\mathscr{H}_2^\perp} = 0, \quad P_{\mathscr{H}_1} R_{\mathscr{H}_2} = S, \quad \|R\| = \|S\|.$$

Conversely if $R : \mathscr{K}_2 \to \mathscr{K}_1$ is a bounded linear operator such that $V_1 R = R V_2$ and $P_{\mathscr{H}_1} R_{\mathscr{H}_2^\perp} = 0$, then $S := P_{\mathscr{H}_1} R_{\mathscr{H}_2}$ satisfies $T_1 S = S T_2$.

Next theorem gives a characterization which gives a condition that a given operator in a larger space becomes a dilation of compression of it to a smaller space.

Theorem 6.1.11. *(Sarason (1965)) (**Sarason's lemma**) Let $\mathscr{H}$ be a closed subspace of a Hilbert space $\mathscr{K}$ and $V : \mathscr{K} \to \mathscr{K}$ be a bounded linear operator. Define $T := P_{\mathscr{H}} V_{|\mathscr{H}}$. Then $T^n = P_{\mathscr{H}} V^n_{|\mathscr{H}}$, for all $n \in \mathbb{N}$ if and only if there are closed subspaces $\mathscr{M} \subseteq \mathscr{N} \subseteq \mathscr{K}$ both are invariant for V such that*

$$\mathscr{H} = \mathscr{N} \ominus \mathscr{M},$$

where $\mathscr{N} \ominus \mathscr{M}$ denotes the orthogonal complement of $\mathscr{M}$ in $\mathscr{N}$.

Following Theorems 6.1.3, 6.1.4, 6.1.6, 6.1.8, 6.1.10 and 6.1.11, extension of contractions on Hilbert spaces became an active area of research, known as dilation theory (Agler and McCarthy (2002); Ambrozie and Muller (2014); Arveson (2010); Levy and Shalit (2014); Paulsen (2002); Pisier (2001); Shalit (2021); Sz.-Nagy et al. (2010)). This study of contractions motivated the study of contractions and other classes of operators not only on Hilbert spaces, but also on Banach spaces (Akcoglu and Sucheston (1977); Fackler and Gluck (2019); Stroescu (1973)).

Recently, Bhat, De, and Rakshit abstracted the key ingredients in Halmos and Sz. Nagy dilation theorem and set up a set theoretic version of dilation theory. Following is the fundamental observation which lead Bhat, De, and Rakshit to set up a set theoretic notion of dilation theory.

(i) **There is an embedding i of the given space in a larger space**.

(ii) **There is a nice map in the larger space**.

(iii) **There is an idempotent from the larger space onto the given space**.

These observations can be picturized using the following commutative diagram.

$$\begin{array}{ccc} \mathscr{K} & \xrightarrow{U^n} & \mathscr{K} \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle P_{\mathscr{H}}} \\ \mathscr{H} & \xrightarrow{T^n} & \mathscr{H} \end{array}$$

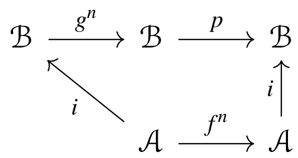
Definition 6.1.12. *(Bhat et al. (2021)) Let $\mathcal{A}$ be a (non empty) set and $f : \mathcal{A} \to \mathcal{A}$ be a map. An* ***injective power dilation*** *of f is a quadruple $(\mathcal{B}, i, g, p)$, where $\mathcal{B}$ is a set, $i : \mathcal{A} \to \mathcal{B}$, $v : \mathcal{B} \to \mathcal{B}$ are injective maps, $p : \mathcal{B} \to \mathcal{B}$ is an idempotent map such that $p(\mathcal{B}) = i(\mathcal{A})$ and*

$$i(f^n(a)) = p(g^n(i(a))), \quad \forall a \in \mathcal{A}, \forall n \in \mathbb{Z}_+. \tag{6.1.1}$$

A dilation $(\mathcal{B}, i, g, p)$ of f is said to be ***minimal*** *if*

$$\mathcal{B} = \bigcup_{n=0}^{\infty} g^n(i(\mathcal{A})).$$

Equation 6.1.1 says that the following diagram commutes for all n.



Bhat, De, and Rakshit succeeded in obtaining fundamental theorems of dilations. We now recall these results.

Definition 6.1.13. *(Bhat et al. (2021)) (**Set shifts**) Let $\mathcal{A}$ be a set. A map $f : \mathcal{A} \to \mathcal{A}$ is said to be **shift** if $\cap_{n=0}^{\infty} f^n(\mathcal{A}) = \emptyset$.*

Theorem 6.1.14. *(Bhat et al. (2021)) (**Wold decomposition for sets**) Let $f : \mathcal{A} \to \mathcal{A}$ be an injective map. Then $\mathcal{A}$ decomposes uniquely as $\mathcal{A} = \mathcal{A}_b \sqcup \mathcal{A}_s$, where $\mathcal{A}_b$, $\mathcal{A}_s$ are invariant for f, $f_{|\mathcal{A}_b}$ is a bijection and $f_{|\mathcal{A}_s}$ is a shift.*

Theorem 6.1.15. *(Bhat et al. (2021)) (**Halmos dilation for sets**) Let $f : \mathcal{A} \to \mathcal{A}$ be a map. Define $\mathcal{B} := \mathcal{A} \times \{0,1\}$,*

$$i : \mathcal{A} \ni a \mapsto (a,0) \in \mathcal{B}$$
$$g : \mathcal{B} \ni (a,m) \mapsto (a,1-m) \in \mathcal{B}$$

and $p : \mathcal{B} \to \mathcal{B}$ by

$$p(a,m) := \begin{cases} (a,0) & \text{if } m = 0 \\ (f(a),0) & \text{if } m = 1. \end{cases}$$

Then i is injective, g is bijective, p is idempotent and

$$i(f(a)) = p(g(i(a))), \quad \forall a \in \mathcal{A}.$$

Theorem 6.1.16. *(Bhat et al. (2021)) (**Sz. Nagy dilation for sets**) Every map $f : \mathcal{A} \to \mathcal{A}$ admits a minimal injective power dilation.*

In Bhat et al. (2021), a particular type of minimal injective dilation, called as **standard dilation** was defined. This dilation is defined as follows. Let $f : \mathcal{A} \to \mathcal{A}$ be a map. Define

$$\mathcal{B} := \mathcal{A} \times \mathbb{Z}_+,$$
$$i(a) := (a,0), \quad \forall a \in \mathcal{A},$$
$$g(a,m) := (a,m+1), \quad \forall (a,m) \in \mathcal{B},$$

$$p(a,m) := (f^m(a), 0), \quad \forall (a,m) \in \mathcal{B}.$$

Then $(\mathcal{B}, i, g, p)$ is a minimal dilation of f.

Theorem 6.1.17. *(Bhat et al. (2021)) (**Inter-twining lifting theorem for sets**) Let $f_1 : \mathcal{A}_1 \to \mathcal{A}_1$, $f_2 : \mathcal{A}_2 \to \mathcal{A}_2$ be maps and $(\mathcal{B}_1, i_1, g_1, p_1)$, $(\mathcal{B}_2, i_2, g_2, p_2)$ be their standard dilations, respectively. Suppose $s : \mathcal{A}_2 \to \mathcal{A}_1$ is a function such that $sf_2 = f_1 s$. Then there exists a map $r : \mathcal{B}_2 \to \mathcal{B}_1$ such that*

$$rg_2 = g_1 r, \quad rp_2 = p_1 r, \quad ri_2 = i_1 s.$$

Conversely if $r : \mathcal{B}_2 \to \mathcal{B}_1$is a map such that $rg_2 = g_1 r$, $rp_2 = p_1 r$, then there exists a map $s : \mathcal{A}_2 \to \mathcal{A}_1$ such that $ri_2 = i_1 s$ and $sf_2 = f_1 s$.

Theorem 6.1.18. *(Bhat et al. (2021)) (**Ando dilation for sets**) Let $\mathbb{J}$ be an index set, $\{f_j\}_{j \in \mathbb{J}}$ be a family of commuting functions on $\mathcal{A}$. Then there exists a quadruple $(\mathcal{B}, i, \{g_j\}_{j \in \mathbb{J}}, p)$, where $\mathcal{B}$ is a set, $i : \mathcal{A} \to \mathcal{B}$ is an injective map, $\{v_j\}_{j \in \mathbb{J}}$ be a family of commuting functions on $\mathcal{B}$, $p : \mathcal{B} \to \mathcal{B}$ is idempotent such that*

$$i(f_{j_1} f_{j_2} \cdots f_{j_k}(a)) = p(g_{j_1} f_{g_2} \cdots f_{g_k}(i(a))), \quad \forall j_1, \dots, j_k \in \mathbb{J}, \forall a \in \mathcal{A}.$$

Theorem 6.1.19. *(Bhat et al. (2021)) (**Sarason's lemma for sets**) Let $g : \mathcal{B} \to \mathcal{B}$ be an injective map and let $\mathcal{A} \subseteq \mathcal{B}$. Suppose $f : \mathcal{A} \to \mathcal{A}$ is a map such that $f(a) = g(a)$ for all $a \in \mathcal{A}$ with $g(a) \in \mathcal{A}$. Suppose $\mathcal{A} = \mathcal{A}_2 \setminus \mathcal{A}_1$, where $\mathcal{A}_1$, and $\mathcal{A}_2$ are invariant under g. Then there exists a map $p : \mathcal{B} \to \mathcal{B}$ such that $p^2 = p$, $p(\mathcal{B}) = \mathcal{A}$ and*

$$pg^n(a) = f^n(a), \quad \forall n \in \mathbb{N}, \forall a \in \mathcal{A}.$$

6.2 WOLD DECOMPOSITION, HALMOS DILATION AND N-DILATION FOR VECTOR SPACES

In this appendix we consider vector spaces (need not be finite dimensional) over arbitrary fields. We note that the Definition 6.1.1 of shift of an operator on a Hilbert space does not use the Hilbert space structure. Thus it can be formulated for vector spaces without modifications.

Definition 6.2.1. *Let $\mathcal{V}$ be a vector space and $T : \mathcal{V} \to \mathcal{V}$ be a linear map. The map T is said to be a **shift** if $\cap_{n=0}^{\infty} T^n(\mathcal{V}) = \{0\}$.*

Theorem 6.2.2. ***(Wold decomposition for vector spaces)*** *Let T be an injective linear map on a vector space $\mathscr{V}$. Then $\mathscr{V}$ decomposes as $\mathscr{V} = \mathscr{V}_b \oplus \mathscr{V}_s$, where $\mathscr{V}_b$ is a T-invariant subspace of $\mathscr{V}$, $T_{|\mathscr{V}_b} : \mathscr{V}_b \to \mathscr{V}_b$ is a bijection and $T_{|\mathscr{V}_s} : \mathscr{V}_s \to \mathscr{V}$ is a shift.*

Proof. Define $\mathscr{V}_b := \cap_{n=0}^{\infty} T^n(\mathscr{V})$ and let $\mathscr{V}_s$ be a vector space complement of $\mathscr{V}_b$ in $\mathscr{V}$. We clearly have $\mathscr{V} = \mathscr{V}_b \oplus \mathscr{V}_s$. Now $T(\mathscr{V}_b) = T(\cap_{n=0}^{\infty} T^n(\mathscr{V})) \subseteq \cap_{n=0}^{\infty} T^n(\mathscr{V}) = \mathscr{V}_b$. Thus $\mathscr{V}_b$ is a T-invariant subspace of $\mathscr{V}$. We now try to show that $T_{|\mathscr{V}_b}$ is a bijection. Since T is already injective, it suffices to show that $T_{|\mathscr{V}_b}$ is surjective. Let $y \in \mathscr{V}_b$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathscr{V}$ such that $y = Tx_1 = T^2x_2 = T^3x_3 = \cdots$. Since T is injective, we then have $x_1 = Tx_2 = T^2x_2 = \cdots$. Therefore $y = Tx_1$ and $x_1 \in \mathscr{V}_b$. Thus $T_{|\mathscr{V}_b}$ is surjective. We are now left with proving that $T_{|\mathscr{V}_s}$ is a shift. Let $y \in \cap_{n=0}^{\infty} (T_{|\mathscr{V}_s})^n(\mathscr{V}_s) \subseteq (\cap_{n=0}^{\infty} T^n(\mathscr{V})) \cap \mathscr{V}_s = \mathscr{V}_b \cap \mathscr{V}_s$. Hence $y = 0$ which completes the proof. □

Since vector space complements are not unique, we do not have uniqueness in Wold decomposition for vector spaces. We next derive Halmos dilation for vector spaces.

Theorem 6.2.3. ***(Halmos dilation for vector spaces)*** *Let $\mathscr{V}$ be a vector space and $T : \mathscr{V} \to \mathscr{V}$ be a linear map. Then the operator*

$$U := \begin{pmatrix} T & I \\ I & 0 \end{pmatrix}$$

is invertible on $\mathscr{V} \oplus \mathscr{V}$. In other words,

$$T = P_{\mathscr{V}} U_{|\mathscr{V}},$$

where $P_{\mathscr{V}} : \mathscr{V} \oplus \mathscr{V} \to \mathscr{V} \oplus \mathscr{V}$ is the first coordinate projection onto $\mathscr{V}$.

Proof. It suffices to produce inverse map for U. A direct calculation says that

$$V := \begin{pmatrix} 0 & I \\ I & -T \end{pmatrix}$$

is the inverse of U. □

In the sequel, any invertible operator of the form

$$\begin{pmatrix} T & B \\ C & D \end{pmatrix},$$

where $B, C, D : \mathscr{V} \to \mathscr{V}$ are linear operators, will be called as a **Halmos dilation** of T. Now we observe that Halmos dilation for vector spaces is not unique. Using the theory of **block matrices** (Lu and Shiou (2002)) we can produce a variety of Halmos dilations for a given operator. Following are some classes of Halmos dilations.

(i) If $T : \mathscr{V} \to \mathscr{V}$ is an invertible linear map and the linear operators $B, C, D : \mathscr{V} \to \mathscr{V}$ are such that $D - CT^{-1}B$ is invertible, then the operator

$$U := \begin{pmatrix} T & B \\ C & D \end{pmatrix} \text{ is a Halmos dilation of } T \text{ on } \mathscr{V} \oplus \mathscr{V} \text{ whose inverse is}$$

$$\begin{pmatrix} T^{-1} + T^{-1}B(D - CT^{-1}B)^{-1} & -T^{-1}B(D - CT^{-1}B)^{-1} \\ -(D - CT^{-1}B)^{-1}CT^{-1} & (D - CT^{-1}B)^{-1} \end{pmatrix}.$$

(ii) $D : \mathscr{V} \to \mathscr{V}$ is an invertible linear map and the linear operators $B, C : \mathscr{V} \to \mathscr{V}$ are such that $T - BD^{-1}C$ is invertible, then the operator

$$\begin{pmatrix} T & B \\ C & D \end{pmatrix} \text{ is a Halmos dilation of } T \text{ on } \mathscr{V} \oplus \mathscr{V} \text{ whose inverse is}$$

$$\begin{pmatrix} (T - BD^{-1}C)^{-1} & -(T - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(T - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(T - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}.$$

(iii) $B : \mathscr{V} \to \mathscr{V}$ is an invertible linear map and the linear operators $C, D : \mathscr{V} \to \mathscr{V}$ are such that $C - DB^{-1}T$ is invertible, then the operator

$$\begin{pmatrix} T & B \\ C & D \end{pmatrix} \text{ is a Halmos dilation of } T \text{ on } \mathscr{V} \oplus \mathscr{V} \text{ whose inverse is}$$

$$\begin{pmatrix} -(C - DB^{-1}T)^{-1}DB^{-1} & (C - DB^{-1}T)^{-1} \\ B^{-1} + B^{-1}T(C - DB^{-1}T)^{-1}DB^{-1} & -B^{-1}T(C - DB^{-1}T)^{-1} \end{pmatrix}.$$

(iv) $C : \mathscr{V} \to \mathscr{V}$ is an invertible linear map and the linear operators $B, D : \mathscr{V} \to \mathscr{V}$ are such that $B - TC^{-1}D$ is invertible, then the operator

$$\begin{pmatrix} T & B \\ C & D \end{pmatrix} \text{ is a Halmos dilation of } T \text{ on } \mathscr{V} \oplus \mathscr{V} \text{ whose inverse is}$$

$$\begin{pmatrix} -C^{-1}D(B-TC^{-1}D)^{-1} & C^{-1}+C^{-1}D(B-TC^{-1}D)^{-1}TC^{-1} \\ (B-TC^{-1}D)^{-1} & -(B-TC^{-1}D)^{-1}TC^{-1} \end{pmatrix}.$$

Recently, Bhat and Mukherjee (2020) proved that there is certain kind of uniqueness of Halmos dilation for strict contractions in Hilbert spaces, as shown below.

Theorem 6.2.4. *(Bhat and Mukherjee (2020)) Let $\mathscr{H}$ be a finite dimensional Hilbert space and $T : \mathscr{H} \to \mathscr{H}$ be a strict contraction (i.e., $\|T\| < 1$). Then Halmos dilation of T on $\mathscr{H} \oplus \mathscr{H}$ is unitarily equivalent to*

$$\begin{pmatrix} T & -\sqrt{I-TT^*}W \\ \sqrt{I-T^*T} & T^*W \end{pmatrix}, \quad \textit{for some unitary operator } W : \mathscr{H} \to \mathscr{H}.$$

We next derive a negative result to Theorem 6.2.4 for Halmos dilation in vector spaces.

Theorem 6.2.5. *Let $\mathscr{V}$ be a finite dimensional vector space and $T : \mathscr{V} \to \mathscr{V}$ be a linear operator with nonzero trace. Then there are Halmos dilations of T which are not similar.*

Proof. Note that

$$\begin{pmatrix} T & T-I \\ T+I & T \end{pmatrix}$$

is an invertible operator and hence is a Halmos dilation of T. It is now enough to show that the matrices

$$\begin{pmatrix} T & T-I \\ T+I & T \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T & I \\ I & 0 \end{pmatrix}$$

are not similar. Since $\mathscr{V}$ is finite dimensional, we can use the property of trace map to conclude that these matrices are not similar. □

Theorem 6.2.3 can be generalized which gives vector space version of Theorem 6.1.6.

Theorem 6.2.6. *(**N-dilation for vector spaces**) Let $\mathscr{V}$ be a vector space and $T : \mathscr{V} \to \mathscr{V}$*

be a linear map. Let N be a natural number. Then the operator

$$U := \begin{pmatrix} T & 0 & 0 & \cdots & 0 & I \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & I & 0 \end{pmatrix}_{(N+1)\times(N+1)}$$

is invertible on $\oplus_{k=1}^{N+1}\mathcal{V}$ *and*

$$T^k = P_{\mathcal{V}}U^k_{|\mathcal{V}}, \quad \forall k = 1, \ldots, N, \tag{6.2.1}$$

where $P_{\mathcal{V}} : \oplus_{k=1}^{N+1}\mathcal{V} \to \oplus_{k=1}^{N+1}\mathcal{V}$ *is the first coordinate projection onto* $\mathcal{V}$*.*

Proof. A direct calculation of power of U gives Equation (6.2.1). To complete the proof, now we need show that U is invertible. Define

$$V := \begin{pmatrix} 0 & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & I \\ I & -T & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{(N+1)\times(N+1)}.$$

Then $UV = VU = I$. Thus V is the inverse of U. ☐

Note that the Equation (6.2.1) holds only upto N and not for $N+1$ and higher natural numbers. We next derive vector space version of Theorem 6.1.5. In the following theorem, $\oplus_{n=-\infty}^{\infty}\mathcal{V}$ is the vector space defined by

$$\oplus_{n=-\infty}^{\infty}\mathcal{V} := \left\{\{x_n\}_{n=-\infty}^{\infty}, x_n \in \mathcal{V}, \forall n \in \mathbb{Z}, x_n \neq 0 \text{ only for finitely many } n'\text{s}\right\}$$

with respect to natural operations.

Theorem 6.2.7. ***(Sz. Nagy dilation for vector spaces)*** *Let* $\mathcal{V}$ *be a vector space and* $T : \mathcal{V} \to \mathcal{V}$ *be a linear map. Let* $U := [u_{n,m}]_{-\infty \leq n,m \leq \infty}$ *be the operator defined on*

$\oplus_{n=-\infty}^{\infty}\mathscr{V}$ given by the infinite matrix defined as follows:

$$u_{0,0} := T, \quad u_{n,n+1} := I, \quad \forall n \in \mathbb{Z}, \quad u_{n,m} := 0 \quad \textit{otherwise},$$

i.e.,

$$U = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & I & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & I & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \underline{T} & I & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}_{\infty\times\infty}$$

where T is in the $(0,0)$ position (which is underlined), which is invertible on $\oplus_{n=-\infty}^{\infty}\mathscr{V}$ and

$$T^n = P_{\mathscr{V}} U^n_{\mathscr{V}}, \quad \forall n \in \mathbb{N}, \tag{6.2.2}$$

where $P_{\mathscr{V}} : \oplus_{n=-\infty}^{\infty}\mathscr{V} \to \oplus_{n=-\infty}^{\infty}\mathscr{V}$ is the first coordinate projection onto $\mathscr{V}$.

Proof. We get Equation (6.2.2) by calculation of powers of operator U. The matrix $V := [v_{n,m}]_{-\infty<n,m<\infty}$ defined by

$$v_{0,0} := 0, \quad v_{1,-1} := -T, \quad v_{n,n-1} := I, \quad \forall n \in \mathbb{Z}, \quad v_{n,m} := 0 \quad \text{otherwise},$$

i.e.,

$$V = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & I & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I & \underline{0} & 0 & 0 & \cdots \\ \cdots & 0 & -T & I & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & I & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}_{\infty\times\infty}$$

where 0 is in the $(0,0)$ position (which is underlined), satisfies $UV = VU = I$ and hence U is invertible which completes the proof. □

6.3 MINIMAL DILATION, INTERTWINING LIFTING THEOREM AND VARIANT OF ANDO DILATION FOR VECTOR SPACES

An important observation associated with Theorems 6.2.5, 6.2.6 and 6.2.7 is that the dilation is not optimal, i.e., even if the given operator is invertible, then also U is not same as T. To overcome this, next we move on with the definition of dilation given by Bhat, De, and Rakshit (Bhat et al. (2021)). Set theoretic definition of dilation, given in Definition 6.1.12 motivated Bhat, De, and Rakshit, to introduce the dilation of linear maps on vector spaces.

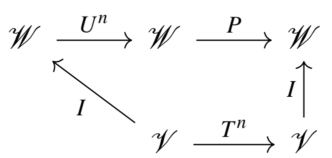
Definition 6.3.1. *(Bhat et al. (2021)) Let $\mathscr{V}$ be a vector space and $T : \mathscr{V} \to \mathscr{V}$ be a linear map. A **linear injective dilation** of T is a quadruple $(\mathscr{W}, I, U, P)$, where $\mathscr{W}$ is a vector space, and $I : \mathscr{V} \to \mathscr{W}$ is an injective linear map, $U : \mathscr{W} \to \mathscr{W}$ is an injective linear map, $P : \mathscr{W} \to \mathscr{W}$ is an idempotent linear map such that $P(\mathscr{W}) = I(\mathscr{W})$ and*

$$\textit{(\textbf{Dilation equation})} \quad IT^n x = PU^n Ix, \quad \forall n \in \mathbb{Z}_+, \forall x \in \mathscr{V}.$$

*A dilation $(\mathscr{W}, I, U, P)$ of T is said to be **minimal** if*

$$\mathscr{W} = \operatorname{span}\{U^n Ix : n \in \mathbb{Z}_+, x \in \mathscr{V}\}.$$

An easier way to remember the dilation equation is the following commutative diagram.



Following result is the vector space version of Theorem 6.1.16.

Theorem 6.3.2. *(Bhat et al. (2021)) (**Minimal Sz. Nagy dilation for sets**) Every linear map $T : \mathscr{V} \to \mathscr{V}$ admits minimal injective linear dilation.*

Proof. We reproduce the proof given by Bhat et al. (2021) for the sake of future use. Define

$$\mathscr{W} := \left\{ (x_n)_{n=0}^{\infty} : x_n \in \mathscr{V}, \forall n \in \mathbb{Z}_+, x_n \neq 0 \text{ only for finitely many } n'\text{s} \right\}.$$

Clearly $\mathscr{W}$ is a vector space w.r.t. natural operations. Now define

$$
\begin{aligned}
&I : \mathscr{V} \ni x \mapsto (x, 0, \dots) \in \mathscr{W}, \\
&U : \mathscr{W} \ni (x_n)_{n=0}^{\infty} \mapsto (0, x_0, \dots) \in \mathscr{W}, \\
&P : \mathscr{W} \ni (x_n)_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} IT^n x_n \in \mathscr{W}.
\end{aligned}
$$

Then $(\mathscr{W}, I, U, P)$ is a minimal injective linear dilation of T. □

We call the dilation $(\mathscr{W}, I, U, P)$ constructed in Theorem 6.3.2 as the **standard dilation** of T. We next consider inter-twining lifting theorem.

Theorem 6.3.3. ***(Inter-twining lifting theorem for vector spaces)*** *Let $\mathscr{V}_1$, $\mathscr{V}_2$ be vector spaces, $T_1 : \mathscr{V}_1 \to \mathscr{V}_1$, $T_2 : \mathscr{V}_2 \to \mathscr{V}_2$ be linear maps. Let $(\mathscr{W}_1, I_1, U_1, P_1)$, $(\mathscr{W}_2, I_2, U_2, P_2)$ be standard dilations of T_1, T_2, respectively. If $S : \mathscr{V}_2 \to \mathscr{V}_1$ is a linear map such that $T_1 S = S T_2$, then there exists a linear map $R : \mathscr{W}_2 \to \mathscr{W}_1$ such that*

$$
U_1 R = R U_2, \quad R P_2 = P_1 R, \quad R I_2 = I_1 S. \tag{6.3.1}
$$

Conversely if $R : \mathscr{W}_2 \to \mathscr{W}_1$ is a linear map such that $U_1 R = R U_2, R P_2 = P_1 R$, then there exists a linear map $S : \mathscr{V}_2 \to \mathscr{V}_1$ such that

$$
R I_2 = I_1 S, \quad T_1 S = S T_2. \tag{6.3.2}
$$

Proof. Define $R : \mathscr{W}_2 \ni (x_n)_{n=0}^{\infty} \mapsto (S x_n)_{n=0}^{\infty} \in \mathscr{W}_1$. We now verify three equalities in Equation (6.3.1). Let $(x_n)_{n=0}^{\infty} \in \mathscr{W}_2$. Then

$$
\begin{aligned}
U_1 R (x_n)_{n=0}^{\infty} &= U_1 (S x_n)_{n=0}^{\infty} = (0, S x_0, S x_1, \dots), \\
R U_2 (x_n)_{n=0}^{\infty} &= R(0, x_0, x_1, \dots) = (0, S x_0, S x_1, \dots),
\end{aligned}
$$

$$
\begin{aligned}
R P_2 (x_n)_{n=0}^{\infty} &= R \left(\sum_{n=0}^{\infty} I_2 T_2^n x_n \right) = \sum_{n=0}^{\infty} R I_2 T_2^n x_n \\
&= \sum_{n=0}^{\infty} R(T_2^n x_n, 0, 0, \dots) = \sum_{n=0}^{\infty} (S T_2^n x_n, 0, 0, \dots), \\
P_1 R (x_n)_{n=0}^{\infty} &= P_1 (S x_n)_{n=0}^{\infty} = \sum_{n=0}^{\infty} I_1 T_1^n S x_n \\
&= \sum_{n=0}^{\infty} I_1 S T_2^n x_n = \sum_{n=0}^{\infty} (S T_2^n x_n, 0, 0, \dots),
\end{aligned}
$$

$$RI_2x = R(x,0,0,\dots) = (Sx,0,0,\dots), \quad I_1Sx = (Sx,0,0,\dots).$$

We now consider the converse part. For this, first we have to define linear map S. Let $y \in \mathscr{V}_2$. Now $RP_2(y,0,\dots) = P_1R(y,0,\dots) \in I_1(\mathscr{V}_1)$ and I_1 is injective implies that there exists a unique $x \in \mathscr{V}_2$ such that $RP_2(y,0,\dots) = P_1R(y,0,\dots) = I_1(x)$. We now define $Sy := x$. Then S is well-defined and linear. Let $y \in \mathscr{V}_2$ and $x \in \mathscr{V}_2$ be such that $Sy = x$. Then $I_1Sy = RP_2(y,0,\dots) = RI_2y$. Thus we verified first equality in (6.3.2). We are left with verification of second equality. We now calculate

$$RP_2U_2(x,0,\dots) = RP_2(0,x,0,\dots) = RI_2T_2x \tag{6.3.3}$$

and

$$P_1U_1R(x,0,\dots) = P_1RU_2(x,0,\dots) = P_1R(0,x,0,\dots) \tag{6.3.4}$$
$$= RP_2(0,x,0,\dots) = RI_2T_2x, \quad \forall x \in \mathscr{V}_2. \tag{6.3.5}$$

Given conditions produce

$$RP_2U_2 = P_1RU_2 = P_1U_1R. \tag{6.3.6}$$

Equation (6.3.6) says that (6.3.3) and (6.3.4) are equal which completes the proof. □

Following is a variant of Ando dilation for vector spaces.

Theorem 6.3.4. ***(Ando like dilation for vector spaces)*** *Let $\mathscr{V}$ be a vector space and $T,S : \mathscr{V} \to \mathscr{V}$ be commuting linear maps. Then there are dilations $(\mathscr{W},I,U_1,P)$ and $(\mathscr{W},I,U_2,P)$ of T,S respectively, such that*

$$\begin{pmatrix} 0_c & U \end{pmatrix} = \begin{pmatrix} 0_r \\ V \end{pmatrix}$$

and

$$IT^nS^mx = PU^nV^mIx, \quad \forall n,m \in \mathbb{Z}_+, \forall x \in \mathscr{V},$$

where 0_c denotes the infinite column matrix of zero vectors and 0_r denotes the infinite row matrix of zero vectors.

Proof. We extend the construction in the proof of Theorem 6.3.2. Define

$$\mathscr{W} := \left\{ \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\infty\times\infty} : x_{n,m} \in \mathscr{V}, \forall n,m \in \mathbb{Z}_+, x_{n,m} \neq 0 \right.$$
$$\left. \text{only for finitely many } (n,m)'\text{s} \right\}.$$

Then $\mathscr{W}$ becomes a vector space with respect to natural operations. We now define the following four linear maps:

$$I : \mathscr{V} \ni x \mapsto \begin{pmatrix} x & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathscr{W}$$

$$U : \mathscr{W} \ni \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & \\ x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathscr{W}$$

$$V : \mathscr{W} \ni \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mapsto \begin{pmatrix} 0 & x_{0,0} & x_{0,1} & \cdots \\ 0 & x_{1,0} & x_{1,1} & \cdots \\ 0 & x_{2,0} & x_{2,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathscr{W}$$

$$P : \mathscr{W} \ni \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mapsto \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} IT^n S^m x_{n,m} \in \mathscr{W}.$$

We then have

$$\begin{pmatrix} 0_c & U \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ 0 & x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ 0 & x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ 0 & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0_r \\ V \end{pmatrix}.$$

Now $PU^nIx = IT^nx$, $PV^nIx = IS^mx$, $\forall x \in \mathscr{V}$, $\forall n,m \in \mathbb{Z}_+$. Hence $(\mathscr{W}, I, U_1, P)$ and $(\mathscr{W}, I, U_2, P)$ are dilations of T, S, respectively. A calculation now shows that $IT^nS^mx = PU^nV^mIx, \forall n,m \in \mathbb{Z}_+, \forall x \in \mathscr{V}$. □

Conclusion and future work : In this appendix we derived some basic results on dilation of linear maps. Since vector spaces are more general than Hilbert spaces and tools of Hilbert spaces will not work in vector space, we are interested to explore algebraic aspects of dilation theory.

APPENDIX B: COMMUTATORS CLOSE TO THE IDENTITY

6.4 C*-ALGEBRAS

Israel M. Gelfand defined abstractly the notion of a complete algebra (Gelfand (1941)). These are Banach spaces in which we can multiply the elements and the multiplication enjoys continuity.

Definition 6.4.1. *(cf. Zhu (1993)) A Banach space $\mathscr{A}$ over $\mathbb{C}$ is said to be a unital* ***Banach algebra*** *if it is a unital algebra and the multiplication satisfies the following:*

(i) $\|xy\| \leq \|x\|\|y\|$, $\forall x, y \in \mathscr{A}$.

(ii) $\|e\| = 1$*, where e is the multiplicative identity of* $\mathscr{A}$.

Example 6.4.2. *(cf. Zhu (1993))*

(i) If K is a compact Hausdorff space, then the space $\mathscr{C}(K)$ of all complex-valued continuous functions on K is a commutative unital Banach algebra w.r.t. sup-norm and pointwise multiplication.

(ii) If $\mathscr{X}$ is a Banach space, then the collection $\mathscr{B}(\mathscr{X})$ of all bounded linear operators on $\mathscr{X}$ is a noncommutative unital Banach algebra w.r.t. operator-norm and operator composition.

Proposition 6.4.3. *(cf. Allan (2011)) Every unital Banach algebra $\mathscr{A}$ can be isometrically embedded in $\mathscr{B}(\mathscr{A})$.*

One of the most important notion associated with the study of Banach algebras is the notion of spectrum.

Definition 6.4.4. *(cf. Zhu (1993)) Let $\mathscr{A}$ be a unital Banach algebra with the identity e.* ***Spectrum*** *of an element x in $\mathscr{A}$ is the set of all complex numbers λ such that $\lambda e - x$ is not invertible.*

Theorem 6.4.5. *(cf. Zhu (1993)) Spectrum of every element of a unital Banach algebra is a nonempty compact subset of $\mathbb{C}$.*

Following is the first fundamental theorem in the study of Banach algebras which characterizes Banach algebras using the information of spectrum.

Theorem 6.4.6. *(cf. Zhu (1993)) (**Gelfand-Mazur theorem**) If every nonzero element of a Banach algebra is invertible, then it is isometrically isomorphic to $\mathbb{C}$.*

A subclass of Banach algebras known as C*-algebras allows to do most of the things which hold good for complex numbers. Notion of C*-algebras, for first time, appeared in the work of Gelfand and Neumark (1943).

Definition 6.4.7. *(cf. Zhu (1993)) A unital Banach algebra $\mathscr{A}$ is called a unital **C*-algebra** if there exists a map $*: \mathscr{A} \ni x \mapsto x^* \in \mathscr{A}$ such that following conditions hold.*

(i) $((x)^*)^* = x, \forall x \in \mathscr{A}$.

(ii) $(x+y)^* = x^* + y^*, \forall x, y \in \mathscr{A}$.

(iii) $(\alpha x)^* = \overline{\alpha} x^*$, $\forall \alpha \in \mathbb{K}, \forall x \in \mathscr{A}$.

(iv) $(xy)^* = y^* x^*, \forall x, y \in \mathscr{A}$.

(v) $\|x^* x\| = \|x\|^2, \forall x \in \mathscr{A}$.

*A map $*: \mathscr{A} \ni x \mapsto x^* \in \mathscr{A}$ satisfying (i)-(iii) is called as **involution**.*

Segal called the term C*-algebra; the letter 'C' stands for uniformly closed. C*-algebras are also known as Gelfand-Naimark algebras (cf. Pietsch (2007)).

Example 6.4.8. *(cf. Zhu (1993))*

(i) If K is a compact Hausdorff space, then $\mathscr{C}(K)$ is a commutative unital C-algebra w.r.t. involution $f^*(x) := \overline{f(x)}, \forall x \in K$.*

(ii) If $\mathscr{H}$ is a Hilbert space, then $\mathscr{B}(\mathscr{H})$ is a noncommutative unital C-algebra w.r.t. operator adjoint.*

(iii) If $\mathscr{H}$ is a Hilbert space, the the space $\mathscr{K}(\mathscr{H})$ of compact operators is a noncommutative C-subalgebra of $\mathscr{B}(\mathscr{H})$. If $\mathscr{H}$ is infinite dimensional, then this algebra is non unital.*

Following two results characterize unital C*-algebras.

Theorem 6.4.9. *(cf. Zhu (1993)) (**Gelfand-Naimark theorem**) If $\mathscr{A}$ is a commutative unital C*-algebra, then $\mathscr{A}$ is isometrically $*$-isomorphic to $\mathscr{C}(K)$ for some compact Hausdorff space K.*

Theorem 6.4.10. *(cf. Zhu (1993)) (**Gelfand-Naimark-Segal theorem**) Let $\mathscr{A}$ be a unital C*-algebra. Then there exists a Hilbert space $\mathscr{H}$ such that $\mathscr{A}$ is isometrically $*$-isomorphic to a C*-subalgebra of $\mathscr{B}(\mathscr{H})$.*

Example 6.4.11. *(cf. Kadison and Ringrose (1997)) Consider the unital C*-algebra $\mathscr{C}[0,1]$. The map*

$$\pi : \mathscr{C}[0,1] \ni f \mapsto \pi(f) \in \mathscr{B}(\mathscr{L}^2[0,1]);$$
$$\pi(f) : \mathscr{L}^2[0,1] \ni g \mapsto (\pi(f))(g) := fg \in \mathscr{L}^2[0,1]$$

is an isometric $*$-isomorphism to a C-subalgebra of $\mathscr{B}(\mathscr{L}^2[0,1])$.*

6.5 COMMUTATORS CLOSE TO THE IDENTITY IN $\mathscr{B}(\mathscr{H})$

Let $n \in \mathbb{N}$ and $M_n(\mathbb{K})$ be the ring of n by n matrices over $\mathbb{K}$. Using the property of trace map we easily get that there does not exist $D,X \in M_n(\mathbb{K})$ such that $DX - XD = 1_{M_n(\mathbb{K})}$ (Halmos (1982)). This argument will not work for bounded linear operators on infinite dimensional Hilbert space since the map trace is not defined on the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on an infinite dimensional Hilbert space $\mathscr{H}$ (it is defined for a proper subalgebra of $\mathscr{B}(\mathscr{H})$ known as the trace class operators (Schatten (1960))). Operators of the form $DX - XD$ are called as **commutator** of D and X and are denoted by $[D,X]$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be a commutator if $T = [D,X]$, for some $D,X \in \mathscr{B}(\mathscr{H})$.
Using the property of spectrum of bounded linear operator, Winter in 1947 proved that the following result.

Theorem 6.5.1. *(Wintner (1947)) Let $\mathscr{H}$ be an infinite dimensional Hilbert space. Then there does not exist $D,X \in \mathscr{B}(\mathscr{H})$ such that*

$$[D,X] = 1_{\mathscr{B}(\mathscr{H})}. \tag{6.5.1}$$

After two years, Wielandt (1949) gave a simple proof for the failure of Equation 6.5.1. We note that the boundedness of operators is crucial in Theorem 6.5.1. Following example shows that Theorem 6.5.1 fails for unbounded operators

Example 6.5.2. *(Halmos (1982)) Let $\mathscr{H} := \mathscr{L}^2(\mathbb{R})$ and define*

$$(Df)(x) := \frac{d}{dx}f, \quad (Xf)(x) := xf(x)$$

Then $[D,X] = 1_{\mathscr{B}(\mathscr{H})}$.

Theorem 6.5.1 leads to the question that which operators on infinite dimensional Hilbert spaces can be written as commutators of operators? First partial answer was given by Halmos.

Theorem 6.5.3. *(cf. Putnam (1967)) Let $\mathscr{H}$ be an infinite dimensional Hilbert space. If $C \in \mathscr{B}(\mathscr{H})$ is compact, then $I_{\mathscr{B}(\mathscr{H})} + C$ is not a commutator.*

Brown and Pearcy (1965) characterized the set of bounded operators which can be written as commutators.

Theorem 6.5.4. *(Brown and Pearcy (1965)) Let $\mathscr{H}$ be an infinite dimensional separable Hilbert space. Then an operator in $\mathscr{B}(\mathscr{H})$ is a commutator if and only if it is not of the form $\lambda I_{\mathscr{B}(\mathscr{H})} + C$, where λ is a nonzero scalar and C is a compact operator.*

Following the paper of Brown and Pearcy (1965) there is a series of papers devoted to the study of commutators on sequence spaces, $\mathscr{L}^p$-spaces, Banach spaces, C*-algebras, von Neumann algebras, Banach *-algebras etc (Dosev and Johnson (2010); Dosev et al. (2013); Dosev (2009); Dykema et al. (2004); Dykema and Skripka (2012); Kadison et al. (2020); Kaftal et al. (2014); Laustsen (2002); Marcoux (2006, 2010); Schneeberger (1971); Stasinski (2016); Yood (2008)).

It was Popa (1982) who started a quantitative study of commutators close to the identity operator. He gave the following quantitative bound given by Popa for the product of norm of operators whenever the commutator is close to the identity.

Theorem 6.5.5. *(Popa (1982)) Let $\mathscr{H}$ be an infinite dimensional Hilbert space. Let $D, X \in \mathscr{B}(\mathscr{H})$ be such that*

$$\|[D,X] - 1_{\mathscr{B}(\mathscr{H})}\| \leq \varepsilon$$

for some $\varepsilon > 0$. Then

$$\|D\|\|X\| \geq \frac{1}{2}\log\frac{1}{\varepsilon}.$$

Now the problem in Theorem 6.5.5, is the existence of $D, X \in \mathscr{B}(\mathscr{H})$ such that the commutator $[D,X]$ is close to the identity operator. This was again obtained by Popa which is stated in the following result. Given real r and positive s, by $r = O(s)$ we mean that there is positive γ such that $|r| \leq \gamma s$.

Theorem 6.5.6. *(Popa (1982); Tao (2019)) Let $\mathscr{H}$ be an infinite dimensional Hilbert space. Then for each $0 < \varepsilon \leq 1$, there exist $D, X \in \mathscr{B}(\mathscr{H})$ with*

$$\|[D,X] - 1_{\mathscr{B}(\mathscr{H})}\| \leq \varepsilon$$

and

$$\|D\|\|X\| = O(\varepsilon^{-2}).$$

Terence Tao improved Theorem 6.5.6 and obtained the following theorem.

Theorem 6.5.7. *(Tao (2019)) Let $\mathscr{H}$ be an infinite dimensional Hilbert space. Then for each $0 < \varepsilon \leq 1/2$, there exist $D, X \in \mathscr{B}(\mathscr{H})$ with*

$$\|[D,X] - 1_{\mathscr{B}(\mathscr{H})}\| \leq \varepsilon$$

such that

$$\|D\|\|X\| = O\left(\log^5 \frac{1}{\varepsilon}\right).$$

In (Popa (1982)) there is another result about commutators. Let $\mathscr{K}(\mathscr{H})$ be the ideal of compact operators in $\mathscr{B}(\mathscr{H})$ and define

$$\mathbb{C} + \mathscr{K}(\mathscr{H}) := \{\lambda.1_{\mathscr{B}(\mathscr{H})} + T : \lambda \in \mathbb{C}, T \in \mathscr{K}(\mathscr{H})\}.$$

Theorem 6.5.8. *(Popa (1982)) If $K \in \mathscr{B}(\mathscr{H})$ is such that*

$$\|A\| = O(1), \quad \|A\| = O(dist(A, \mathbb{C} + \mathscr{K}(\mathscr{H}))^{\frac{2}{3}}),$$

then there exist $D, X \in \mathscr{B}(\mathscr{H})$ with

$$\|D\|\|X\| = O(1) \quad \textit{such that} \quad A = [D,X].$$

6.6 COMMUTATORS CLOSE TO THE IDENTITY IN UNITAL C*-ALGEBRAS

We recall fundamentals of matrices over unital C*-algebras as given in (Murphy (1990)).

Let $\mathscr{A}$ be a unital C*-algebra. For $n \in \mathbb{N}$, $M_n(\mathscr{A})$ is defined as the set of all n by n matrices over $\mathscr{A}$. It is clearly an algebra with respect to natural matrix operations. We define the involution of an element $A := [a_{i,j}]_{1 \leq i,j \leq n} \in M_n(\mathscr{A})$ as $A^* := [a_{j,i}^*]_{1 \leq i,j \leq n}$. Then $M_n(\mathscr{A})$ is a *-algebra. From the Gelfand-Naimark-Segal theorem (Theorem 6.4.10) there exists unique universal representation $(\mathscr{H}, \pi)$, where $\mathscr{H}$ is a Hilbert space, $\pi : M_n(\mathscr{A}) \to M_n(\mathscr{B}(\mathscr{H}))$ is an isometric *-homomorphism. This gives a norm on $M_n(\mathscr{A})$ defined as

$$\|A\| := \|\pi(A)\|, \quad \forall A \in M_n(\mathscr{A}).$$

This norm makes $M_n(\mathscr{A})$ as a C*-algebra.

In the sequel, $\mathscr{A}$ is a unital C*-algebra. We first derive a lemma followed by a corollary for unital C*-algebras. Proof of the lemma is a direct algebraic calculation.

Lemma 6.6.1. ***(Commutator calculation)*** *Let $u, v, b_1, \ldots, b_n \in \mathscr{A}$ and $\delta > 0$. Let*

$$X := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \delta b_1 \\ 1_{\mathscr{A}} & 0 & 0 & \cdots & 0 & \delta b_2 \\ 0 & 1_{\mathscr{A}} & 0 & \cdots & 0 & \delta b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \delta b_{n-1} \\ 0 & 0 & 0 & \cdots & 1_{\mathscr{A}} & \delta b_n \end{pmatrix} \in M_n(\mathscr{A})$$

and

$$D := \begin{pmatrix} \frac{v}{\delta} & 1_{\mathscr{A}} & 0 & \cdots & 0 & \delta b_1 u \\ \frac{u}{\delta} & \frac{v}{\delta} & 2.1_{\mathscr{A}} & \cdots & 0 & \delta b_2 u \\ 0 & \frac{u}{\delta} & \frac{v}{\delta} & \cdots & 0 & \delta b_3 u \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{v}{\delta} & (n-1)1_{\mathscr{A}} + \delta b_{n-1} u \\ 0 & 0 & 0 & \cdots & \frac{u}{\delta} & \frac{v}{\delta} + \delta b_n u \end{pmatrix} \in M_n(\mathscr{A}).$$

Then

$$[D, X] = 1_{M_n(\mathscr{A})} + \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & [v, b_1] + 0 + \delta b_2 + \delta b_1 [u, b_n] \\ 0 & 0 & 0 & \cdots & 0 & [v, b_2] + [u, b_1] + 2\delta b_3 + \delta b_2 [u, b_n] \\ 0 & 0 & 0 & \cdots & 0 & [v, b_3] + [u, b_2] + 3\delta b_4 + \delta b_3 [u, b_n] \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & [v, b_{n-1}] + [u, b_{n-2}] + (n-1)\delta b_n + \delta b_{n-1} [u, b_n] \\ 0 & 0 & 0 & \cdots & 0 & [v, b_n] + [u, b_{n-1}] + 0 + \delta b_n [u, b_n] - n.1_{\mathscr{A}} \end{pmatrix}.$$

Corollary 6.6.2. *Let $u, v, b_1, \ldots, b_n \in \mathscr{A}$. Assume that for some $\delta > 0$, we have equations*

$$[v, b_i] + [u, b_{i-1}] + i\delta b_{i+1} + \delta b_i [u, b_n] = 0, \quad \forall i = 2, \ldots, n-1 \tag{6.6.1}$$

and

$$[v, b_n] + [u, b_{n-1}] + \delta b_n [u, b_n] = n \cdot 1_{M_n(\mathscr{A})}. \tag{6.6.2}$$

Then for any $\mu > 0$*, there exist matrices* $D_\mu, X_\mu \in M_n(\mathscr{A})$ *such that*

$$\|D_\mu\| \leq \frac{\|u\|}{\mu^2\delta} + \frac{\|v\|}{\mu\delta} + (n-1) + \delta\sum_{i=1}^{n}\mu^{n-i-1}\|b_i\|\|u\|,$$
$$\|X_\mu\| \leq 1 + \delta\sum_{i=1}^{n}\mu^{n-i+1}\|b_i\| \quad \textit{and}$$
$$\|[D_\mu, X_\mu] - 1_{M_n(\mathscr{A})}\| \leq \mu^{n-1}\|[v,b_1] + \delta b_2 + \delta b_1[u,b_n]\|.$$

Proof. Let D and X be as in Lemma 6.6.1. Define

$$S_\mu := \begin{pmatrix} \mu^{n-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu^{n-2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mu^{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in M_n(\mathbb{K}),$$
$$D_\mu := \frac{1}{\mu}S_\mu D S_\mu^{-1}, \quad X_\mu := \mu S_\mu X S_\mu^{-1}.$$

Then

$$\|D_\mu\| = \left\| \begin{pmatrix} \frac{v}{\mu\delta} & 1_{\mathscr{A}} & 0 & \cdots & 0 & \mu^{n-2}\delta b_1 u \\ \frac{u}{\mu^2\delta} & \frac{v}{\mu\delta} & 2.1_{\mathscr{A}} & \cdots & 0 & \mu^{n-3}\delta b_2 u \\ 0 & \frac{u}{\mu^2\delta} & \frac{v}{\mu\delta} & \cdots & 0 & \mu^{n-4}\delta b_3 u \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{v}{\mu\delta} & (n-1)1_{\mathscr{A}} + \delta b_{n-1}u \\ 0 & 0 & 0 & \cdots & \frac{u}{\mu^2\delta} & \frac{v}{\mu\delta} + \mu^{-1}\delta b_n u \end{pmatrix} \right\|$$
$$\leq \left\|\frac{u}{\mu^2\delta}\right\| + \left\|\frac{v}{\mu\delta}\right\| + \|(n-1)1_{\mathscr{A}}\| + \left\| \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \mu^{n-2}\delta b_1 u \\ 0 & 0 & 0 & \cdots & 0 & \mu^{n-3}\delta b_2 u \\ 0 & 0 & 0 & \cdots & 0 & \mu^{n-4}\delta b_3 u \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \delta b_{n-1}u \\ 0 & 0 & 0 & \cdots & 0 & \mu^{-1}\delta b_n u \end{pmatrix} \right\|$$
$$\leq \frac{\|u\|}{\mu^2\delta} + \frac{\|v\|}{\mu\delta} + (n-1) + \delta\sum_{i=1}^{n}\mu^{n-i-1}\|b_i\|\|u\|$$

and

$$\|X_\mu\| = \left\| \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \mu^n \delta b_1 \\ 1_{\mathscr{A}} & 0 & 0 & \cdots & 0 & \mu^{n-1}\delta b_2 \\ 0 & 1_{\mathscr{A}} & 0 & \cdots & 0 & \mu^{n-2}\delta b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mu^2\delta b_{n-1} \\ 0 & 0 & 0 & \cdots & 1_{\mathscr{A}} & \mu\delta b_n \end{pmatrix} \right\|$$

$$\leq \|1_{\mathscr{A}}\| + \left\| \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \mu^n \delta b_1 \\ 0 & 0 & 0 & \cdots & 0 & \mu^{n-1}\delta b_2 \\ 0 & 0 & 0 & \cdots & 0 & \mu^{n-2}\delta b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mu^2\delta b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \mu\delta b_n \end{pmatrix} \right\|$$

$$\leq 1 + \delta \sum_{i=1}^{n} \mu^{n-i+1} \|b_i\|.$$

Now using (6.6.1) and (6.6.2) we get

$$\|[D_\mu, X_\mu] - 1_{M_n(\mathscr{A})}\| = \left\| \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \mu^{n-1}([v,b_1] + \delta b_2 + \delta b_1[v,b_n]) \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right\|$$

$$\leq \mu^{n-1}\|[v,b_1] + \delta b_2 + \delta b_1[u,b_n]\|.$$

□

Let $\mathscr{A}$ be a unital C*-algebra. Assume that there are isometries $u, v \in \mathscr{A}$ such that

$$u^*u = v^*v = uu^* + vv^* = 1_{\mathscr{A}} \quad \text{and} \quad u^*v = v^*u = 0. \tag{6.6.3}$$

Examples of such unital C*-algebras are $\mathscr{B}(\mathscr{H})$ (where $\mathscr{H}$ is an infinite dimensional Hilbert space) as well as any unital C*-algebra which contains the Cuntz algebra $\mathscr{O}_2$ (Cuntz (1977)). Note that whenever a unital C*-algebra admits a trace map there are no isometries satisfying Equation (6.6.3). In particular, any finite dimensional unital C*-algebra does not have such elements. It is also clear that no commutative unital

C*-algebra can have isometries satisfying Equation (6.6.3).

It is shown in Tao (2019) that whenever $\mathscr{H}$ is an infinite dimensional Hilbert space, then the Banach algebras $\mathscr{B}(\mathscr{H})$ and $M_2(\mathscr{B}(\mathscr{H}))$ are isometrically isomorphic. We now do these results for C*-algebras whenever they have isometries satisfying Equation (6.6.3). To do so we first need a result from the theory of C*-algebras.

Theorem 6.6.3. *(cf. Pedersen (2018); Takesaki (2002))*

*(i) Every *-homomorphism between C*-algebras is norm decreasing.*

*(ii) If a *-homomorphism between C*-algebras is injective, then it is isometric.*

Theorem 6.6.4. *Let $\mathscr{A}$ be a unital C*-algebra. If there are isometries $u, v \in \mathscr{A}$ such that Equation (6.6.3) holds, then the map*

$$\phi : \mathscr{A} \ni x \mapsto \begin{pmatrix} u^*xu & u^*xv \\ v^*xu & v^*xv \end{pmatrix} \in M_2(\mathscr{A}) \tag{6.6.4}$$

is a C-algebra isomorphism with the inverse map*

$$\psi : M_2(\mathscr{A}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto uau^* + ubv^* + vcu^* + vdv^* \in \mathscr{A}. \tag{6.6.5}$$

Proof. Using Equation (6.6.3), a direct computation gives

$$\begin{aligned}
\phi\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \phi(uau^* + ubv^* + vcu^* + vdv^*) \\
&= \begin{pmatrix} u^*(uau^* + ubv^* + vcu^* + vdv^*)u & u^*(uau^* + ubv^* + vcu^* + vdv^*)v \\ v^*(uau^* + ubv^* + vcu^* + vdv^*)u & v^*(uau^* + ubv^* + vcu^* + vdv^*)v \end{pmatrix} \\
&= \begin{pmatrix} 1_{\mathscr{A}}a1_{\mathscr{A}} + 1_{\mathscr{A}}b0 + 0c1_{\mathscr{A}} + 0d0 & 1_{\mathscr{A}}a0 + 1_{\mathscr{A}}b1_{\mathscr{A}} + 0c0 + 0d1_{\mathscr{A}} \\ 0a1_{\mathscr{A}} + 0b0 + 1_{\mathscr{A}}c1_{\mathscr{A}} + 1_{\mathscr{A}}d0 & 0a0 + 0b1_{\mathscr{A}} + 1_{\mathscr{A}}c0 + 1_{\mathscr{A}}d1_{\mathscr{A}} \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathscr{A})
\end{aligned}$$

and

$$\begin{aligned}
\psi\phi x = \psi \begin{pmatrix} u^*xu & u^*xv \\ v^*xu & v^*xv \end{pmatrix} \\
&= u(u^*xu)u^* + u(u^*xv)v^* + v(v^*xu)u^* + v(v^*xv)v^* \\
&= uu^*x(uu^* + vv^*) + vv^*x(uu^* + vv^*)
\end{aligned}$$

$$= uu^* x 1_{\mathscr{A}} + vv^* x 1_{\mathscr{A}} = (uu^* + vv^*)x = x, \quad \forall x \in \mathscr{A}.$$

Further,

$$(\phi(x))^* = \begin{pmatrix} u^*xu & u^*xv \\ v^*xu & v^*xv \end{pmatrix}^* = \begin{pmatrix} u^*x^*u & (v^*xu)^* \\ (u^*xv)^* & v^*x^*v \end{pmatrix}$$
$$= \begin{pmatrix} u^*x^*u & u^*x^*v \\ v^*x^*u & v^*x^*v \end{pmatrix} = \phi(x^*), \quad \forall x \in \mathscr{A}.$$

Hence ϕ is a *-isomorphism. Using Theorem 6.6.3, to show ϕ is a C*-algebra isomorphism (i.e., isometric isomorphism), it suffices to show that ϕ is injective. Let $x \in \mathscr{A}$ be such that $\phi x = 0$. Then

$$u^*xu = u^*xv = 0, \quad v^*xv = v^*xu = 0.$$

Using the first equation we get $uu^*xuu^* = uu^*xvv^* = 0$ which implies $uu^*x = uu^*x(uu^* + vv^*) = 0$. Similarly using the second equation we get $vv^*x = 0$. Therefore $x = (uu^* + vv^*)x = 0$. Hence ϕ is injective which completes the proof. □

Along with the lines of Theorem 6.6.4 we can easily derive the following result.

Theorem 6.6.5. *Let $\mathscr{A}$ be a unital C*-algebra and $n \in \mathbb{N}$. If there are isometries $u, v \in \mathscr{A}$ such that Equation (6.6.3) holds, then the map*

$$\phi : M_n(\mathscr{A}) \ni X \mapsto \begin{pmatrix} u^*Xu & u^*Xv \\ v^*Xu & v^*Xv \end{pmatrix} \in M_{2n}(\mathscr{A})$$

is a C-algebra isomorphism with the inverse map*

$$\psi : M_{2n}(\mathscr{A}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto uAu^* + uBv^* + vCu^* + vDv^* \in M_n(\mathscr{A}),$$

where if $X := [x_{i,j}]_{i,j}$ is a matrix, and $a, b \in \mathscr{A}$, by aXb we mean the matrix $[ax_{i,j}b]_{i,j}$. In particular, the C-algebras $\mathscr{A}, M_2(\mathscr{A}), M_4(\mathscr{A}), \dots, M_{2n}(\mathscr{A}), \dots$ are all *-isometrically isomorphic.*

In the rest of this chapter, we assume that unital C*-algebra $\mathscr{A}$ has isometries u, v satisfying Equation (6.6.3). In the next result we use the following notation. Given a vector $x \in \mathscr{A}^n$, x_i means its i^{th} coordinate.

Proposition 6.6.6. *Let $n \geq 2$ and $T : \mathscr{A}^n \to \mathscr{A}^{n-1}$ be the bounded linear operator defined by*

$$T(b_i)_{i=1}^n := ([v,b_i]+[u,b_{i-1}])_{i=2}^n, \quad \forall (b_i)_{i=1}^n \in \mathscr{A}^n.$$

Then there exists a bounded linear right-inverse $R : \mathscr{A}^{n-1} \to \mathscr{A}^n$ for T such that

$$\begin{aligned}\|Rb\| = \sup_{1\leq i\leq n} \|(Rb)_i\| &\leq 8\sqrt{2}n^2 \sup_{2\leq i\leq n} \|b_i\| \\ &\leq 8\sqrt{2}n^2 \sup_{1\leq i\leq n} \|b_i\| = 8\sqrt{2}n^2\|b\|, \quad \forall b \in \mathscr{A}^n.\end{aligned}$$

Proof. Define

$$L : \mathscr{A}^{n-1} \ni (x_i)_{i=2}^n \mapsto \left(-\frac{1}{2}x_i v^* - \frac{1}{2}x_{i+1}u^*\right)_{i=1}^n \in \mathscr{A}^n, \quad \text{where } x_1 := 0, x_{n+1} := 0$$

and

$$E : \mathscr{A}^{n-1} \ni (x_i)_{i=2}^n \mapsto \left(\frac{1}{2}(vx_iv^* + vx_{i+1}u^* + ux_{i-1}v^* + ux_iu^*)\right)_{i=2}^n \in \mathscr{A}^{n-1}.$$

Then

$$\begin{aligned}
TL(x_i)_{i=2}^n &= T\left(-\frac{1}{2}x_iv^* - \frac{1}{2}x_{i+1}u^*\right)_{i=1}^n = -\frac{1}{2}(T(x_iv^*)_{i=2}^n + T(x_{i+1}u^*)_{i=2}^n) \\
&= -\frac{1}{2}(([v,x_iv^*]+[u,x_{i-1}v^*])_{i=2}^n + ([v,x_{i+1}u^*]+[u,x_iu^*])_{i=2}^n) \\
&= -\frac{1}{2}(vx_iv^* - x_iv^*v + ux_{i-1}v^* - x_{i-1}v^*u + vx_{i+1}u^* - x_{i+1}u^*v + ux_iu^* - x_iu^*u)_{i=2}^n \\
&= -\frac{1}{2}(vx_iv^* - x_i + ux_{i-1}v^* - x_{i-1}v^*u + vx_{i+1}u^* + ux_iu^* - x_i)_{i=2}^n \\
&= (x_i)_{i=2}^n - \frac{1}{2}(vx_iv^* + vx_{i+1}u^* + ux_{i-1}v^* + ux_iu^*)_{i=1}^n \\
&= (1-E)(x_i)_{i=2}^n, \quad \forall (x_i)_{i=2}^n \in \mathscr{A}^{n-1}, \quad \text{where } 1(x_i)_{i=2}^n := (x_i)_{i=2}^n.
\end{aligned}$$

We next try to show that the operator $1-E$ is bounded invertible with the help of Neumann series. First step is to change the norm on $\mathscr{A}^n$ to an equivalent norm so that invertibility property will not affect in both norms. Define a new norm on $\mathscr{A}^{n-1}$ by

$$\|(x_i)_{i=2}^n\|' := \sup_{2\leq i\leq n} \left(2 - \frac{i^2}{n^2}\right)^{\frac{-1}{2}} \|x_i\|.$$

Let $x = (x_i)_{i=2}^n \in \mathscr{A}^{n-1}$ be such that $\|(x_i)_{i=2}^n\|' \leq 1$. Then

$$\left(2 - \frac{i^2}{n^2}\right)^{\frac{-1}{2}} \|x_i\| \leq \sup_{2\leq i\leq n} \left(2 - \frac{i^2}{n^2}\right)^{\frac{-1}{2}} \|x_i\| \leq 1, \quad \forall 2 \leq i \leq n.$$

Hence $\|x_i\| \leq \left(2 - \frac{i^2}{n^2}\right)^{\frac{1}{2}}$ for all $2 \leq i \leq n$. Using Theorem 6.6.4 we now get

$$\begin{aligned}
\|(Ex)_i\| &= \frac{1}{2}\|vx_iv^* + vx_{i+1}u^* + ux_{i-1}v^* + ux_iu^*\| \\
&= \frac{1}{2}\left\|\begin{pmatrix} x_i & x_{i+1} \\ x_{i-1} & x_i \end{pmatrix}\right\| \leq \frac{1}{2}\left\|\begin{pmatrix} \|x_i\| & \|x_{i+1}\| \\ \|x_{i-1}\| & \|x_i\| \end{pmatrix}\right\| \\
&\leq \frac{1}{2}\left(\|x_i\|^2 + \|x_{i+1}\|^2 + \|x_{i-1}\|^2 + \|x_i\|^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\left(\left(2 - \frac{i^2}{n^2}\right) + \left(2 - \frac{(i+1)^2}{n^2}\right) + \left(2 - \frac{(i-1)^2}{n^2}\right) + \left(2 - \frac{i^2}{n^2}\right)\right)^{\frac{1}{2}} \\
&= \left(2 - \frac{i^2}{n^2} - \frac{1}{2n^2}\right)^{\frac{1}{2}} \leq \left(1 - \frac{1}{8n^2}\right)^{\frac{1}{2}} \left(2 - \frac{i^2}{n^2}\right)^{\frac{1}{2}} \\
&\leq \left(1 - \frac{1}{8n^2}\right)\left(2 - \frac{i^2}{n^2}\right)^{\frac{1}{2}}, \quad \forall 2 \leq i \leq n.
\end{aligned}$$

Hence

$$\|Ex\|' = \sup_{2\leq i\leq n} \left(2 - \frac{i^2}{n^2}\right)^{\frac{-1}{2}} \|(Ex)_i\| \leq \left(1 - \frac{1}{8n^2}\right)\|x\|', \quad \forall x \in \mathscr{A}^{n-1}.$$

Since $1 - \frac{1}{8n^2} < 1$, $1 - E$ is invertible and $\|(1-E)^{-1}x\|' \leq 8n^2\|x\|'$. Now going back to the original norm, we get

$$\begin{aligned}
\frac{1}{\sqrt{2}}\|((1-E)^{-1}x)_i\| &\leq \sup_{2\leq i\leq n} \left(2 - \frac{i^2}{n^2}\right)^{\frac{-1}{2}} \|((1-E)^{-1}x)_i\| \\
&= \|(1-E)^{-1}x\|' \leq 8n^2\|x\|' \\
&= 8n^2 \sup_{2\leq i\leq n} \left(2 - \frac{i^2}{n^2}\right)^{\frac{-1}{2}} \|x_i\| \\
&\leq 8n^2 \sup_{2\leq i\leq n} \|x_i\| = 8n^2\|x\|, \quad \forall x \in \mathscr{A}^{n-1}.
\end{aligned}$$

Define $R := L(1-E)^{-1}$. Then $TR = TL(1-E)^{-1} = (1-E)(1-E)^{-1} = 1$ and

$$\begin{aligned}\|Rb\| &= \sup_{1\le i\le n} \|(Rb)_i\| = \|L(1-E)^{-1}b\| \le \|L\|\|(1-E)^{-1}b\| \le \|(1-E)^{-1}b\| \\ &= \sup_{2\le i\le n} \|((1-E)^{-1}b)_i\| \le 8\sqrt{2}n^2\|b\| = 8\sqrt{2}n^2 \sup_{2\le i\le n} \|b_i\|, \quad \forall b \in \mathscr{A}^{n-1}.\end{aligned}$$

□

As given in Tao (2019) we try to shift from the systems of equations (6.6.1) and (6.6.2) to the solution of single equation. Let $n \ge 2$. Define $a := (0,\dots,n) \in \mathscr{A}^n$,

$$F : \mathscr{A}^n \ni (b_i)_{i=1}^n \mapsto (-2b_3,\dots,-(n-1)b_n,0) \in \mathscr{A}^{n-1}$$

and

$$G : \mathscr{A}^n \times \mathscr{A}^n \ni ((b_i)_{i=1}^n,(c_i)_{i=1}^n) \mapsto (-b_2[u,c_n],\dots,-b_n[u,c_n]) \in \mathscr{A}^{n-1}.$$

We then have $\|F\| \le n-1$ and $\|G\| \le 2$.

Proposition 6.6.7. *Systems (6.6.1) and (6.6.2) have a solution b if and only if*

$$Tb = a + \delta F(b) + \delta G(b,b). \tag{6.6.6}$$

Proof. Systems (6.6.1) and (6.6.2) have a solution b if and only if

$$[v,b_i] + [u,b_{i-1}] = -i\delta b_{i+1} - \delta b_i[u,b_n], \quad \forall i = 2,\dots,n-1$$

and

$$[v,b_n] + [u,b_{n-1}] = -\delta b_n[u,b_n] + n \cdot 1_{M_n(\mathscr{A})}$$

if and only if

$$\begin{aligned}&([v,b_i] + [u,b_{i-1}])_{i=2}^n = \\ &\quad (0,\dots,n) + \delta(-2b_3,\dots,-(n-1)b_n,0) + \delta(-b_2[u,b_n],\dots,-b_n[u,b_n])\end{aligned}$$

if and only if

$$Tb = a + \delta F(b) + \delta G(b,b).$$

□

The above proposition reduces the work of solving systems (6.6.1) and (6.6.2) to a single operator equation. To solve (6.6.6) we need an abstract lemma from Tao (2019).

Lemma 6.6.8. *(Tao (2019)) Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces, $T, F : \mathscr{X} \to \mathscr{Y}$ be bounded linear operators, and let $G : \mathscr{X} \times \mathscr{X} \to \mathscr{Y}$ be a bounded bilinear operator with bound $r > 0$ and let $a \in \mathscr{Y}$. Suppose that T has a bounded linear right inverse $R : \mathscr{Y} \to \mathscr{X}$. If $\delta > 0$ is such that*

$$\delta(2\|F\|\|R\| + 4r\|R\|^2\|a\|) < 1, \tag{6.6.7}$$

then there exists $b \in \mathscr{X}$ with $\|b\| \leq 2\|R\|\|a\|$ that solves the equation

$$Tb = a + \delta F(b) + \delta G(b,b).$$

Theorem 6.6.9. *For each $n \geq 2$, there exists a solution b to Equation (6.6.6) such that $\|b\| \leq 16\sqrt{2}n^3$.*

Proof. We apply Lemma 6.6.8 for

$$\delta := \frac{1}{2000n^5}.$$

Then using Proposition 6.6.6, we get

$$\begin{aligned} \delta(2\|F\|\|R\| + 4r\|R\|^2\|a\|) &\leq \frac{1}{2000n^5}(2(n-1)8\sqrt{2}n^2 + 4.2.128.n^4.n) \\ &\leq \frac{1}{2000n^5}(16\sqrt{2}n^3 + 1024n^5) < 1. \end{aligned}$$

Lemma 6.6.8 now says that there exists a b which satisfies (6.6.6). □

Theorem 6.6.10. *For each $n \geq 2$, let b be an element satisfying Equation (6.6.6) and $\|b\| \leq 16\sqrt{2}n^3$. Then for $\mu = \frac{1}{2}$, $D_\mu, X_\mu \in M_n(\mathscr{A})$ such that*

$$\|D_\mu\| = O(n^5), \quad \|X_\mu\| = O(1), \quad \|[D_\mu, X_\mu] - 1_{M_n(\mathscr{A})}\| = O(n^3 2^{-n}).$$

Proof. Let $D_\mu, X_\mu \in M_n(\mathscr{A})$ be as in Corollary 6.6.2. We then have

$$\begin{aligned} \|D_\mu\| &\leq 4.2000n^5\|u\| + 2.2000n^5\|v\| + (n-1) + \frac{1}{2000n^5}\sum_{i=1}^{n}\frac{1}{2^{n-i-1}}16\sqrt{2}n^3\|u\| \\ &= O(n^5), \\ \|X_\mu\| &\leq 1 + \frac{1}{2000n^5}\sum_{i=1}^{n}\frac{1}{2^{n-i-1}}16\sqrt{2}n^3 = O(1), \end{aligned}$$

$$
\begin{aligned}
\|[D_\mu, X_\mu] - 1_{M_n(\mathscr{A})}\| &\le 2\mu^{n-1}(\|v\|\|b_1\| + \delta\|b_2\| + \delta\|b_1\|\|u\|\|b_n\|). \\
&\le 2\frac{1}{2^{n-1}}(\|u\|16\sqrt{2}n^3 + \frac{1}{2000n^5}16\sqrt{2}n^3 + \frac{1}{2000n^5}16\sqrt{2}n^3\|u\|16\sqrt{2}n^3) \\
&\le 2\frac{n^3}{2^{n-1}}(\|v\|16\sqrt{2} + \frac{1}{2000n^5}16\sqrt{2} + \frac{1}{2000n^5}16\sqrt{2}n^3\|u\|16\sqrt{2}) = O(n^3 2^{-n}).
\end{aligned}
$$

□

Theorem 6.6.11. *Let $0 < \varepsilon \le 1/2$. Then there exist an even integer n and $D, X \in M_n(\mathscr{A})$ with*

$$\|[D,X] - 1_{M_n(\mathscr{A})}\| \le \varepsilon$$

such that

$$\|D\|\|X\| = O\left(\log^5 \frac{1}{\varepsilon}\right).$$

Proof. Let $D_\mu, X_\mu \in M_n(\mathscr{A})$ be as in Corollary 6.6.2. Theorem 6.6.10 says that there are $\alpha, \beta, \gamma > 0$ be such that

$$\|D_\mu\| \le \alpha n^5, \quad \|X_\mu\| \le \beta, \quad \|[D_\mu, X_\mu] - 1_{M_n(\mathscr{A})}\| \le \gamma n^3 2^{-n}.$$

Since $2^n > n^4$ all but finitely many n's, $\gamma n^3 2^{-n} < \varepsilon$ all but finitely many n's. We now choose real c such that $n = c \log \frac{1}{\varepsilon}$ is even $\gamma n^3 2^{-n} < \varepsilon$. We then have $\|D_\mu\| = O(\log^5(\frac{1}{\varepsilon}))$ and $\|[D_\mu, X_\mu] - 1_{M_n(\mathscr{A})}\| \le \varepsilon$. □

Theorem 6.6.11 and Theorem 6.6.5 easily give the following.

Theorem 6.6.12. *Let $\mathscr{A}$ be a unital C*-algebra. Suppose there are isometries $u, v \in \mathscr{A}$ such that Equation (6.6.3) holds. Then for each $0 < \varepsilon \le 1/2$, there exist $d, x \in \mathscr{A}$ with*

$$\|[d,x] - 1_{\mathscr{A}}\| \le \varepsilon$$

such that

$$\|d\|\|x\| = O\left(\log^5 \frac{1}{\varepsilon}\right).$$

Remark 6.6.13. *Let $\mathscr{A}$ be a finite dimensional unital C*-algebra. From the structure theory (Davidson (1996)) we have*

$$\mathscr{A} \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C}),$$

for unique (upto permutation) natural numbers $n_1, \ldots, n_r$. This result says that normalized trace map (a trace map Tr such that $Tr(1_{\mathscr{A}}) = 1$) exists on $\mathscr{A}$. Using this we make the following two observations.

(i) $\mathscr{A}$ can not have isometries satisfying Equation (6.6.3). Suppose that there are such isometries. Then

$$1 = Tr(uu^* + vv^*) = Tr(uu^*) + Tr(vv^*) = Tr(u^*u) + Tr(v^*v) = 2$$

which is impossible.

(ii) In (Tao (2019)), Tao observed that if $\mathscr{H}$ is a finite dimensional Hilbert space, then there are no $D, X \in \mathscr{B}(\mathscr{H})$ satisfying $\|[D,X] - 1_{\mathscr{B}(\mathscr{H})}\| < 1$. We elaborate this for any finite dimensional unital C-algebra $\mathscr{A}$, namely, there do not exist $d, x \in \mathscr{A}$ satisfying $\|[d,x] - 1_{\mathscr{A}}\| < 1$. In other words, Theorem 6.6.12 fails for every finite dimensional unital C*-algebra. Let $d, x \in \mathscr{A}$ be arbitrary. From the structure theory, we identify that d as D and x as X for some matrices $D, X \in M_n(\mathbb{C})$ and for some n. Using the commutativity of trace we then have $Tr([D,X]) = 0$. Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of $[D,X]$. Then $\sum_{j=1}^{n} \lambda_j = Tr([D,X]) = 0$. This gives*

$$n = \left| \sum_{j=1}^{n} (\lambda_j - 1) \right| \leq \sum_{j=1}^{n} |\lambda_j - 1|.$$

Previous inequality says that there is atleast one j such that $|\lambda_j - 1| \geq 1$. We next see that all the eigenvalues of $[D,X] - 1_{M_n(\mathbb{C})}$ are $\lambda_1 - 1, \ldots, \lambda_n - 1$. Using the property of operator norm we finally get

$$\|[d,x] - 1_{\mathscr{A}}\| = \|[D,X] - 1_{M_n(\mathbb{C})}\| \geq \sup_{1 \leq j \leq n} |\lambda_j - 1| \geq 1.$$

Conclusion and future work : In this appendix we showed that the result of Tao's is valid in more general spaces. One of the future objectives is to improve the bounds in Theorem 6.6.12 and Theorem 6.5.8.

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LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbb{N}$	:	Set of natural numbers
$\mathbb{R}$	:	Field of real numbers
$\mathbb{C}$	:	Field of complex numbers
$\{\cdot\}_n$	:	Collections/sequences indexed by $\mathbb{N}$
$\mathbb{M}$	:	Subset of natural numbers
$\mathbb{M}^c$	:	Complement of $\mathbb{M}$
$\mathbb{K}$	:	$\mathbb{R}$ or $\mathbb{C}$
α, β, γ	:	Elements of $\mathbb{K}$
$\mathcal{H}, \mathcal{H}_0, \dots$	:	Separable Hilbert spaces
$h, h_0, \dots$	:	Elements of Hilbert spaces
$\mathcal{H} \otimes \mathcal{H}_0, \dots$	:	Tensor product of $\mathcal{H}$ and $\mathcal{H}_0$
$\langle \cdot, \cdot \rangle$	:	Inner product which is linear in first variable and conjugate linear in second variable
$\mathcal{X}, \mathcal{Y}, \dots$	:	Separable Banach spaces
$\|\cdot\|$	:	Norm
$\mathcal{X}^*$	:	Dual of Banach space $\mathcal{X}$ equipped with operator norm
$I_{\mathcal{X}}$	:	Identity operator on $\mathcal{X}$
$\mathcal{B}(\mathcal{X}, \mathcal{Y})$	:	Banach space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ equipped with operator-norm
cf.	:	Cross reference (reference may not be the first reference where the notion/result arose)
ASF	:	Approximate Schauder frame
OVF	:	Operator-valued frame
p	:	A real number in $[1, \infty)$
$\ell^p(\mathbb{N})$	:	$\{\{a_n\}_n : a_n \in \mathbb{K}, \forall n \in \mathbb{N}, \sum_{n=1}^\infty \|a_n\|^p < \infty\}$
$\ell^\infty(\mathbb{N})$	:	$\{\{a_n\}_n : a_n \in \mathbb{K}, \forall n \in \mathbb{N}, \sup_{n\in\mathbb{N}} \|a_n\| < \infty\}$
$\mathcal{L}^p(\mathbb{R})$	:	$\{f : \mathbb{R} \in \mathbb{C}, f$ measurable , $\int_{\mathbb{R}} \|f(x)\|^p \, dx < \infty\}$
$\{e_n\}_n$	:	Standard Schauder basis for $\ell^p(\mathbb{N})$
$\mathcal{A}$	:	C*-algebra
$\mathcal{M}, \mathcal{N}$	:	Metric spaces
$d(\cdot, \cdot)$	:	Metric on a metric space

G	:	Locally compact group
μ_G	:	Left Haar measure on G
π	:	Unitary representation of G
$\mathscr{U}$	:	Group-like unitary system
$\mathscr{X}_d$	:	BK-space
$\operatorname{Lip}(\cdot)$	:	Lipschitz number
$(\mathscr{M},0)$	:	Pointed metric space
$\|\cdot\|_{\operatorname{Lip}_0}$	:	Lipschitz norm
$\operatorname{Lip}(\mathscr{M},\mathscr{X})$	:	Space of Lipschitz functions from $\mathscr{M}$ to $\mathscr{X}$ equipped with Lipschitz number
$\operatorname{Lip}_0(\mathscr{M},\mathscr{X})$	:	Banach space of base point preserving Lipschitz functions from $\mathscr{M}$ to $\mathscr{X}$ equipped with Lipschitz norm
$\operatorname{Lip}_0(\mathscr{X})$		$\operatorname{Lip}_0(\mathscr{X},\mathscr{X})$
$\mathscr{F}(\mathscr{M})$	:	Lipschitz free Banach space of $\mathscr{M}$
$c_0(\mathbb{N})$	:	$\{\{a_n\}_n : a_n \in \mathbb{K}, \forall n \in \mathbb{N}, \lim_{n\to\infty} a_n = 0\}$
$c(\mathbb{N})$	:	$\{\{a_n\}_n : a_n \in \mathbb{K}, \forall n \in \mathbb{N}, \{a_n\}_n$ converges in $\mathbb{K}\}$
T^*	:	Adjoint of the operator T
θ_τ	:	Analysis operator for frame $\{\tau_n\}_n$ for Hilbert space
θ_τ^*	:	Synthesis operator for frame $\{\tau_n\}_n$ for Hilbert space
S_τ	:	Frame operator for frame $\{\tau_n\}_n$ for Hilbert space
θ_f	:	Analysis operator for p-ASF $(\{f_n\}_n, \{\tau_n\}_n)$ for Banach space
$S_{f,\tau}$	:	Frame operator for p-ASF $(\{f_n\}_n, \{\tau_n\}_n)$ for Banach space
θ_A	:	Analysis operator for OVF $\{A_n\}_n$
θ_A^*	:	Synthesis operator for OVF $\{A_n\}_n$
S_A	:	Frame operator for OVF $\{A_n\}_n$
$S_{A,\Psi}$	:	Frame operator for weak OVF $(\{A_n\}_n, \{\Psi_n\}_n)$
$M_{\lambda,f,\tau}$	:	Multiplier
$\tau \otimes f$	:	Map defined by $(\tau\otimes f)(x) := f(x)\tau$
f^{-1}	:	Inverse of map f
$\chi_{[0,1]}$	:	Characteristic function on $[0,1]$
dim	:	Dimension of a subspace of a vector space
$P^\perp$	:	$I_{\mathscr{H}} - P$, whenever P is a projection on $\mathscr{H}$
$[\cdot,\cdot]$	:	Semi-inner product
$A^\dagger$	:	Generalized adjoint of the operator A in a semi-inner product space
$\mathscr{A}'$	:	Commutant of a set in an algebra

ker	:	Kernel of a linear operator
$W^{\perp}$	:	Orthogonal complement of a subspace W of a Hilbert space
$\mathbb{T}$	:	Unit circle group centered at origin
group$(\cdot)$	:	Group generated by a subset of a group
$\mathscr{B}(\mathscr{H})$	:	$\mathscr{B}(\mathscr{H},\mathscr{H})$
$\mathscr{H} \oplus \mathscr{H}_0$	:	Direct sum of Hilbert spaces $\mathscr{H}$ and $\mathscr{H}_0$
$\overline{\text{span}}\, W$	:	Closure of span of subset W of a Banach space $\mathscr{X}$
$M_n(\mathscr{A})$	:	C*-algebra of n by n matrices over unital C*-algebra $\mathscr{A}$
$[A,B]$	:	$AB-BA$, commutator of A and B
$\mathscr{O}_2$	:	Cuntz algebra generated by two isometries
$\mathscr{K}(\mathscr{H})$	:	C*-algebra of compact operators in $\mathscr{B}(\mathscr{H})$
dist(x,Y)	:	Distance between an element x and a subset Y of a metric space
$r=O(s)$	:	Asymptotic notation
$\mathscr{C}(K)$	:	C*-algebra of all complex-valued continuous functions on compact Hausdorff space K
$\mathcal{A}$	:	Non empty set
$\mathscr{V}$	:	Vector space
$\mathbb{J}$	:	Index set
$\mathbb{Z}_+$	:	$\{0\}\cup\mathbb{N}$
$\delta_{n,m}$	:	Kronecker delta

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