ON DYNAMICS OF CONTINUOUS FUNCTIONS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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In the memory of my beloved father

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **On Dynamics of Continuous Functions** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

Place: NITK, Surathkal.

Date: 10-02-2021

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CERTIFICATE

This is to *certify* that the Research Thesis entitled **On Dynamics of Continuous Functions** submitted by **Mr. Chaitanya G K** (Register Number: 155030MA15F01) as the record of the research work carried out by him is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. V. Murugan
Research Supervisor

Chairman - DRPC
(Signature with Date and Seal)

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ABSTRACT

The discrete dynamical system of a continuous self-map is generated by iteration of the map; however, the iteration itself, being an operator on the space of continuous self-maps, may generate unusual dynamical behaviours. In this thesis, we prove that the iteration operator is continuous on the space of all continuous self-maps of a compact metric space and therefore induces a discrete dynamical system on the space. We also show how its fixed points and periodic points are determined, and characterize them in the case that the compact metric space is a compact interval or the unit circle by discussing the Babbage equation. Furthermore, we prove that all orbits of the iteration operator are bounded, but most fixed points are not stable. The boundedness and instability exhibit a complex behaviour of the iteration operation, but we prove that this complex behaviour is not chaotic in Devaney's sense.

Another complicated yet critical discrete dynamical system is that which emanates due to a continuous piecewise monotone self-map on an interval. In the kneading theory developed by Milnor and Thurston, it is proved that the kneading matrix and the kneading determinant associated with such a map are invariants under orientation-preserving conjugacy. We consider whether this result is valid for orientation-reversing conjugacy. We also present applications of obtained results towards the computational complexity of kneading matrices and the classification of maps up to topological conjugacy. Furthermore, a relation between kneading matrices of maps and their iterates for a class of chaotic maps is described.

Closely related is the theory of iterative equations. There are obtained many results on solutions of such equations involving a linear combination of iterates, called polynomial-like iterative equations. We investigate an iterative equation with multiplication, a nonlinear combination of iterates, and give results on the existence, uniqueness, stability, and construction of its continuous solutions.

Our study not only addresses essential problems in the theory of dynamical systems and iterative equations but also exhibits subtle interplay between these two areas.

Keywords: Iteration operator, Babbage equation, dynamical system, chaos, piecewise monotone map, turning point, topological conjugacy, kneading matrix, kneading determinant, iterative equation, Banach contraction principle.

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List of Notations

$\mathbb{N}$	Set of all natural numbers
$\mathbb{Z}$	Set of all integers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}_+$	Set of all positive real numbers
$\mathbb{R}_-$	Set of all negative real numbers
$\mathbb{C}$	Set of all complex numbers
S^1	The unit circle in $\mathbb{C}$
$\mathscr{C}^0(X,Y)$	Set of all continuous maps of X into Y
$\mathscr{C}(X)$	$\mathscr{C}^0(X,X)$
f^k	The k^{th} order iterate of f
id	The identity map
$\mathscr{R}(f)$	The range of f
$f\|_A$	The restriction of f to A
$\mathrm{Fix}(f;X)$	Set of all fixed points of f in X
$\mathrm{Per}(f;X)$	Set of all periodic points of f in X
$\mathscr{J}_n$	The iteration operator of order n
$\mathscr{M}(I)$	Set of all piecewise monotone self-maps of I
$T(f)$	Set of all turning points of f
$L(f)$	Set of all laps of f

$\#T(f)$	Number of turning points of f
$f \nearrow I$	$f\|_I$ is strictly increasing
$f \searrow I$	$f\|_I$ is strictly decreasing
$N(f;t)$	The kneading matrix of f
$D(f;t)$	The kneading determinant of f
$\mathbb{O}_{k\times l}$	The zero matrix of order $k \times l$
$\mathbb{I}_k$	The identity matrix of order k

CHAPTER 1

INTRODUCTION

"The only impossible journey is the one you never begin."
- Anthony Robbins

Iteration theory is one of the important areas of research in nonlinear analysis, and much of the modern research is focused on the study of dynamical systems and functional equations. Targoński (1995) also has stated that "*Iteration can be considered as a field of research bordering on functional equations as well as on dynamics*". *Dynamical systems*, being models describing the temporal evolution of systems, have applications to a wide variety of fields, both in and outside mathematics. The growth of many physical processes and the stability of their long-term behaviour can be studied in terms of dynamical systems. For example, population biologists use logistic maps to set up a mathematical model for population growth, whereas physicists use the so-called Duffing equation to model damped oscillators. Another typical application is that any problem whose solving technique involves iteration scheme, such as the problem of finding roots of a polynomial using the Newton-Raphson method, can be considered as a problem in dynamical systems.

On the other hand, the search for solutions of *functional equations*, in particular *iterative equations*, is also an old, frequent and significant problem for many applications in science and engineering. The *iterative root problem*, rooted in the classical works of Babbage (1815), can be used to solve the *embedding flow problems* (Fort (1955)) and the *invariant curve problem* (Kuczma et al. (1990)). The *Schröder's equation* (Schröder (1870)) is useful in analysing discrete dynamical systems by finding a new coordinate in which the system looks simpler. Moreover, functional equations also have applications to economics, game theory, geometry, neural networks, and artificial intelligence (Aczél (1966); Iannella and Kindermann (2005)). Additionally, *fixed point theory* (Zeidler (1986)), *phantom iterates* (Targoński (1985)), and *fractals* (Mandelbrot (1982)) are some other interesting topics in iteration theory. This thesis considers certain aspects involving iteration operation, which confines mainly to the theory of dynamical systems and iterative equations.

1.1 PRELIMINARIES

The following three subsections give a brief account of certain notions in the theory of dynamical systems, Milnor-Thurston's kneading theory for continuous piecewise monotone maps, and the theory of iterative equations, which together constitute the preliminaries of our work.

1.1.1 Discrete Dynamical Systems

A continuous map $f : X \to X$, where X is a metric space equipped with the metric d, defines a *discrete dynamical system* (X, f) (see e.g. Brin and Stuck (2002); Holmgren (1994)) with its iteration semigroup $\{f^k : k \geq 0\}$ and the *orbit* of any $x \in X$ under f is the sequence $x, f(x), f^2(x), \ldots$, where f^k denotes the k^{th} order iterate of f, which is defined by the composition $f^k := f \circ f \circ \cdots (k \text{ times}) \cdots \circ f$, i.e., recursively by

$$f^k(x) := f(f^{k-1}(x)), \quad f^0(x) := \mathrm{id}(x), \quad \forall x \in X,$$

id being the identity map on X. A point $x_0 \in X$ is called a *periodic point* of f if $f^k(x_0) = x_0$ for some $k \in \mathbb{N}$ and the least such k is called the *period* of x_0. Periodic points of period 1 are called *fixed points* of f. In order to emphasize the dependence on the space X, we use $\mathrm{Fix}(f;X)$ and $\mathrm{Per}(f;X)$ to denote the set of all fixed points and the set of all periodic points of f in X, respectively. For real maps, we have the following classical result.

Theorem 1.1.1. (Sarkovskiĭ (1965)) *Consider the relation $\triangleleft$ on $\mathbb{N}$ defined as follows: $n_1 \triangleleft n_2$ if for any continuous map of $\mathbb{R}$ into itself the existence of a periodic point of period n_2 follows from the existence of a periodic point of period n_1. Then $\triangleleft$ transforms $\mathbb{N}$ into an ordered set, ordered in the following way:*

$$3 \triangleleft 5 \triangleleft 7 \triangleleft 9 \triangleleft 11 \triangleleft \cdots \triangleleft 3 \cdot 2 \triangleleft 5 \cdot 2 \triangleleft \cdots \triangleleft 3 \cdot 2^2 \triangleleft 5 \cdot 2^2 \triangleleft \cdots \triangleleft 2^3 \triangleleft 2^2 \triangleleft 2 \triangleleft 1.$$

Let (X, f) be a discrete dynamical system, where X is a metric space endowed with metric d. As defined in Holmgren (1994), f is said to be *topologically transitive* if for every pair of open sets U, V in X there exist $x \in U$ and $n \in \mathbb{N}$ such that $f^n(x) \in V$. f is said to be *sensitively dependent on initial conditions* if there exists $\delta > 0$ such that for every $x \in X$ and every $\varepsilon > 0$ there exist $y \in X$ and $n \in \mathbb{N}$ such that $d(x,y) < \varepsilon$ and $d\big(f^n(x), f^n(y)\big) > \delta$. As defined in Devaney (2003), f is said to be *chaotic* in Devaney's sense if **(i)** the set of periodic points of f is dense in X, **(ii)** f is topologically transitive, and **(iii)** f exhibits sensitive dependence on initial conditions. An account of various elementary concepts in dynamical systems can be found in, for example, Collet and Eckmann (1980); Devaney (2003); Brin and Stuck (2002); Lind and Marcus (1995); Milnor (2006); de Vries (2014) and Ruette (2017).

Although it is Devaney's definition that is most popular now, it was Li and Yorke (1975) who gave the first mathematical definition of chaos. A pair $(x,y) \in X \times X$ is said to be

(i) *proximal* if $\liminf_{k\to\infty} d(f^k(x), f^k(y)) = 0$,

(ii) *asymptotic* if $\lim_{k\to\infty} d(f^k(x), f^k(y)) = 0$,

(iii) *Li-Yorke* if it is proximal but not asymptotic, i.e., the orbits of x and y get arbitrarily close to each other, but infinitely often, they are at a positive distance.

A subset S of X is said to be *scrambled* if the pair (x,y) is Li-Yorke whenever $x, y \in S$ such that $x \neq y$. The system (X, f) is said to be *Li-Yorke chaotic* if X has an uncountable scrambled set. These two definitions of chaos, in some sense, describe the complexity of a system using the behaviours of points under iteration. Moreover, it is proved in Huang and Ye (2002) that for compact spaces the chaos in the sense of Devaney is stronger than that of Li–Yorke.

Example 1.1.2. The *Logistic map* $L : [0,1] \to [0,1]$ defined by $L(x) = 4x(1-x)$ is chaotic in the sense of Devaney and Li-Yorke.

More generally, for interval maps, we have the following celebrated result.

Theorem 1.1.3. (Theorem 1 of Li and Yorke (1975)) *Let J be an interval in $\mathbb{R}$. If $f : J \to J$ is continuous and has a periodic point of period three, then it is Li-Yorke chaotic.*

1.1.2 Milnor-Thurston's Kneading Theory

Continuous piecewise monotone self-maps of a compact interval in the real line provide interesting examples of discrete dynamical systems (Devaney (2003); Holmgren (1994); Preston (1983, 1988)), however their behaviour can be very complex. Milnor and Thurston have developed the so-called *kneading theory* (Milnor and Thurston

(1977, 1988)) to analyse the iterates of such maps, which makes use of advanced techniques from combinatorics and analysis. They associated with each piecewise monotone map a matrix and an unusual determinant, called the kneading matrix and kneading determinant, respectively. In some sense, this matrix contains most of the crucial combinatorial information of the map and its iterates.

Let $I = [a,b]$ be a compact interval in $\mathbb{R}$ such that $a < b$ and $\mathscr{C}(I) := C^0(I,I)$ consist of all continuous self-maps of I. As defined in Milnor and Thurston (1988), an element $f \in \mathscr{C}(I)$ is said to be *piecewise monotone* if there exists a partition $a = c_0 < c_1 < \cdots < c_m < c_{m+1} = b$ of I such that the restriction of f to subintervals $I_j = [c_{j-1}, c_j]$ is strictly monotone for $1 \le j \le m+1$. Let $f \in \mathscr{M}(I)$, the set of all piecewise monotone maps in $\mathscr{C}(I)$, and suppose that the minimal choice for the c_i's is made so that f is not monotone in any neighbourhood of c_i for $1 \le i \le m$. Then the points $c_1, c_2, \ldots, c_m$ are called the *turning points* of f and the subintervals I_j, $j = 1, 2, \ldots, m+1$, the *laps* of f. A map f in $\mathscr{M}(I)$ with exactly one turning point is called a *unimodal map*. For $f \in \mathscr{M}(I)$, let $T(f)$ denote the set of turning points of f, $\#T(f)$ the number of turning points of f and $L(f)$ the set of laps of f.

The set $\mathscr{M}(I)$ is closed with respect to composition of maps. In fact,

$$T(f \circ g) = \big(T(g) \cup g^{-1}\big(T(f)\big)\big) \cap (a,b). \tag{1.1.1}$$

So, in particular, if $f \in \mathscr{M}(I)$, then $f^k \in \mathscr{M}(I)$ satisfying that

$$T(f^k) = \{x \in (a,b) : f^l(x) \in T(f) \text{ for some } 0 \le l \le k-1\} \tag{1.1.2}$$

for each $k \in \mathbb{N}$.

Let V be the $(m+1)$-dimensional vector space over $\mathbb{Q}$ with an ordered basis the set of formal symbols $I_1, I_2, \ldots, I_{m+1}$ and $V[[t]]$ be the $\mathbb{Q}[[t]]$-module consisting of all formal power series with coefficients in V. For each $x \in I$ and $k \ge 0$, let

$$A(f^k(x)) := \begin{cases} I_j & \text{if} \quad f^k(x) \in I_j \text{ and } f^k(x) \notin T(f),\ 1 \le j \le m+1, \\ C_i & \text{if} \quad f^k(x) = c_i,\ 1 \le i \le m, \end{cases}$$

where $C_i := \frac{1}{2}(I_i + I_{i+1})$ for $1 \le i \le m$, and let

$$A(x, f; t) := \sum_{k \ge 0} A(f^k(x)) t^k.$$

The symbol $A(x)$ is called the *address* of x. For $k \ge 0$, we denote $A(f^k(x))$ by $A_k(x, f)$.

Given any subinterval I' of I, we write $f \nearrow I'$ (resp. $f \searrow I'$) if the restriction of f

to I' is strictly increasing (resp. strictly decreasing). For each symbol I_j, we define its *sign* by

$$\varepsilon(I_j) = \begin{cases} +1 & \text{if } f \nearrow I_j, \\ -1 & \text{if } f \searrow I_j, \end{cases}$$

and for each of the vector C_j corresponding to the turning point c_j, let $\varepsilon(C_j) := 0$. For each $x \in I$, let

$$\varepsilon(x, f^k) := \varepsilon(A_k(x, f)) \text{ for } k \geq 0,$$

$$\theta(x, f^0) := A_0(x, f), \quad \text{and} \quad \theta(x, f^k) := \left(\prod_{l=0}^{k-1} \varepsilon(x, f^l) \right) A_k(x, f) \text{ for } k \geq 1.$$

The corresponding formal power series are defined by

$$\varepsilon(x, f; t) = \sum_{k \geq 0} \varepsilon_k(x, f) t^k \quad \text{and} \quad \theta(x, f; t) = \sum_{k \geq 0} \theta_k(x, f) t^k,$$

where $\varepsilon_k(x, f)$ and $\theta_k(x, f)$ denote $\varepsilon(x, f^k)$ and $\theta(x, f^k)$, respectively.

Consider $V[[t]]$ in the formal power series topology in which the submodules $t^k V[[t]]$ form a basis for the neighbourhoods of zero. For each $x \in [a, b)$ and $k \geq 0$, let

$$A(f^k(x+)) := \lim_{y \downarrow x} A_k(y, f), \quad \varepsilon_k(x+, f) := \lim_{y \downarrow x} \varepsilon_k(y, f)$$

and

$$\theta_k(x+, f) := \lim_{y \downarrow x} \theta_k(y, f).$$

Also, for each $x \in (a, b]$ and $k \geq 0$, the limits $A(f^k(x-))$, $\varepsilon_k(x-, f)$ and $\theta_k(x-, f)$ are defined similarly. Then it follows that

$$\varepsilon_k(x+, f) = \varepsilon(A_k(x+, f)) \text{ for } x \in [a, b),\ k \geq 0,$$

and

$$\varepsilon_k(x-, f) = \varepsilon(A_k(x-, f)) \text{ for } x \in (a, b],\ k \geq 0,$$

where $A_k(x+, f)$ and $A_k(x-, f)$ denote $A(f^k(x+))$ and $A(f^k(x-))$, respectively. Moreover,

$$A_k(c_i+, f) = A_k(c_i-, f) \text{ for } 1 \leq i \leq m \text{ and } k \in \mathbb{N}. \tag{1.1.3}$$

For each $x \in [a,b)$, let $\theta(x+,f) := \lim_{y \downarrow x} \theta(y,f)$, and for each $x \in (a,b]$, let $\theta(x-,f) := \lim_{y \uparrow x} \theta(y,f)$. Then

$$\theta(x+,f;t) = \sum_{k \geq 0} \theta_k(x+,f)t^k \text{ for } x \in [a,b)$$

and

$$\theta(x-,f;t) = \sum_{k \geq 0} \theta_k(x-,f)t^k \text{ for } x \in (a,b].$$

As defined in Milnor and Thurston (1988), the measure of discontinuity $\theta(c_i+,f;t) - \theta(c_i-,f;t)$, evaluated at c_i, is called the i^{th} *kneading increment* $\nu(c_i,f;t)$ of f for $1 \leq i \leq m$. The matrix $N(f;t) = [N_{ij}(f;t)]$ of order $m \times (m+1)$, with entries in $\mathbb{Z}[[t]]$, obtained by setting

$$\nu(c_i,f;t) = N_{i1}(f;t)I_1 + N_{i2}(f;t)I_2 + \cdots + N_{i,m+1}(f;t)I_{m+1}, \text{ for } 1 \leq i \leq m$$

is called the *kneading matrix* of f. We can write the matrix $N(f;t)$ as a power series $\sum_{k \geq 0}[N_{ij}^k(f;t)]t^k$, where the coefficients $[N_{ij}^0(f;t)]$, $[N_{ij}^1(f;t)]$, ... are matrices with integer entries. For $k \geq 1$, the entry $N_{ij}^k(f;t)$ of $[N_{ij}^k(f;t)]$ is non-zero if and only if $A_k(c_i+,f) = I_j$, each non-zero entry being either $+2$ or -2 according as c_i is a local minimum or a local maximum point of f^k. On the other hand, for $k = 0$, the matrix $[N_{ij}^0(f;t)]$ is given by

$$[N_{ij}^0(f;t)] = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{m \times (m+1)}$$

and in fact, it is independent of the map f. Let $N_k(f;t)$ denote the matrix $[N_{ij}^k(f;t)]$ for $k \geq 0$, and $M(f;t) := \sum_{k \geq 1} N_k(f;t)t^k$.

For $1 \leq j \leq m+1$, let $N^{(j)}(f;t)$ denote the $m \times m$ matrix obtained by deleting the j^{th} column of $N(f;t)$. Then the power series

$$(-1)^{j+1}\big(1 - \varepsilon(I_j)t\big)^{-1} \det\big(N^{(j)}(f;t)\big)$$

is indeed independent of the choice of j for $1 \leq j \leq m+1$ and this common expression, denoted by $D(f;t)$, is called the *kneading determinant* of f. The power series $D(f;t)$ has leading coefficient $+1$, and hence is a unit in the ring $\mathbb{Z}[[t]]$. Moreover, it is proved in Milnor and Thurston (1988) that

$$D(f;t) = 1 + \sum_{k\geq 1} \left(\prod_{l=1}^{k} \varepsilon_l(c+,f) \right) t^k,$$

whenever f is a unimodal map with turning point c.

Example 1.1.4. For the *Tent map* $\mathsf{T} : [0,1] \to [0,1]$ defined by $\mathsf{T}(x) = 1 - |1-2x|$, we have

$$D(\mathsf{T};t) = 1 - t - t^2 - t^3 - \cdots$$

and $N(\mathsf{T};t) = [N_{11}(\mathsf{T};t),\ N_{12}(\mathsf{T};t)]$, where

$$N_{11}(\mathsf{T};t) = -1 + 2t^2 + 2t^3 + \cdots \quad \text{and} \quad N_{12}(\mathsf{T};t) = 1 - 2t.$$

Example 1.1.5. Let g be defined by

$$g(x) = 64x^4 - 128x^3 + 80x^2 - 16x + 1, \quad \forall x \in [0,1].$$

Then $g \in \mathscr{M}([0,1])$ such that

$$T(g) = \left\{ d_1 = \tfrac{2-\sqrt{2}}{4},\ d_2 = \tfrac{1}{2},\ d_3 = \tfrac{2+\sqrt{2}}{4} \right\}$$

and

$$L(g) = \left\{ J_1 = \left[0, \tfrac{2-\sqrt{2}}{4}\right],\ J_2 = \left[\tfrac{2-\sqrt{2}}{4}, \tfrac{1}{2}\right],\ J_3 = \left[\tfrac{1}{2}, \tfrac{2+\sqrt{2}}{4}\right],\ J_4 = \left[\tfrac{2+\sqrt{2}}{4}, 1\right] \right\}.$$

Also, $\nu(d_i,g;t) = 2tJ_1 - J_i + J_{i+1} + (-2t^2 - 2t^3 - \cdots)J_4$ for $i = 1,3$ and $\nu(d_2,g;t) = -J_i + J_{i+1} + (-2t - 2t^2 + \cdots)J_4$. Therefore

$$N(g;t) = \begin{bmatrix} -1+2t & 1 & 0 & -2t^2 - 2t^3 - \cdots \\ 0 & -1 & 1 & -2t - 2t^2 - \cdots \\ 2t & 0 & -1 & 1 - 2t^2 - 2t^3 - \cdots \end{bmatrix}_{3\times 4}, \tag{1.1.4}$$

and hence $D(g;t) = 1 - 3t - 3t^2 - \cdots$.

The kneading theory has slight modifications made by Preston (1989), wherein the kneading matrix is indeed a square matrix of order m and the corresponding kneading determinant is the usual determinant. Moreover, being an important area of research in symbolic dynamics, this theory has been developed in various aspects. Preston (1989) extended this theory for piecewise monotone maps which have discontinuities at their turning points. Alves and Sousa Ramos (1999), using a functorial approach to this theory, have given explicit methods to compute the lap numbers and periodic points of piecewise monotone maps. Mendes and Ramos (2004) have developed a kneading theory for two-dimensional triangular maps and thereby exhibited adequate techniques for rigorous computation of the topological entropy of such maps. The other advancements in this theory also include kneading with weights (Rugh and Lei (2015)), and kneading theory for tree maps (Alves and Sousa Ramos (2004)).

1.1.3 Iterative Equations

Being an important operation in the present era of informatics, iteration gets more and more attractive to researchers and attentions were paid to those functional equations involving iteration, called *iterative equations* (Kuczma et al. (1990); Baron and Jarczyk (2001); Zdun and Solarz (2014)). The general form of such equations can be presented as

$$\Phi(f(x), f^2(x), \ldots, f^m(x)) = F(x), \quad x \in X, \tag{1.1.5}$$

where X is a non-empty set, F and Φ are given functions, and f is unknown. Some special cases of this equation, for example, dynamics of a quadratic map (Devaney (2003)) and *Feigenbaum's equation* related to period doubling bifurcations (McCarthy (1983)), are interesting topics in dynamical systems.

Although there can be found several papers (Murugan and Subrahmanyam (2006); Wang and Si (2001)) on the general Lipschitzian Φ, more efforts were still made to the basic form

$$f^m(x) = F(x), \tag{1.1.6}$$

called the *iterative root problem*. After Babbage (1815) initiated the research of solving

$$f^m(x) = \mathrm{id(x)}, \tag{1.1.7}$$

usually called the *Babbage equation*, (1.1.6) has been studied extensively in various aspects, see for instance, for continuous maps on intervals (Kuczma (1968); Targoński (1981); Baron and Jarczyk (2001); Li and Zhang (2018); Li and Liu (2019)), continuous complex maps (Zdun (2000); Jarczyk (2003)), continuous maps on planes and $\mathbb{R}^n$

(Leśniak (2002)), and set valued maps (Nikodem and Zhang (2004); Li (2009); Li et al. (2009)).

Usually, a solution ϕ of (1.1.7) is referred to as an m^{th} *order unit iterative root* if m is the smallest positive integer such that (1.1.7) is satisfied. In particular, as in Kuczma (1968) (p. 290), for $m = 2$ every solution of (1.1.7), whose inverse is itself, is called an *involutory function*. The following lemma gives a necessary condition on the exponent m for existence of solutions.

Lemma 1.1.6. *If f is an m^{th} order unit iterative root and $f^k = \mathrm{id}$, then $m \mid k$.*

Proof. Let k be a positive integer such that $f^k = \mathrm{id}$. Then, by division algorithm, $k = mq + r$ for some $q \in \mathbb{N}$ and $r \in \{0, 1, \ldots, m-1\}$. Also,

$$f^r = \mathrm{id} \circ f = (f^m)^q \circ f^r = f^{mq+r} = f^k = \mathrm{id}.$$

Therefore, by our assumption on m, we must have $r = 0$. Hence $m \mid k$. □

Let $S^1 := \{e^{i\theta} : 0 \leq \theta < 2\pi\}$ denote the unit circle in $\mathbb{C}$. Given $z_0, z_1, \ldots, z_{m-1} \in S^1$ with $m \geq 2$, we write $z_0 \prec z_1 \prec \cdots \prec z_{m-1}$ if there exist $t_1, t_2, \ldots, t_{m-1} \in \mathbb{R}$ such that $0 < t_1 < t_2 < \ldots < t_{m-1} < 1$ and $z_j = z_0 e^{2\pi i t_j}$ for $1 \leq j \leq m-1$. In this case, we have

$$z_{j(\text{mod m})} \prec z_{j+1(\text{mod m})} \prec \cdots \prec z_{j+m-1(\text{mod m})}, \ \forall j \in \mathbb{N},$$

so that "$\prec$" is indeed a *cyclic order* on S^1. For any two distinct points $z_1, z_2 \in S^1$, define the *arcs* (z_1, z_2), $[z_1, z_2)$ and $(z_1, z_2]$ by

$$(z_1, z_2) = \{z \in S^1 : z_1 \prec z \prec z_2\},$$

$$[z_1, z_2) = (z_1, z_2) \cup \{z_1\} \quad \text{and} \quad (z_1, z_2] = (z_1, z_2) \cup \{z_2\}.$$

Then we have

$$(z_1, z_2) = \{e^{2\pi i t} \in S^1 : t \in (t_1, t_2)\},$$

where t_1, t_2 are unique reals such that $z_1 = e^{2\pi i t_1}, z_2 = e^{2\pi i t_2}$ and $0 \leq t_1 < t_2 < t_1 + 1 < 2$.

It is known (cf. Block and Coppel (1992); Wall (1993)) that for every homeomorphism $F : S^1 \to S^1$ there exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ satisfying one of the *Abel equations*

$$f(t+1) = f(t) + 1 \text{ if } f \text{ is strictly increasing},$$
$$f(t+1) = f(t) - 1 \text{ if } f \text{ is strictly decreasing}$$

such that

$$F(e^{2\pi it}) = e^{2\pi if(t)},\ \forall t \in \mathbb{R}.$$

Every such f is called a *lift* of F. We say that F *preserves* (or *reverses*) orientation accordingly as f is strictly increasing (or decreasing) on $\mathbb{R}$.

On a general set E, the m^{th} order unit iterative roots are formulated in Kuczma (1968).

Lemma 1.1.7. (Theorem 15.1 of Kuczma (1968)) *Let* $\{m_0,...,m_r\}$*, where* $1 = m_0 < m_1 < \cdots < m_r = m$*, be the complete set of divisors of* m *and let*

$$E = \bigcup_{i=0}^{r} \bigcup_{j=1}^{m_i} U_j^i$$

be a decomposition of E *into disjoint sets such that the sets* $U_1^i, U_{m_1}^i, \ldots, U_{m_i}^i$ *have the same cardinality for each* $1 \le i \le r$*. For* $1 \le i \le r$ *and* $1 \le j \le m_i - 1$*, let* f_{ij} *be an arbitrary one-to-one map of* U_j^i *onto* U_{j+1}^i*. Then the formula*

$$f(x) := \begin{cases} x & \text{for} \quad x \in U_1^0, \\ f_{ij}(x) & \text{for} \quad x \in U_j^i, j = 1,2,\ldots,m_i - 1, \quad i \ge 1, \\ f_{i1}^{-1}\big(\cdots\big(f_{i,m_i-1}^{-1}(x)\big)\cdots\big) & \text{for} \quad x \in U_{m_i}^i, \quad i \ge 1 \end{cases}$$

defines the general solution of $\phi^m = \mathrm{id}$ *on* E*.*

The following three lemmas together describe the general continuous solutions of (1.1.7) on the compact interval $[a,b]$ in $\mathbb{R}$ and S^1, each of which can also be deduced from Lemma 1.1.7.

Lemma 1.1.8. (McShane (1961); Vincze (1959)) *If* $f \in \mathscr{C}([a,b])$ *is a solution of* (1.1.7)*, then either* $f = \mathrm{id}$ *or* f *is a decreasing involutory function on* $[a,b]$ *(i.e.,*

$$f(x) = \begin{cases} \phi_0(x) & \text{if} \quad x \in [a,x_0], \\ \phi_0^{-1}(x) & \text{if} \quad x \in [x_0,b], \end{cases}$$

where $x_0 \in [a,b]$ *and* $\phi_0 : [a,x_0] \to [x_0,b]$ *is a decreasing bijective map).*

Lemma 1.1.9. (Jarczyk (2003)) *Let* $f \in \mathscr{C}(S^1)$ *be a solution of* (1.1.7) *and has a fixed point in* S^1*.*

(i) *If* f *is orientation-preserving, then* f *is the identity map.*

(ii) *If f is orientation-reversing, then f is an involution (i.e.,*

$$f(z) = \begin{cases} \phi_0(z) & \text{if } z \in [z_0, z_1), \\ z_1 & \text{if } z = z_1, \\ \phi_0^{-1}(z) & \text{if } z \in (z_1, z_0), \end{cases}$$

where $z_0, z_1 \in S^1$ and $\phi_0 : [z_0, z_1) \to (z_1, z_0]$ is an arbitrary homeomorphism such that $\phi_0(z_0) = z_0$).

Lemma 1.1.10. (Jarczyk (2003)) *All m^{th} order iterative roots of identity in $\mathscr{C}(S^1)$ having no fixed points in S^1 are given by*

$$f(z) = \begin{cases} \phi_0(z) & \text{if } z \in [z_0, z_{m-1}), \\ (\phi_1 \circ \phi_2 \circ \cdots \circ \phi_{m-1})^{-1}(z) & \text{if } z \in [z_{m-1}, z_0) \end{cases} \tag{1.1.8}$$

with

$$\phi_j := \phi_0|_{[z_{j(m-k)-1}, z_{j(m-k)})} \text{ for } 1 \le j \le m-1,$$

where k is an integer in $\{1, 2, \ldots, m-1\}$ relatively prime to m and $z_0, z_1, \ldots, z_{m-1}$ are some points in S^1 such that $z_0 \prec z_1 \prec \cdots \prec z_{m-1}$, and $\phi_0 : [z_0, z_{m-1}) \to [z_k, z_{k-1})$ is any arbitrary homeomorphism such that

$$\phi_0([z_{j-1}, z_j)) = [z_{j-1+k}, z_{j+k}) \text{ for } 1 \le j \le m-1$$

with $z_j := z_{j(\text{mod m})}$ for $j \ge m$.

As indicated in Kuczma (1968), every decreasing involutory function on $[a, b]$ has a graph symmetric with respect to the diagonal of $[a, b] \times [a, b]$. More precisely, any solution of (1.1.7) on $[a, b]$ for general m is also a 2^{nd} order unit iterative root. If m is odd, then the solution is uniquely the identity id; if m is even, then the solution is either id or a decreasing involutory function on $[a, b]$. As seen in Lemma 1.1.9, similar conclusions hold if the solution of (1.1.7) on S^1 has a fixed point.

Usually, a solution f of (1.1.6) is called an m^{th} *order iterative root* of F. For interval maps, we have the following classical result.

Theorem 1.1.11. (Theorem 11.2.2 of Kuczma et al. (1990)) *Let J be an interval in $\mathbb{R}$. If $F \in \mathscr{C}(J)$ is strictly increasing, then it has strictly increasing iterative roots of all orders.*

Recently, the generalized form

$$\alpha_1 f(x) + \cdots + \alpha_n f^n(x) = F(x) \tag{1.1.9}$$

of (1.1.6) with Φ in a linear combination, called the *polynomial-like iterative equation*, is also being investigated for more concrete properties.

Let $\mathscr{C}_b(\mathbb{R})$ be the Banach space of all bounded continuous self-maps of $\mathbb{R}$ with the uniform norm $\|\cdot\|$, defined by $\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}$. Let $\mathbb{I} := |a,b|$, where $|a,b|$ denotes either an open interval (a,b), a semi-closed interval $[a,b)$ or $(a,b]$, or a closed interval $[a,b]$ in $\mathbb{R}$, and one or both of the endpoints of $\mathbb{I}$ may be infinite. For $\zeta \in \bar{\mathbb{I}}$, the closure of $\mathbb{I}$, and $\lambda \in [0,1)$, let

$$R_{\zeta,\lambda}[\mathbb{R};\mathbb{I}] := \{f \in \mathscr{C}_b(\mathbb{R}) : f|_{\mathbb{I}} \text{ is strictly increasing and satisfies } \textbf{(A1)} \text{ and } \textbf{(A2)}\},$$

where

(A1) $\quad (f(x) - (1-\lambda)x)(\zeta - x) > 0$ for $x \neq \zeta$,

(A2) $\quad (f(x) - (1-\lambda)\zeta)(\zeta - x) < 0$ for $x \neq \zeta$.

Then we have the following results on construction of solutions for

$$f^n(x) = \sum_{k=1}^{n-1} \lambda_k f^k(x) + F(x), \tag{1.1.10}$$

which is indeed equivalent to (1.1.9).

Lemma 1.1.12. (Xu and Zhang (2007a)) *Let $\lambda \in [0,1)$ and $F \in R_{a,\lambda}[\mathbb{I};\mathbb{I}]$. Then for arbitrary $x_0 \in (a,b|$,* (1.1.10) *has a solution in $R_{a,0}[I_1;I_1]$, where $I_1 = [a,x_0]$. More concretely, for every arbitrary $x_0 \in (a,b|$, there exists a strictly decreasing sequence $(x_1,x_2,\ldots,x_{n-1})$ in (a,x_0) such that the sequence (x_m) defined recursively by*

$$x_{n+m} = \sum_{j=1}^{n-1} \lambda_j x_{j+m} + F(x_m) \quad \textit{for } m \geq 0 \tag{1.1.11}$$

satisfies the conditions **(i)** $x_{m+1} \in (a,x_m)$ *for* $m \geq 1$, **(ii)** $(a,x_0] = \bigcup_{m=1}^{\infty} [x_m,x_{m-1}]$, *and*

$$f(x) := \begin{cases} a & \text{if } \ x = a, \\ f_m(x) & \text{if } \ x \in [x_m,x_{m-1}],\ m \geq 1 \end{cases}$$

is a solution of (1.1.10) *in $R_{a,0}[I_1;I_1]$, where $f_j : [x_j,x_{j-1}] \to [x_{j+1},x_j]$ is an arbitrary* *order-preserving homeomorphism for* $1 \leq j \leq n-1$ *and* $f_m : [x_m, x_{m-1}] \to [x_{m+1}, x_m]$ *is the order-preserving homeomorphism defined recursively by*

$$\begin{aligned} f_m(x) &= \lambda_{n-1}x + \lambda_{n-2} f_{m-1}^{-1}(x) + \cdots + \lambda_1 f_{m-n+2}^{-1} \circ f_{m-n+3}^{-1} \circ \cdots \circ f_{m-1}^{-1}(x) \\ &\quad + F \circ f_{m-n+1}^{-1} \circ f_{m-n+2}^{-1} \circ \cdots \circ f_{m-1}^{-1}(x), \quad x \in [x_m, x_{m-1}], \quad \text{for } m \geq n. \end{aligned}$$

Lemma 1.1.13. (Xu and Zhang (2007a)) *Let* $\lambda \in [0,1)$ *and* $F \in R_{b,\lambda}[\mathbb{I};\mathbb{I}]$. *Then for every arbitrary* $x_0 \in |a,b)$, (1.1.10) *has a solution in* $R_{b,0}[I_2; I_2]$, *where* $I_2 = [x_0, b]$. *More concretely, for every arbitrary* $x_0 \in |a,b)$, *there exists a strictly increasing sequence* $(x_1, x_2, \ldots, x_{n-1})$ *in* (x_0, b) *such that the sequence* (x_m) *defined recursively by* (1.1.11) *satisfies the conditions* **(i)** $x_{m+1} \in (x_m, b)$ *for* $m \geq 1$, **(ii)** $[x_0, b) = \bigcup_{m=1}^{\infty} [x_{m-1}, x_m]$, *and*

$$f(x) := \begin{cases} f_m(x) & \text{if } x \in [x_{m-1}, x_m],\ m \geq 1, \\ b & \text{if } x = b, \end{cases}$$

is a solution of (1.1.10) *in* $R_{b,0}[I_2; I_2]$, *where* $f_j : [x_{j-1}, x_j] \to [x_j, x_{j+1}]$ *is an arbitrary order-preserving homeomorphism for* $1 \leq j \leq n-1$ *and* $f_m : [x_{m-1}, x_m] \to [x_m, x_{m+1}]$ *is the order-preserving homeomorphism defined recursively by*

$$\begin{aligned} f_m(x) &= \lambda_{n-1}x + \lambda_{n-2} f_{m-1}^{-1}(x) + \cdots + \lambda_1 f_{m-n+2}^{-1} \circ f_{m-n+3}^{-1} \circ \cdots \circ f_{m-1}^{-1}(x) \\ &\quad + F \circ f_{m-n+1}^{-1} \circ f_{m-n+2}^{-1} \circ \cdots \circ f_{m-1}^{-1}(x), \quad x \in [x_{m-1}, x_m], \quad \text{for } m \geq n. \end{aligned}$$

Further, differentiable solutions, convex solutions and decreasing solutions, and equivariant solutions of (1.1.9) are discussed in Zhang (1990); Xu and Zhang (2007b), and Zhang (2000), respectively.

1.2 ORGANIZATION OF THE THESIS

In this section, we give a more detailed outline of the contents of this thesis. The present work is focused mainly on investigating certain dynamical behaviours of various essential classes of continuous maps. Indeed, we study the dynamical systems generated by iteration operators on function spaces, and those by continuous maps on compact intervals. We also discuss continuous solutions for a class of iterative equations with multiplication. The proposed thesis consists of five chapters, which we have organized as follows.

This introductory chapter summarizes the context, motivation, and main contributions of this thesis. More precisely, to make the discussions self-contained, a concise introduction to the theory of discrete dynamical systems, the theory of iterative equations, and Milnor-Thurston's kneading theory are presented. Further, a detailed review of the literature related to the present work is conducted.

As known from Li and Yorke (1975), even for interval maps, the existence of some periodic points may result in complex dynamical behaviours, in which more concepts such as topological transitivity, sensitive dependence, and chaos are involved. In those dynamic behaviours, the operation of iteration plays an important role.

Let (K,d) be a compact metric space and $\mathscr{C}(K)$ consist of all continuous self-maps of K, which is also a complete metric space equipped with the supremum metric

$$\rho(f,g) := \sup\left\{d\big(f(x),g(x)\big) : x \in K\right\}. \tag{1.2.1}$$

For each $n \in \mathbb{N}$, the n^{th} order iteration defines the map $\mathscr{J}_n : \mathscr{C}(K) \to \mathscr{C}(K)$ such that

$$\mathscr{J}_n f = f^n. \tag{1.2.2}$$

The map $\mathscr{J}_n$, called the *iteration operator* of order n on $\mathscr{C}(K)$, is nonlinear. Many interesting and complicated properties of $\mathscr{J}_n$ were investigated by discussion of iterates (Blokh (1992); Milnor and Thurston (1988)), iterative roots (Babbage (1815); Bödewadt (1944); Fort (1955); Zhang (1997)) and solutions of iterative equations (Jarczyk (1996); Kuczma (1960)). It is proved in Zhang and Zhang (2011) that $\mathscr{J}_n$ is continuous on $\mathscr{C}([a,b])$ for any closed interval $[a,b]$ of $\mathbb{R}$, which proposes a further study on dynamics of $\mathscr{J}_n$.

In Chapter 2, we investigate some dynamical behaviours of $\mathscr{J}_n$. For an arbitrary compact metric space K, we generally prove the continuity of $\mathscr{J}_n$ on the space $\mathscr{C}(K)$ of continuous self-maps, implying that the operator $\mathscr{J}_n$ defines a discrete dynamical system on $\mathscr{C}(K)$. Then we show how a fixed point or periodic point of $\mathscr{J}_n$ is determined. In section 2.2 we characterize all fixed points and periodic points of the system in the case that K is a compact interval by discussing the Babbage equation. We do the same in the case that K is the unit circle S^1 in section 2.3. Further, we prove that every orbit of $\mathscr{J}_n$ is bounded and every fixed point of $\mathscr{J}_n$ which equals the identity on its range is not Lyapunov stable. The boundedness and instability exhibit a complex behaviour of $\mathscr{J}_n$, but we prove that $\mathscr{J}_n$ is not topologically transitive and therefore is not chaotic on $\mathscr{C}(K)$ in Devaney's sense (see Theorem 2.5.1).

Let $f \in \mathscr{C}(I)$ and $g \in \mathscr{C}(J)$, where $J = [c,d]$ is a compact interval in $\mathbb{R}$ such that $c < d$. As in Holmgren (1994), we say that f is *topologically h-conjugate* (or simply *conjugate*) to g if there exists a homeomorphism $h : I \to J$ such that $h \circ f = g \circ h$. In this case, h is called a *topological conjugacy*. It is proved in Milnor and Thurston (1988) that the kneading matrix and the kneading determinant associated with a continuous piecewise monotone map are invariant under orientation-preserving topological conjugacy. In Chapter 3, we consider if this result is valid for orientation-reversing conjugacy. More concretely, in section 3.1, we prove that the kneading matrix is no longer an invariant under orientation-reversing conjugacy while the kneading determinant is (see Theorem 3.1.7 and Corollary 3.1.9). Then, in section 3.2, we present two applications of our results- these to the reduction of computational complexity, and the nonexistence of topological conjugacy between continuous piecewise monotone maps.

As seen in section 1.1.2, the kneading matrix of an $f \in \mathscr{M}(I)$ with m turning points is an $m \times (m+1)$ matrix with entries from the ring of formal power series with integer coefficients. Moreover, the iterates of f satisfy the ascending relation

$$\#T(f) \leq \#T(f^2) \leq \#T(f^3) \leq \cdots. \tag{1.2.3}$$

Therefore the process of finding the kneading matrices of higher order iterates of f involves tedious computations. In section 3.4, we describe a relation between kneading matrices of maps and their iterates for tent-like maps, the family of chaotic maps each of which is onto on its every lap (see Theorem 3.4.2). We also define the modified kneading matrix for such maps and describe a relationship between the corresponding determinant and the usual kneading determinant.

As observed in section 1.1.3, plentiful results were obtained by various researchers on the solutions of (1.1.5) whenever Φ is a linear combination of iterates of f. However, it is also interesting to discuss Φ of nonlinear combinations, for instance, considered as in Zdun and Zhang (2007) on the unit circle.

In Chapter 4, we consider the iterative equation with multiplication

$$(g(x))^{\alpha_1}(g^2(x))^{\alpha_2}\cdots(g^n(x))^{\alpha_n} = G(x), \tag{1.2.4}$$

i.e., $\Phi(u_1, u_2, \ldots, u_n) = \prod_{k=1}^{n} u_k^{\alpha_k}$, where G is given and g is unknown. Unlike those Murugan and Subrahmanyam (2006); Si (1995); Wang and Si (2001); Si and Zhang (1998); Zhang (1988, 1989) on compact intervals, our work to (1.2.4) is concentrated in solving (1.1.9) on the whole $\mathbb{R}$. Our strategy is to restrict our discussion of (1.2.4) on $\mathbb{R}_+ := (0, +\infty)$ and use an exponential function to reduce in conjugation to the well-known form of polynomial-like iterative equation (1.1.9) on the whole $\mathbb{R}$ (see Proposition 4.1.1). Note that all found results on the solutions of (1.1.9) are given either on a compact interval or near a fixed point, none of which can be applied to our case. We generally discuss (1.1.9) on the whole $\mathbb{R}$ and use obtained result to give solutions of equation (1.2.4) on $\mathbb{R}_+$ and $\mathbb{R}_- := (-\infty, 0)$. Our approach here is twofold. First, using the Banach contraction principle, we give sufficient conditions for existence and uniqueness of continuous solutions for (1.2.4). We also prove that the obtained solution depends on G continuously. Then, using the second method we construct its solutions, sewing piece by piece as done in Xu and Zhang (2007a); Zhang et al. (2013).

The main focus of the present work is to study the dynamical behaviours of continuous maps. More precisely, we consider the discrete dynamical systems of iteration operators, the iterates of continuous piecewise monotone maps on intervals, and an iterative equation with multiplication. Chapter 5 presents the conclusions of this thesis describing the scope for future research in these areas.

CHAPTER 2

THE ITERATION OPERATOR $\mathscr{J}_n$

"Few ideas work on the first try. Iteration is key to innovation."
- Sebastian Thrun

In this chapter, we investigate some critical dynamical properties of the iteration operator $\mathscr{J}_n$, defined as in (1.2.2), on the space $\mathscr{C}(K)$ of continuous self-maps of a compact metric space K.

2.1 DYNAMICAL SYSTEM OF ITERATION

Let K be a compact metric space with metric d. In this section we prove $(\mathscr{C}(K), \mathscr{J}_n)$ to be a discrete dynamical system indeed by showing the continuity of $\mathscr{J}_n$ on $\mathscr{C}(K)$.

Theorem 2.1.1. *$\mathscr{J}_n$ is continuous on $\mathscr{C}(K)$ for each $n \in \mathbb{N}$.*

Proof. Since $\mathscr{J}_1$ is the identity operator on $\mathscr{C}(K)$, clearly it is continuous. In what follows, we assume that $n \geq 2$. Let $f \in \mathscr{C}(K)$ and $(f_k)_{k\in\mathbb{N}}$ be any sequence in $\mathscr{C}(K)$ converging to f. Then for each $x \in K$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} d\big(f_k^n(x), f^n(x)\big) &\leq d\big(f_k^n(x), (f^{n-1}\circ f_k)(x)\big) + d\big((f^{n-1}\circ f_k)(x), f^n(x)\big) \\ &\leq \rho(f_k^{n-1}, f^{n-1}) + \rho(f^{n-1}\circ f_k, f^n) \quad \text{(by using (1.2.1))} \\ &\leq \rho(f_k^{n-1}, f^{n-2}\circ f_k) + \rho(f^{n-2}\circ f_k, f^{n-1}) + \rho(f^{n-1}\circ f_k, f^n) \\ &\leq \rho(f_k^{n-2}, f^{n-2}) + \rho(f^{n-2}\circ f_k, f^{n-1}) + \rho(f^{n-1}\circ f_k, f^n). \end{aligned}$$

Proceeding as above, by induction we obtain

$$d\big(f_k^n(x), f^n(x)\big) \leq \rho(f_k, f) + \rho(f\circ f_k, f^2) + \cdots + \rho(f^{n-1}\circ f_k, f^n)$$

for every $x \in K$ and $k \in \mathbb{N}$. This implies

$$\rho(f_k^n, f^n) \le \rho(f_k, f) + \rho(f \circ f_k, f^2) + \cdots + \rho(f^{n-1} \circ f_k, f^n) \tag{2.1.1}$$

for every $k \in \mathbb{N}$.

Claim: $f^j \circ f_k \to f^{j+1}$ as $k \to \infty$ uniformly on K for each $j = 1, 2, \ldots, n-1$.

Let $1 \le j \le n-1$ and $\varepsilon > 0$. Since f^j is uniformly continuous on K, there exists $\delta > 0$ such that

$$d\big(f^j(x), f^j(y)\big) < \varepsilon \tag{2.1.2}$$

whenever $x, y \in K$ with $d(x,y) < \delta$. Since $(f_k)_{k\in\mathbb{N}}$ converges to f uniformly on K, there exists $n_0 \in \mathbb{N}$ such that

$$d\big(f_k(x), f(x)\big) < \delta, \ \forall k \ge n_0 \text{ and } \forall x \in K.$$

It follows from (2.1.2) that

$$d\big((f^j \circ f_k)(x), (f^j \circ f)(x)\big) < \varepsilon, \ \forall k \ge n_0 \text{ and } \forall x \in K.$$

This proves the claim.

Hence, by the claimed fact, we see from (2.1.1) that $\rho(f_k^n, f^n) \to 0$ as $k \to \infty$, i.e., $\mathscr{J}_n f_k \to \mathscr{J}_n f$ as $k \to \infty$, which shows that $\mathscr{J}_n$ is continuous on $\mathscr{C}(K)$. □

The most fundamental problems on dynamical systems are concerning fixed points and periodic points. Since $\mathscr{J}_n^k f = f^{n^k}$ by (1.2.2), we see that $f \in \mathscr{C}(K)$ is a fixed point of $\mathscr{J}_n$ if and only if f satisfies the functional equation

$$f^n = f, \tag{2.1.3}$$

and $f \in \mathscr{C}(K)$ is a k-periodic point of $\mathscr{J}_n$ if and only if f satisfies

$$f^{n^k} = f, \ f^{n^{k-1}} \ne f \quad \text{and} \quad f^{n^i} \ne f, \ \forall i = 2, ..., k-2 \text{ with } i \nmid (k-1). \tag{2.1.4}$$

So, the fixed points and periodic points of $\mathscr{J}_n$ are related to solutions of the Babbage equation (1.1.7). The functional equations in (2.1.3) and (2.1.4) cannot be simply treated as the Babbage equation (1.1.7) because the range of f may not be the whole K. In what follows, we need to consider restriction of maps. For any $f \in \mathscr{C}(K)$ and $A \subseteq K$, let $\mathscr{R}(f)$ denote the range of f and $f|_A$ be the restriction of f to A.

Lemma 2.1.2. *Let $n \in \mathbb{N}$. Then $f \in \mathscr{C}(K)$ is a solution of the equation*

$$\phi^n = \phi \tag{2.1.5}$$

on K if and only if f is a continuous extension of the solution $g|_{\mathscr{R}(g)}$ of the Babbage equation

$$\phi^{n-1} = \mathrm{id} \tag{2.1.6}$$

on $\mathscr{R}(g)$ to K for some $g \in \mathscr{C}(K)$ such that $\mathscr{R}(f) = \mathscr{R}(g)$.

Proof. If $f \in \mathscr{C}(K)$ is a solution of (2.1.5), then $f^{n-1}\big(f(x)\big) = f(x)$ for all $x \in K$, i.e., $f^{n-1}(y) = y$ for all $y \in \mathscr{R}(f)$. Therefore $f|_{\mathscr{R}(f)}$ is a solution of (2.1.6) on $\mathscr{R}(f)$. Let $g := f$. Then f is a continuous extension of the solution $g|_{\mathscr{R}(g)}$ on $\mathscr{R}(g)$ to K such that $\mathscr{R}(f) = \mathscr{R}(g)$.

Conversely, let $g \in \mathscr{C}(K)$ be such that $g|_{\mathscr{R}(g)}$ is a solution of (2.1.6) on $\mathscr{R}(g)$ and f be any continuous extension of $g|_{\mathscr{R}(g)}$ to K such that

$$\mathscr{R}(f) = \mathscr{R}(g). \tag{2.1.7}$$

Then $f \in \mathscr{C}(K)$ clearly and, since f is an extension of $g|_{\mathscr{R}(g)}$ to K, we have

$$f|_{\mathscr{R}(g)} = g|_{\mathscr{R}(g)}. \tag{2.1.8}$$

Therefore, for every $x \in K$ we have

$$f^n(x) = f^{n-1}\big(f(x)\big) = f^{n-1}|_{\mathscr{R}(f)}\big(f(x)\big) = \big(f|_{\mathscr{R}(f)}\big)^{n-1}\big(f(x)\big) = \big(g|_{\mathscr{R}(g)}\big)^{n-1}\big(f(x)\big)$$

by (2.1.7) and (2.1.8). As $g|_{\mathscr{R}(g)}$ satisfies (2.1.6) on $\mathscr{R}(g)$, using (2.1.7) we get

$$f^n(x) = \mathrm{id}\big(f(x)\big) = f(x), \quad \forall x \in K.$$

Hence f is a solution of (2.1.5) on K. □

Having Lemma 1.1.7, which describe all the m^{th} order unit iterative roots on a general set E, we are ready to define

$$\mathscr{U}_E^m := \{f \in \mathscr{C}(K) : f|_E \text{ is an } m^{\text{th}} \text{ order unit iterative root on } E \text{ and } \mathscr{R}(f) = E\}$$

for any subset E of K and $m \in \mathbb{N}$. Since $\mathrm{Fix}(\mathscr{J}_1;\mathscr{C}(K)) = \mathscr{C}(K)$, i.e., the problem of fixed points of $\mathscr{J}_1$ is trivial, we focus on $\mathscr{J}_n$ with $n \geq 2$. We give the following results on fixed points and periodic points.

Theorem 2.1.3. *Let $n, k \geq 2$ be integers. Then*

(i) *$f \in \mathscr{C}(K)$ is a fixed point of $\mathscr{J}_n$ if and only if $f \in \mathscr{U}_E^m$ for a compact subset E of K and an integer $m \geq 1$ dividing $n-1$ exactly.*

(ii) *$f \in \mathscr{C}(K)$ is a k-periodic point of $\mathscr{J}_n$ if and only if $f \in \mathscr{U}_E^m$ for a compact subset E of K and an integer $m > 1$ satisfying that*

$$m \mid (n^k - 1) \text{ and } m \nmid (n^j - 1) \text{ for } 1 \leq j \leq k-1. \tag{2.1.9}$$

Proof. If $f \in \mathscr{C}(K)$ is a fixed point of $\mathscr{J}_n$, then $f^n = f$ on K. By Lemma 2.1.2, there exists $g \in \mathscr{C}(K)$ such that $g|_{\mathscr{R}(g)}$ satisfies $\phi^{n-1} = \mathrm{id}$ on $\mathscr{R}(g)$ and f is a continuous extension of $g|_{\mathscr{R}(g)}$ to K with $\mathscr{R}(f) = \mathscr{R}(g)$. In fact, we take $g = f$. Then $E := \mathscr{R}(f)$ is a compact subset of K. Let m be the least positive integer such that $(f|_E)^m = \mathrm{id}$. Such an m exists since $(f|_E)^{n-1} = \mathrm{id}$, and moreover m divides $n-1$ by Lemma 1.1.6. Therefore, $f \in \mathscr{U}_E^m$.

Conversely, let $f \in \mathscr{U}_E^m$ for a compact subset E of K and an integer $m \geq 1$ dividing $n-1$. Then there exists $l \in \mathbb{N}$ such that $n-1 = ml$ and

$$(f|_{\mathscr{R}(f)})^{n-1} = (f|_E)^{n-1} = (f|_E)^{ml} = ((f|_E)^m)^l = \mathrm{id}^l = \mathrm{id}.$$

This implies that $f^n = f$ on K, and therefore f is a fixed point of $\mathscr{J}_n$. This proves result **(i)**.

In order to prove result **(ii)**, assume $n > 1$ since $\mathscr{J}_1$ does not have a k-periodic point for $k \geq 2$. Let $f \in \mathscr{C}(K)$ be a k-periodic point of $\mathscr{J}_n$. Then

$$f^{n^k} = f \text{ and } f^{n^j} \neq f \text{ for } 1 \leq j \leq k-1 \tag{2.1.10}$$

on K. Since f is a fixed point of $\mathscr{J}_{n^k}$, by result **(i)** there exist a compact subset E of K and an $m \in \mathbb{N}$ dividing $n^k - 1$ such that $f \in \mathscr{U}_E^m$. If $m = 1$, then $f|_{\mathscr{R}(f)} = \mathrm{id}$ so that $f^n = f$ on K, a contradiction to (2.1.10). So $m > 1$. If m divides $n^j - 1$ for some $1 \leq j \leq k-1$, then $(f|_{\mathscr{R}(f)})^{n^j-1} = \mathrm{id}$ so that $f^{n^j} = f$ on K, a contradiction to (2.1.10). So $m \nmid (n^j - 1)$ for $1 \leq j \leq k-1$. This proves the direct implication in result **(ii)**. For the converse, let $f \in \mathscr{U}_E^m$ for a compact subset E of K and an integer $m > 1$ satisfying (2.1.9). Choose $l \in \mathbb{N}$ such that $n^k - 1 = ml$. Then

$$(f|_{\mathscr{R}(f)})^{n^k-1} = (f|_E)^{n^k-1} = (f|_E)^{ml} = ((f|_E)^m)^l = \mathrm{id}^l = \mathrm{id},$$

implying that $f^{n^k} = f$. If $f^{n^j} = f$ for some $1 \le j \le k-1$, then $(f|_E)^{n^j-1} = (f|_{\mathscr{R}(f)})^{n^j-1}$ $=$ id, implying that m divides $n^j - 1$, a contradiction to (2.1.9). So $f^{n^j} \neq f$ for $1 \le j \le k-1$, and thus f is a k-periodic point of $\mathscr{J}_n$. $\square$

Example 2.1.4. Consider $f(x) = |x|$. Then $f \in \mathscr{C}([-1,1])$. Clearly, $f \in \mathscr{U}^1_{[0,1]}$. By result **(i)** of Theorem 2.1.3, f is a fixed point of $\mathscr{J}_n$ on $[-1,1]$ for each $n \ge 2$.

Example 2.1.5. For any $n \ge 2$ and $k \in \mathbb{N}$, consider the discrete metric space $F_{n,k} := \{1,2,\ldots,n^k-1\}$. Then the map $f : F_{n,k} \to F_{n,k}$ defined by

$$f(j) = \begin{cases} j+1 & \text{if} \quad j = 1,2,\ldots,n^k-2, \\ 1 & \text{if} \quad j = n^k - 1 \end{cases}$$

lies in $\mathscr{U}^{n^k-1}_{F_{n,k}}$. By result **(ii)** of Theorem 2.1.3, f is a k-periodic point of $\mathscr{J}_n$ in $\mathscr{C}(F_{n,k})$.

Example 2.1.6. The map $f : S^1 \to S^1$ defined by $f(e^{i\theta}) = e^{i(\theta+\frac{\pi}{2})}$ lies in $\mathscr{U}^4_{S^1}$. Therefore, by result **(ii)** of Theorem 2.1.3, f is a 2-periodic point of $\mathscr{J}_3$.

The above results are given for general K. If we consider K to be $K = [a,b]$ or $K = S^1$ concretely, we may obtain more details on fixed points and periodic points of $\mathscr{J}_n$, as seen in next two sections.

2.2 DISCUSSION ON $[a,b]$

For more detailed results, in this section we focus on the case $K = I := [a,b]$, a compact interval, and discuss fixed points and periodic points of iteration operators on $\mathscr{C}(I)$.

In what follows, we refer to monotonically increasing (or decreasing) functions satisfying (2.1.5) as *monotonically increasing* (or *decreasing*) *fixed point* of $\mathscr{J}_n$.

Theorem 2.2.1. *Every monotonic fixed point f of $\mathscr{J}_2$ in $\mathscr{C}(I)$ is of the first form in (2.2.1). Every monotonic fixed point f of $\mathscr{J}_3$ in $\mathscr{C}(I)$ is of one of the following two forms*

$$f(x) = \begin{cases} c & \text{if} \quad a \le x \le c, \\ x & \text{if} \quad c \le x \le d, \\ d & \text{if} \quad d \le x \le b, \end{cases} \qquad f(x) = \begin{cases} d & \text{if} \quad a \le x \le c, \\ g(x) & \text{if} \quad c \le x \le d, \\ c & \text{if} \quad d \le x \le b, \end{cases} \tag{2.2.1}$$

where $c,d \in I$ satisfy $c \le d$ and $g \in \mathscr{C}([c,d])$ is a decreasing involutory map.

Proof. Let f be a fixed point of $\mathscr{J}_3$. Since $f^3 = f$, by result **(i)** of Theorem 2.1.3, there exists a closed subinterval $J = [c,d]$ of I such that $f \in \mathscr{U}^m_J$ for a certain integer m

dividing 2. Then

$$(f|_J)^2 = \mathrm{id} \quad \text{and} \quad \mathscr{R}(f) = J. \tag{2.2.2}$$

Case (a): If f is monotonically increasing, then by Lemma 1.1.8 we see from the first result of (2.2.2) that $f|_J = \mathrm{id}$. Outside J we see from the second result of (2.2.2) that $f(x) = c$ for all $x \in [a,c]$ and $f(x) = d$ for all $x \in [d,b]$. Hence f is of the first form.
Case (b): If f is monotonically decreasing, then again by Lemma 1.1.8 we see from the first result of (2.2.2) that $f|_J$ is a decreasing involutory function g on J. Outside J we see from the second result of (2.2.2) that $f(x) = d$ for all $x \in [a,c]$ and $f(x) = c$ for all $x \in [d,b]$. Therefore f is of the second form. □

The above theorem is only applicable to monotonic fixed points. There exist fixed points of $\mathscr{J}_2$ or $\mathscr{J}_3$ which are not monotonic. For example, as considered in Example 2.1.4, the function $f(x) = |x|$ on $[-1,1]$ is not a monotonic map on $[-1,1]$ but a fixed point of $\mathscr{J}_2$.

The 'monotone' in Theorem 2.2.1 need not mean 'strict monotone'. For instance, it may happen that $[c,d] \subsetneqq I$ with $a < c$ in the first form of (2.2.1). In that case, we have $f(x) = c$ for $x \in [a,c]$, implying that f is not strictly monotone. For a specific example, consider the map f on $[-1,1]$ defined by $f(x) = 0$ if $x \in [-1,0]$ and $f(x) = x$ if $x \in [0,1]$. One can check that f is a fixed point of $\mathscr{J}_2$. Moreover, f is monotonic but not strictly monotonic. For strictly monotonic ones, we have the following.

Corollary 2.2.2. *If f is a strictly monotonic fixed point of $\mathscr{J}_3$ in $\mathscr{C}(I)$, then either $f = \mathrm{id}$ or f is a strictly decreasing involutory function on I.*

Proof. Since f is monotonic fixed point of $\mathscr{J}_3$, it follows that f is either of the forms given in Theorem 2.2.1. In any case $c = a$ and $d = b$, as f is strictly monotone on I. □

Theorem 2.2.1 gives results only for $\mathscr{J}_2$ and $\mathscr{J}_3$, but not for the generic $\mathscr{J}_n$. In what follows, we show that those monotonic fixed points of $\mathscr{J}_2$ and $\mathscr{J}_3$ are important representatives for the generic $\mathscr{J}_n$. Let $\mathscr{C}_{\mathrm{id}}(I)$ and $\mathscr{C}_{\mathrm{inv}}(I)$ consist of all continuous self-maps of I which are the identity and decreasing involutions on their range, respectively. By Theorem 2.2.1, monotonic fixed points of $\mathscr{J}_3$ are in both classes $\mathscr{C}_{\mathrm{id}}(I)$ and $\mathscr{C}_{\mathrm{inv}}(I)$ but monotonic fixed points of $\mathscr{J}_2$ are all in the same class $\mathscr{C}_{\mathrm{id}}(I)$. The following results (**i**) and (**ii**) of Theorem 2.2.3 together describe all fixed points of $\mathscr{J}_n$ for any $n \geq 2$.

Theorem 2.2.3. *The following statements are true for system $\big(\mathscr{C}(I), \mathscr{J}_n\big)$:*

(i) $\mathrm{Fix}(\mathscr{J}_m; \mathscr{C}(I)) = \mathrm{Fix}(\mathscr{J}_n; \mathscr{C}(I))$ *if integers $m,n \geq 2$ satisfy $m \equiv n (\mathrm{mod}\ 2)$.*

(ii) $\mathrm{Fix}(\mathscr{J}_2;\mathscr{C}(I)) \subsetneq \mathrm{Fix}(\mathscr{J}_3;\mathscr{C}(I))$. *More precisely,* $\mathrm{Fix}(\mathscr{J}_2;\mathscr{C}(I)) = \mathscr{C}_{\mathrm{id}}(I)$ *and* $\mathrm{Fix}(\mathscr{J}_3;\mathscr{C}(I)) = \mathscr{C}_{\mathrm{id}}(I) \cup \mathscr{C}_{\mathrm{inv}}(I)$.

Proof. For result (**i**), it suffices to show that for $n \geq 2$

$$\mathrm{Fix}(\mathscr{J}_n;\mathscr{C}(I)) = \begin{cases} \mathrm{Fix}(\mathscr{J}_2;\mathscr{C}(I)) & \text{if} \quad n \equiv 0 (\mathrm{mod}\ 2), \\ \mathrm{Fix}(\mathscr{J}_3;\mathscr{C}(I)) & \text{if} \quad n \equiv 1 (\mathrm{mod}\ 2). \end{cases}$$

First, consider the case that $n \geq 2$ such that $n \equiv 0(\mathrm{mod}\ 2)$. Clearly, $\mathrm{Fix}(\mathscr{J}_2;\mathscr{C}(I)) \subseteq \mathrm{Fix}(\mathscr{J}_n;\mathscr{C}(I))$. For the reverse inclusion, consider an arbitrary $f \in \mathrm{Fix}(\mathscr{J}_n;\mathscr{C}(I))$. Since $f^{n-1} = \mathrm{id}$ on $\mathscr{R}(f)$ and $n-1$ is odd, by Lemma 1.1.8, $f|_{\mathscr{R}(f)} = \mathrm{id}$. Therefore $f^2 = f$ on I, and hence $\mathrm{Fix}(\mathscr{J}_n;\mathscr{C}(I)) \subseteq \mathrm{Fix}(\mathscr{J}_2;\mathscr{C}(I))$.

Next, consider the case that $n > 2$ such that $n \equiv 1(\mathrm{mod}\ 2)$, i.e., $n = 2l+1$ for some $l \in \mathbb{N}$. Consider an arbitrary $f \in \mathrm{Fix}(\mathscr{J}_3;\mathscr{C}(I))$. Since $f^2 = \mathrm{id}$ on $\mathscr{R}(f)$, we have

$$f^n(x) = f^{2l+1}(x) = f^{2l}(f(x)) = \mathrm{id}(f(x)) = f(x), \quad \forall x \in I,$$

i.e., $f \in \mathrm{Fix}(\mathscr{J}_n;\mathscr{C}(I))$. Therefore $\mathrm{Fix}(\mathscr{J}_3;\mathscr{C}(I)) \subseteq \mathrm{Fix}(\mathscr{J}_n;\mathscr{C}(I))$. For the reverse inclusion, consider any $f \in \mathrm{Fix}(\mathscr{J}_n;\mathscr{C}(I))$. Then $f^{2l} = \mathrm{id}$ on $\mathscr{R}(f)$ and therefore by Lemma 1.1.8, f is either identity or a decreasing involutory map on $\mathscr{R}(f)$. This implies $f^3 = f$ on I, and therefore $f \in \mathrm{Fix}(\mathscr{J}_3;\mathscr{C}(I))$. This proves result (**i**).

If $f \in \mathrm{Fix}(\mathscr{J}_3;\mathscr{C}(I))$, then $f^2 = \mathrm{id}$ on $\mathscr{R}(f)$ and therefore by Lemma 1.1.8, either $f|_{\mathscr{R}(f)}$ is the identity map or a decreasing involution. This implies $f \in \mathscr{C}_{\mathrm{id}}(I) \cup \mathscr{C}_{\mathrm{inv}}(I)$. The reverse inclusion also follows, since $f^3 = f$ whenever $f|_{\mathscr{R}(f)}$ is the identity map or a decreasing involution. Therefore $\mathrm{Fix}(\mathscr{J}_3;\mathscr{C}(I)) = \mathscr{C}_{\mathrm{id}}(I) \cup \mathscr{C}_{\mathrm{inv}}(I)$.

If $f \in \mathrm{Fix}(\mathscr{J}_2;\mathscr{C}(I))$, then clearly $f^3 = f$ and $f|_{\mathscr{R}(f)} = \mathrm{id}$, implying that $f \in \mathscr{C}_{\mathrm{id}}(I)$. On the other hand, if $f \in \mathscr{C}_{\mathrm{id}}(I)$, then $f|_{\mathscr{R}(f)} = \mathrm{id}$, implying that $f^2 = f$. Hence $\mathrm{Fix}(\mathscr{J}_2;\mathscr{C}(I)) = \mathscr{C}_{\mathrm{id}}(I)$. This proves the second result of (**ii**).

The first result of (**ii**) follows from second and noting that the map $f : I \to I$ defined by $f(x) = a+b-x, \forall x \in I$ is a continuous decreasing involutory map. □

Theorem 2.2.1 together with result (**i**) of Theorem 2.2.3 shows that Theorem 2.2.1 is indeed true for every integer $n \geq 2$. More precisely, monotonic fixed points of $\mathscr{J}_n$ coincide with those of $\mathscr{J}_2$ and $\mathscr{J}_3$ accordingly as n is even and odd, respectively. So Theorem 2.2.1 actually gives results for the representatives.

Although we can find many fixed points of $\mathscr{J}_n$, the following result shows that there are no nontrivial periodic points.

Theorem 2.2.4. *For each* $n \in \mathbb{N}$, $\mathscr{J}_n$ *does not have periodic points of period* $k \geq 2$ *in* $\mathscr{C}(I)$.

Proof. Since every element of $\mathscr{C}(I)$ is a fixed point of $\mathscr{J}_1$, we see that $\mathscr{J}_1$ has no periodic point of period k for $k \geq 2$. Let $n \geq 2$ and suppose that $\mathscr{J}_n$ has a periodic point f of period $k \geq 2$ in $\mathscr{C}(I)$. Then f satisfy (2.1.10) on I. Since $k \geq 2$, clearly f is a non-constant map on I. Since $f^{n^k} = f$ on I, we have $f^{n^k-1} = \text{id}$ on $\mathscr{R}(f)$, and therefore by Lemma 1.1.8, either $f|_{\mathscr{R}(f)} = \text{id}$ or $f|_{\mathscr{R}(f)}$ is a decreasing involutory function. We discuss in the two cases.

Case (a): If $f|_{\mathscr{R}(f)} = \text{id}$, then we have

$$f^2(x) = f\big(f(x)\big) = (f|_{\mathscr{R}(f)})\big(f(x)\big) = \text{id}\big(f(x)\big) = f(x),$$

for all $x \in I$ so that $f^2 = f$ and hence $f^{n^{k-1}} = f$ on I, a contradiction to (2.1.10), as $n,k \geq 2$.

Case (b): If $f|_{\mathscr{R}(f)}$ is a decreasing involutory function, then

$$f^j = \begin{cases} f & \text{if} \quad j = 1,3,5,\ldots, \\ f^2 & \text{if} \quad j = 2,4,6,\ldots \end{cases} \tag{2.2.3}$$

on I. If n is odd, then n^j is odd for every $j \in \mathbb{N}$. By (2.2.3), $f^{n^j} = f$ on I for all $j \in \mathbb{N}$, implying in particular that $f^{n^{k-1}} = f$ on I, a contradiction to (2.1.10), since $n,k \geq 2$. Therefore n and hence n^k is even. By (2.2.3), we have $f^{n^k} = f^2$ on I. This implies $f^2 = f$ on I, and therefore $\text{id} = \big(f|_{\mathscr{R}(f)}\big)^2 = f^2|_{\mathscr{R}(f)} = f|_{\mathscr{R}(f)}$, a contradiction, since $f|_{\mathscr{R}(f)}$ is a non-constant decreasing map on $\mathscr{R}(f)$.

Thus $\mathscr{J}_n$ has no periodic point of period k for $k \geq 2$. □

As observed before, every element of $\mathscr{C}(I)$ is a fixed point for $\mathscr{J}_1$. So in particular, $\mathscr{J}_1$ has a dense set of periodic points in $\mathscr{C}(I)$. However, this is not true for $n > 1$, which can be viewed as a consequence of the following.

Theorem 2.2.5. $\text{Fix}(\mathscr{J}_3;\mathscr{C}(\text{I}))$ *is not dense in* $\mathscr{C}(I)$.

Proof. Let $f : I \to I$ be defined by

$$f(x) = a + \frac{(x-a)^2}{b-a}, \ \forall x \in I$$

and let $\varepsilon = \frac{3(b-a)}{16}$.

Claim: $B_\rho(f,\varepsilon) \cap \text{Fix}(\mathscr{J}_3;\mathscr{C}(I)) = \emptyset$.

For an indirect proof, assume that there exists $g \in B_\rho(f,\varepsilon)$ such that $\mathscr{J}_3(g) = g$. Then $\rho(g,f) < \varepsilon$, and $(g|_{\mathscr{R}(g)})^2 = \text{id}$. By Lemma 1.1.8, $g|_{\mathscr{R}(g)}$ is either the identity map or a decreasing involution. We discuss in the two cases.

Case (a): If $g|_{\mathscr{R}(g)} = \mathrm{id}$, then

$$\begin{aligned} 0 \le x - \left(a + \frac{(x-a)^2}{b-a}\right) &= |\mathrm{id}(x) - f(x)| \\ &= |g|_{\mathscr{R}(g)}(x) - f(x)| \le \rho(g,f) < \varepsilon, \quad \forall x \in \mathscr{R}(g), \end{aligned}$$

which implies that $\mathscr{R}(g) \subseteq [a, \frac{3a+b}{4}] \cup [\frac{a+3b}{4}, b]$ and therefore either $\mathscr{R}(g) \subseteq [a, \frac{3a+b}{4}]$ or $\mathscr{R}(g) \subseteq [\frac{a+3b}{4}, b]$ because $\mathscr{R}(g)$ is connected. This is a contradiction; otherwise, the inclusion $\mathscr{R}(g) \subseteq [a, \frac{3a+b}{4}]$ implies that

$$f(b) - g(b) \ge b - \frac{3a+b}{4} = \frac{3(b-a)}{4} > \frac{3(b-a)}{16} = \varepsilon,$$

and the inclusion $\mathscr{R}(g) \subseteq [\frac{a+3b}{4}, b]$ implies that

$$g(a) - f(a) \ge \frac{a+3b}{4} - a = \frac{3(b-a)}{4} > \frac{3(b-a)}{16} = \varepsilon.$$

Case (b): If $g|_{\mathscr{R}(g)}$ is a decreasing involutory map, then $\mathscr{R}(g) = [c,d]$ for some $c,d \in I$ such that $g(c) = d$ and $g(d) = c$. We have

$$\left| a + \frac{(c-a)^2}{b-a} - d \right| = |f(c) - g(c)| < \varepsilon,$$

implying that

$$a + \frac{(c-a)^2}{b-a} - \varepsilon < d < a + \frac{(c-a)^2}{b-a} + \varepsilon. \tag{2.2.4}$$

Also

$$\left| a + \frac{(d-a)^2}{b-a} - c \right| = |f(d) - g(d)| < \varepsilon,$$

implying that

$$a + \frac{(d-a)^2}{b-a} - \varepsilon < c < a + \frac{(d-a)^2}{b-a} + \varepsilon. \tag{2.2.5}$$

It follows from (2.2.4) and (2.2.5) that

$$d - c < \left(a + \frac{(c-a)^2}{b-a} + \varepsilon\right) - \left(a + \frac{(d-a)^2}{b-a} - \varepsilon\right) = \frac{(c+d-2a)(c-d)}{b-a} + 2\varepsilon,$$

which can be simplified as

$$d-c<2\varepsilon\frac{b-a}{(b-a)+(c+d-2a)}<2\varepsilon.$$

It follows that the length of the interval $\mathscr{R}(g)$ is less than 2ε. Thus, either $c-a\geq\varepsilon$ or $b-d\geq\varepsilon$; otherwise, $c-a<\varepsilon$ and $b-d<\varepsilon$, implying that

$$b-a=(b-d)+(d-c)+(c-a)<4\varepsilon=\frac{3(b-a)}{4}<b-a,$$

a contradiction. This implies that either $g(a)-f(a)\geq\varepsilon$ or $f(b)-g(b)\geq\varepsilon$, a contradiction to our assumption that $\rho(f,g)<\varepsilon$.
Therefore the claim is proved and the result follows. □

Theorems 2.2.4, 2.2.5 and 2.2.3 together show that $\mathscr{J}_n$ does not have a dense set of periodic points in $\mathscr{C}(I)$ for $n\geq 2$.

2.3 DISCUSSION ON S^1

The following two theorems characterize fixed points and periodic points of $\mathscr{J}_n$ in $\mathscr{C}(S^1)$, respectively.

Theorem 2.3.1. *Let $n\in\mathbb{N}$. Then $f\in\mathscr{C}(S^1)$ is a fixed point of $\mathscr{J}_n$ if and only if one of the following conditions is satisfied:* **(i)** *$f|_{\mathscr{R}(f)}$ is the identity map,* **(ii)** *$f|_{\mathscr{R}(f)}$ is an orientation-reversing involution, or* **(iii)** *f is of the form* (1.1.8) *for a divisor m of $n-1$.*

Proof. Let $n\in\mathbb{N}$. To find all fixed points of $\mathscr{J}_n$ in $\mathscr{C}(S^1)$, in view of Lemma 2.1.2, it suffices to find all $f\in\mathscr{C}(S^1)$ satisfying the equation $\phi^{n-1}=\mathrm{id}$ on $\mathscr{R}(f)$. So let $f\in\mathscr{C}(S^1)$ be such that $f^{n-1}=\mathrm{id}$ on $\mathscr{R}(f)$. We discuss in the two cases that $\mathscr{R}(f)=S^1$ and $\mathscr{R}(f)\subsetneq S^1$ separately.
Case (a): Suppose that $\mathscr{R}(f)=S^1$. If f has a fixed point, then by Lemma 1.1.9, f is either the identity map or an involution according as f is orientation-preserving or orientation-reversing. If f has no fixed points, then by Lemma 1.1.10, f is of the form (1.1.8) for some divisor m of $n-1$.
Case (b): Suppose that $\mathscr{R}(f)\subsetneq S^1$. Then $\mathscr{R}(f)$ is either a singleton set or an arc in S^1, because S^1 is connected and compact. If $\mathscr{R}(f)$ is a singleton set, then f is a constant map on S^1. If $\mathscr{R}(f)$ is an arc, say $[z_1,z_2]$ for some $z_1,z_2\in S^1$, then consider a homeomorphism $h_f:\mathscr{R}(f)\to[t_1,t_2]$, where t_1,t_2 are unique reals satisfying the conditions $z_1=e^{2\pi i t_1}, z_2=e^{2\pi i t_2}$ and $0\leq t_1<t_2<t_1+1<2$. Define a map

$H_f : \mathscr{C}(\mathscr{R}(f)) \to \mathscr{C}([t_1,t_2])$ by $H_f(g) := h_f \circ g \circ h_f^{-1}$ for all $g \in \mathscr{C}(\mathscr{R}(f))$. Then H_f is a bijective, bi-continuous map such that $H_f \circ \mathscr{J}_n = \mathscr{J}_n \circ H_f$ (i.e., $(\mathscr{C}(\mathscr{R}(f)), \mathscr{J}_n)$ is topologically H_f-conjugate to $(\mathscr{C}([t_1,t_2]), \mathscr{J}_n)$). Now since $f^{n-1} = \mathrm{id}$ on $\mathscr{R}(f)$, we have $H_f(f^{n-1}) = H_f(\mathrm{id}) = \mathrm{id}$ on $[t_1,t_2]$. i.e., $(h_f \circ f \circ h_f^{-1})^{n-1} = \mathrm{id}$ on $[t_1,t_2]$. Therefore by Lemma 1.1.8, $h_f \circ f \circ h_f^{-1}$ is either the identity map or a decreasing involutory map on $[t_1,t_2]$. This implies that $f|_{\mathscr{R}(f)}$ is either identity map or an orientation-reversing involutory map.

Conversely, if $f \in \mathscr{C}(S^1)$ satisfies either of the conditions (**i**) and (**ii**), then $f^n = f$ on S^1 implying that f is a fixed point of $\mathscr{J}_n$. If f satisfies (**iii**), then by Lemma 1.1.10 we have $f^m = \mathrm{id}$ on S^1, and therefore $f^{n-1} = \mathrm{id}$ on S^1 as m divides $n-1$. Therefore f is a fixed point of $\mathscr{J}_n$ □

Theorem 2.3.2. *Let $n,k \geq 2$. Then $f \in \mathscr{C}(S^1)$ is a k-periodic point of $\mathscr{J}_n$ if and only if f is of the form* (1.1.8) *for some $m > 1$ such that $m \mid n^k - 1$ and $m \nmid n^j - 1$ for $1 \leq j \leq k-1$.*

Proof. Let $f \in \mathscr{C}(S^1)$ be a k-periodic point of $\mathscr{J}_n$. Then f satisfies (2.1.10) on S^1 and also by result **(ii)** of Theorem 2.1.3, $f \in \mathscr{U}_E^m$ for some compact subset E of S^1 and $m > 1$ satisfying (2.1.9). In fact, here $E = \mathscr{R}(f)$. We assert that $E = S^1$. Note that E, being the image of connected and compact set S^1 under f, is either a singleton set, an arc or S^1. If E is a singleton, then f is a constant map on S^1, and therefore $f^n = f$, a contradiction to (2.1.10). If E is an arc, then $f|_E$ is either the identity map or an orientation-reversing involution. In any case, we arrive at a contradiction to (2.1.10). Therefore $E = S^1$ so that $f \in \mathscr{U}_{S^1}^m$. This implies by Lemma 1.1.10 that, f is of the form (1.1.8).

Conversely, if f is of the form (1.1.8) for some $m > 1$ such that $m \mid n^k - 1$ and $m \nmid n^j - 1$ for $1 \leq j \leq k-1$, then clearly $f \in \mathscr{U}_{S^1}^m$ with $m > 1$ satisfying (2.1.9) so that by result **(ii)** of Theorem 2.1.3, f is a k-periodic point of $\mathscr{J}_n$. □

2.4 STABILITY IN $\mathscr{J}_n$

In this section we study stability of fixed points in the dynamical system of iteration $\mathscr{J}_n$. We focus on the classes $\mathscr{C}_{\mathrm{id}}(I)$ and $\mathscr{C}_{\mathrm{id}}(S^1)$, but leave the stability problem on $\mathscr{C}_{\mathrm{inv}}(I)$, $\mathscr{C}_{\mathrm{inv}}(S^1)$ and periodic points open in Chapter 5.

Theorem 2.4.1. *All orbits of $\mathscr{J}_n$ are bounded.*

Proof. Since K is assumed to be a compact metric space in the beginning of section 2.1, it is bounded. So there exist $y \in K$ and $M > 0$ such that $d(x,y) \leq \frac{M}{2}$, implying that $d(f^{n^k}(x), f^{n^l}(x)) \leq d(f^{n^k}(x), y) + d(y, f^{n^l}(x)) \leq M$ for all $x \in K$ and $k,l \in \mathbb{N}$. Thus $\rho(f^{n^k}, f^{n^l}) \leq M$ for all $k,l \in \mathbb{N}$, implying that the orbit of f under $\mathscr{J}_n$ is bounded. □

As defined in Milnor (2006), the orbit $\{f^n(x)\}$ of a discrete dynamical system (X, f), where X is a metric space equipped with the metric d, is said to be (*Lyapunov*) *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(f^k(x), f^k(y)) < \varepsilon, \forall k \in \mathbb{N} \text{ whenever } y \in X \text{ satisfies } d(x,y) < \delta.$$

The following two theorems prove that most fixed points of $\mathscr{J}_n$ are not stable.

Theorem 2.4.2. *Let $n \in \mathbb{N}$ and $f \in \mathscr{C}_{\mathrm{id}}(I)$. Then f is stable for $\mathscr{J}_n$ if and only if f is a constant map on I.*

Proof. Let $n \in \mathbb{N}$ and $f \in \mathscr{C}_{\mathrm{id}}(I)$ be a constant map on I. For given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then for every $k \in \mathbb{N}$ and $g \in \mathscr{C}(I)$ with $\rho(f,g) < \delta$, we have

$$|f(x) - g^k(x)| = |f(g^{k-1}(x)) - g(g^{k-1}(x))| \leq \rho(f,g) < \varepsilon, \forall x \in I,$$

implying $\rho(f, g^k) < \varepsilon$, and hence in particular $\rho(f^{n^k}, g^{n^k}) < \varepsilon$. Therefore f is stable.

Conversely, suppose that $f \in \mathscr{C}_{\mathrm{id}}(I)$ is a non-constant map on I. Then there exist $c, d \in I$ with $c < d$ such that $\mathscr{R}(f) = [c,d]$ and $f|_{[c,d]} = \mathrm{id}$. For each $\eta > 0$, let $g_\eta : I \to I$ be the map defined by

$$g_\eta(x) = \begin{cases} f(x) & \text{if } x \in [a,c] \cup [d,b], \\ c + (x-c)(1-\eta) & \text{if } x \in [c, \frac{c+d}{2}], \\ x(1+\eta) - d\eta & \text{if } x \in [\frac{c+d}{2}, d]. \end{cases}$$

Then $g_\eta \in \mathscr{C}(I)$ for each $\eta > 0$. Let $\varepsilon = \frac{d-c}{8}$ and for any $\delta > 0$, choose $\eta_\delta > 0$ such that $\eta_\delta < \min\{\frac{\delta_0}{d-c}, 1\}$ for some $0 < \delta_0 < \delta$.

Claim: $\rho(f, g_{\eta_\delta}) < \delta$ and $\rho(f, g_{\eta_\delta}^{n^{k_0}}) \geq \varepsilon$ for some $k_0 \in \mathbb{N}$.

Consider any $x \in I$. If $x \in [a,c] \cup [d,b]$, then $|f(x) - g_{\eta_\delta}(x)| = |f(x) - f(x)| = 0 < \delta_0$. If $x \in (c, \frac{c+d}{2})$, then

$$|f(x) - g_{\eta_\delta}(x)| = |x - (c + (x-c)(1-\eta_\delta))| = (x-c)\eta_\delta < (x-c)\frac{\delta}{d-c} < \delta_0.$$

If $x \in [\frac{c+d}{2}, d)$, then

$$|f(x) - g_{\eta_\delta}(x)| = |x - (x(1+\eta_\delta) - d\eta_\delta)| = (d-x)\eta_\delta < (d-x)\frac{\delta}{d-c} < \delta_0.$$

Therefore $|f(x) - g_{\eta_\delta}(x)| < \delta_0, \forall x \in I$ and hence $\rho(f, g_{\eta_\delta}) \leq \delta_0 < \delta$.

Now for any $x \in [c, \frac{c+d}{2}]$, we have

$$g_{\eta_\delta}(x) = x(1-\eta_\delta) + c\eta_\delta \geq c(1-\eta_\delta) + c\eta_\delta = c,$$
$$g_{\eta_\delta}(x) \leq \frac{c+d}{2}(1-\eta_\delta) + c\eta_\delta = \frac{c+d}{2} - \frac{d-c}{2}\eta_\delta < \frac{c+d}{2},$$

implying that $g_{\eta_\delta}(x) \in [c, \frac{c+d}{2}]$. Therefore $g_{\eta_\delta}([c, \frac{c+d}{2}]) \subseteq [c, \frac{c+d}{2}]$. By induction,

$$g^k_{\eta_\delta}(x) = c + (x-c)(1-\eta_\delta)^k \tag{2.4.1}$$

for every $x \in [c, \frac{c+d}{2}]$ and $k \in \mathbb{N}$. Let $y = \frac{c+d}{2}$. Since $\eta_\delta \in (0,1)$, (2.4.1) implies that $g^k_{\eta_\delta}(y) \to c$ as $k \to \infty$. So there exists $N \in \mathbb{N}$ such that $|g^k_{\eta_\delta}(y) - c| < \varepsilon, \forall k \geq N$. Choose $k_0 \in \mathbb{N}$ so large that $n^{k_0} > N$. Then $g^{n^{k_0}}_{\eta_\delta}(y) - c < \frac{d-c}{8}$, i.e., $-g^{n^{k_0}}_{\eta_\delta}(y) > -\frac{7c+d}{8}$. Thus

$$f^{n^{k_0}}(y) - g^{n^{k_0}}_{\eta_\delta}(y) = (1-(1-\eta_\delta)^{n^{k_0}})(d-c)/2 > 0,$$

and therefore

$$|f^{n^{k_0}}(y) - g^{n^{k_0}}_{\eta_\delta}(y)| = f^{n^{k_0}}(y) - g^{n^{k_0}}_{\eta_\delta}(y) > \frac{c+d}{2} - \frac{7c+d}{8} = \frac{3(d-c)}{8} > \frac{d-c}{8} = \varepsilon,$$

implying that

$$\rho(f, g^{n^{k_0}}_{\eta_\delta}) = \rho(f^{n^{k_0}}, g^{n^{k_0}}_{\eta_\delta}) \geq |f^{n^{k_0}}(y) - g^{n^{k_0}}_{\eta_\delta}(y)| > \varepsilon.$$

This proves the claim, and therefore f is not stable, a contradiction. Hence f is a constant map on I. □

Theorem 2.4.3. *Let $n \in \mathbb{N}$ and $f \in \mathscr{C}_{\text{id}}(S^1)$, where $\mathscr{C}_{\text{id}}(S^1)$ consists of all continuous self-maps of S^1 which are the identity on their range. Then f is stable for $\mathscr{J}_n$ if and only if f is a constant map on S^1.*

Proof. The proof of "only if" part is similar to that of Theorem 2.4.2. We prove "if" part by the method of contradiction. Let $n \in \mathbb{N}$ and suppose that $f \in \mathscr{C}_{\text{id}}(S^1)$ is a non-constant map on S^1. Then $f|_{\mathscr{R}(f)} = \text{id}$ such that either $\mathscr{R}(f) = S^1$ or $\mathscr{R}(f) = [z_1, z_2]$ for some $z_1 = e^{it_1}, z_2 = e^{it_2} \in S^1$ with $0 \leq t_1 < t_2 < 2\pi$. For each $\eta > 0$, let $g_\eta : S^1 \to S^1$ be the map defined by

$$g_\eta(e^{it}) = \begin{cases} f(e^{it}) & \text{if } t \in [0, 2\pi) \setminus [t_1, t_2], \\ e^{i[t_1 + (t-t_1)(1-\eta)]} & \text{if } t \in [t_1, \frac{t_1+t_2}{2}], \\ e^{i[t(1+\eta) - t_2\eta]} & \text{if } t \in [\frac{t_1+t_2}{2}, t_2]. \end{cases}$$

Then $g_\eta \in \mathscr{C}(S^1)$ for each $\eta > 0$. Let $\varepsilon = \frac{t_2-t_1}{2\sqrt{2}\pi}$ and for any $\delta > 0$, choose $\eta_\delta > 0$ such that $\eta_\delta < \min\{\frac{2\delta_0}{t_2-t_1}, 1\}$ for some $0 < \delta_0 < \delta$.

Claim: $\rho(f, g_{\eta_\delta}) < \delta$ and $\rho(f, g_{\eta_\delta}^{n_{k_0}}) \geq \varepsilon$ for some $k_0 \in \mathbb{N}$.

Consider any $t \in [0, 2\pi)$. If $t \in [0, 2\pi) \setminus [t_1, t_2]$, then

$$|f(e^{it}) - g_{\eta_\delta}(e^{it})| = |f(e^{it}) - f(e^{it})| = 0 < \delta_0.$$

If $t \in (t_1, \frac{t_1+t_2}{2})$, then

$$\begin{aligned} |f(e^{it}) - g_{\eta_\delta}(e^{it})| &= |e^{it} - e^{i[t_1+(t-t_1)(1-\eta_\delta)]}| \\ &= |1 - e^{i\eta_\delta(t_1-t)}| \\ &= 2\left|\sin\left(\frac{\eta_\delta(t_1-t)}{2}\right)\right| \\ &\leq |\eta_\delta(t_1-t)| = (t-t_1)\eta_\delta < \frac{t_2-t_1}{2}\frac{2\delta_0}{t_2-t_1} = \delta_0. \end{aligned}$$

If $t \in [\frac{t_1+t_2}{2}, t_2)$, then by a similar argument, we have $|f(e^{it}) - g_{\eta_\delta}(e^{it})| < \delta_0$. Therefore

$$|f(e^{it}) - g_{\eta_\delta}(e^{it})| < \delta_0, \quad \forall t \in [0, 2\pi)$$

and hence $\rho(f, g_{\eta_\delta}) \leq \delta_0 < \delta$. Now, for any $t \in [t_1, \frac{t_1+t_2}{2}]$, we have

$$\begin{aligned} t_1 = t_1(1-\eta_\delta) + t_1\eta_\delta &\leq t(1-\eta_\delta) + t_1\eta_\delta \\ &= t_1 + (t-t_1)(1-\eta_\delta) \\ &\leq \frac{t_1+t_2}{2}(1-\eta_\delta) + t_1\eta_\delta \\ &= \frac{t_1+t_2}{2} - \frac{t_2-t_1}{2}\eta_\delta < \frac{t_1+t_2}{2}, \end{aligned}$$

implying that $g_{\eta_\delta}(e^{it}) \in [z_1, w]$, where $w := e^{i\frac{t_1+t_2}{2}}$. Therefore $g_{\eta_\delta}([z_1, w]) \subseteq [z_1, w]$. Hence it can be shown by induction that

$$g_{\eta_\delta}^k(e^{it}) = e^{i[t_1+(t-t_1)(1-\eta_\delta)^k]},$$

for every $t \in [t_1, \frac{t_1+t_2}{2}]$ and $k \in \mathbb{N}$. Also

$$\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi} \quad \forall t \in [0, \pi],$$

and therefore

$$|e^{it} - 1| = \sqrt{2}\sin\left(\frac{t}{2}\right) \geq \frac{\sqrt{2}t}{\pi}, \forall t \in [0, \pi]. \tag{2.4.2}$$

Now for each $k \in \mathbb{N}$, we have

$$\begin{aligned} |f(w) - g_{\eta_\delta}^k(w)| &= |e^{i\frac{t_1+t_2}{2}} - e^{i[t_1 + (\frac{t_1+t_2}{2} - t_1)(1-\eta_\delta)^k]}| \\ &= |1 - e^{-i[\frac{t_2-t_1}{2}(1-(1-\eta_\delta)^k)]}| \\ &= |e^{i[\frac{t_2-t_1}{2}(1-(1-\eta_\delta)^k)]} - 1|, \end{aligned} \tag{2.4.3}$$

and

$$0 \leq \frac{t_2 - t_1}{2}[1 - (1-\eta_\delta)^k] < \frac{t_2 - t_1}{2} < \frac{t_2}{2} \leq \frac{2\pi}{2} = \pi,$$

implying by (2.4.2) that

$$\begin{aligned} |e^{i[\frac{t_2-t_1}{2}(1-(1-\eta_\delta)^k)]} - 1| &\geq \frac{\sqrt{2}}{\pi} \cdot \frac{t_2 - t_1}{2}[1 - (1-\eta_\delta)^k] \\ &= \frac{t_2 - t_1}{\sqrt{2}\pi}[1 - (1-\eta_\delta)^k], \end{aligned}$$

for each $k \in \mathbb{N}$. Then (2.4.3) implies that

$$|f(w) - g_{\eta_\delta}^k(w)| \geq \frac{t_2 - t_1}{\sqrt{2}\pi}[1 - (1-\eta_\delta)^k], \tag{2.4.4}$$

for each $k \in \mathbb{N}$. Since $1 - (1-\eta_\delta)^k \to 1$ as $k \to \infty$, there exists $N \in \mathbb{N}$ such that $(1 - \eta_\delta)^k < \frac{1}{2}, \forall k \geq N$. Choose k_0 sufficiently large such that $n^{k_0} > N$. Then $1 - (1-\eta_\delta)^{n^{k_0}} > \frac{1}{2}$, and therefore from (2.4.4) we have

$$|f(w) - g_{\eta_\delta}^{n^{k_0}}(w)| \geq \frac{t_2 - t_1}{\sqrt{2}\pi} \cdot \frac{1}{2} = \frac{t_2 - t_1}{2\sqrt{2}\pi} = \varepsilon,$$

which implies that

$$\rho(f, g_{\eta_\delta}^{n^{k_0}}) = \rho(f^{n^{k_0}}, g_{\eta_\delta}^{n^{k_0}}) \geq |f^{n^{k_0}}(w) - g_{\eta_\delta}^{n^{k_0}}(w)| \geq \varepsilon.$$

This proves the claim, from which it follows that f is not stable, a contradiction. Hence f is a constant map on S^1. □

2.5 $\mathscr{J}_n$ IS NOT CHAOTIC

Although, as seen in section 2.4, all orbits of the iteration operator $\mathscr{J}_n$ are bounded, most of its fixed points are unstable, thereby exhibiting a complex behaviour of $\mathscr{J}_n$. In this section we prove that the complex behaviour is not chaotic in Devaney's sense.

As observed in section 2.2, the set $\mathrm{Per}(\mathscr{J}_n;\mathscr{C}(I))$ is not dense in $\mathscr{C}(I)$, and therefore $\mathscr{J}_n$ is not chaotic on $\mathscr{C}(I)$ for $n \geq 2$. More generally, we have the following result for any compact metric space K.

Theorem 2.5.1. *$\mathscr{J}_n$ is not topologically transitive on $\mathscr{C}(K)$ for each $n \in \mathbb{N}$. Moreover, $\mathscr{J}_n$ does not exhibit sensitive dependence on initial conditions for each $n \in \mathbb{N}$.*

Proof. Let f_1, f_2 be constant functions on K with $f_1 \neq f_2$. Then $f_1, f_2 \in \mathscr{C}(K)$. Since $\mathscr{C}(K)$ is Hausdorff, there exist disjoint open sets U_1 and V_1 in $\mathscr{C}(K)$ containing f_1 and f_2, respectively. Choose $\varepsilon > 0$ such that

$$B_\rho(f_1, \varepsilon) \subseteq U_1 \text{ and } B_\rho(f_2, \varepsilon) \subseteq V_1. \tag{2.5.1}$$

Let $U = B_\rho(f_1, \varepsilon)$ and $V = B_\rho(f_2, \varepsilon)$. Since U_1 and V_1 are disjoint, by (2.5.1) it follows that U and V are disjoint. Also, for each $f \in U$, $x \in K$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} d\big(f_1(x), f^k(x)\big) &= d\big(f_1^k(x), f^k(x)\big) \\ &= d\big(f_1\big(f_1^{k-1}(x)\big), f\big(f^{k-1}(x)\big)\big) \\ &= d\big(f_1\big(f^{k-1}(x)\big), f\big(f^{k-1}(x)\big)\big) \ \text{(since } f_1\big(f_1^{k-1}(x)\big) = f_1\big(f^{k-1}(x)\big)\text{)} \\ &\leq \rho(f_1, f) < \varepsilon, \end{aligned}$$

implying that $\rho(f_1, f^k) < \varepsilon$ for all $f \in U$ and $k \in \mathbb{N}$. So $f^k \in U$, and thus in particular $\mathscr{J}_n^k f = f^{n^k} \in U$ for all $f \in U$ and $k \in \mathbb{N}$. Therefore $\mathscr{J}_n^k f \notin V$ for all $f \in U$ and $k \in \mathbb{N}$. Hence $\mathscr{J}_n$ is not topologically transitive.

In order to prove the second result, consider any $\delta > 0$. Let $\varepsilon = \delta$ and $f \in \mathscr{C}(K)$ be any constant function. Then for each $x \in K$, $k \in \mathbb{N}$ and $g \in \mathscr{C}(K)$ with $\rho(f, g) < \varepsilon$, we have

$$\begin{aligned} d\big(f^k(x), g^k(x)\big) &= d\big(f\big(f^{k-1}(x)\big), g\big(g^{k-1}(x)\big)\big) \\ &= d\big(f\big(g^{k-1}(x)\big), g\big(g^{k-1}(x)\big)\big) \ \text{(since } f\big(f^{k-1}(x)\big) = f\big(g^{k-1}(x)\big)\text{)} \\ &\leq \rho(f, g) < \varepsilon = \delta, \end{aligned}$$

implying that $\rho\big(\mathscr{J}_n^k f, \mathscr{J}_n^k g\big) < \delta$ for all $k \in \mathbb{N}$ and $g \in \mathscr{C}(K)$ with $\rho(f, g) < \varepsilon$. Hence $\mathscr{J}_n$ does not exhibit sensitive dependence on initial conditions. □

As mentioned in the beginning of this section, Theorem 2.5.1 shows that $\mathscr{J}_n$ is not chaotic on $\mathscr{C}(K)$, where K is any compact metric space, but Theorems 2.4.1, 2.4.2 and 2.4.3 tell that the dynamical behaviour of $\mathscr{J}_n$ is complex. The following example shows that how complex an orbit of the iteration operator $\mathscr{J}_n$ can be.

Example 2.5.2. Let f be the identity map on $[0,1]$. Let $\varepsilon = \frac{1}{8}$, $\delta = 0.1$, $\delta_0 = 0.08$, $\eta_\delta = 0.05$ and g_2 be the map g_{η_δ} as defined in Theorem 2.4.2. Define the maps g_1, $g_3 : [0,1] \to [0,1]$ by $g_1(x) = 0.95x$ for all $x \in [0,1]$, and

$$g_3(x) = \begin{cases} 2x & \text{if } \; 0 \le x \le 0.04, \\ x + 0.04 & \text{if } \; 0.04 \le x \le 0.08, \\ -2x + 0.28 & \text{if } \; 0.08 \le x \le 0.12, \\ \frac{1}{11}(12x - 1) & \text{if } \; 0.12 \le x \le 1. \end{cases}$$

Then $f \in \mathrm{Fix}(\mathscr{J}_2; \mathscr{C}([0,1]))$ and $g_1, g_2, g_3 \in \mathscr{C}([0,1])$. An easy computation shows that $\rho(g_j, f) < \delta$ for $j = 1,2,3$. Also,

$$g_1^{2^3}(0.5) = 0.331710, \; g_2^{2^3}(0.5) = 0.331710 \; \text{ and } \; g_3^{2^3}(0.5) = 0.118738,$$

implying that $\rho(g_j^{2^3}, f) > \varepsilon$ for each $j = 1,2,3$. Indeed, the identity map is not stable for $\mathscr{J}_2$.

In order to illustrate the complexity of iteration operator $\mathscr{J}_2$, we investigate the asymptotic behaviour of the orbits of g_1, g_2 and g_3 (see Figures 2.1, 2.2 and 2.3). We have $g_1^k(x) = 0.95^k x$ for all $k \in \mathbb{N}$ and $\forall x \in [0,1]$. Therefore the sequence of maps $(g_1^k)_{k \in \mathbb{N}}$ and hence the orbit $(g_1^{2^k})_{k \in \mathbb{N} \cup \{0\}}$ of g_1 converges uniformly to the zero map on $[0,1]$. As noted in the proof of Theorem 2.4.2, $g_2^k(x) \to 0$ as $k \to \infty$, for each $x \in [0,0.5]$. Also, for each $x \in [0.5,1)$, there exists $k_x \in \mathbb{N}$ such that $g_2^{k_x}(x) \in [0,0.5]$. Moreover, $g_2(1) = 1$. Thus the sequence of maps $(g_2^k)_{k \in \mathbb{N}}$ and hence the orbit $(g_2^{2^k})_{k \in \mathbb{N} \cup \{0\}}$ of g_2 converges pointwise to the discontinuous map $f_2 : [0,1] \to [0,1]$ defined by

$$f_2(x) = \begin{cases} 0 & \text{if } \; 0 \le x < 1, \\ 1 & \text{if } \; x = 1. \end{cases}$$

The point $x = 0.04$ is a 3-periodic point of g_3 and therefore the orbit $(g_3^{2^k})_{k \in \mathbb{N} \cup \{0\}}$ of g_3 does not converge. In fact, by Theorem 1.1.1, g_3 has periodic points of all periods, and moreover by Theorem 1.1.3, it is chaotic in the sense of Li-Yorke. However, $g_3([0, \frac{1}{2}]) \subseteq [0, \frac{1}{2}]$, implying that g_3 is not topologically transitive and therefore is not chaotic in the sense of Devaney.

Thus, although all the orbits of $\mathscr{J}_2$ are bounded, it is possible that an orbit may not converge or, even if it converges, the limit function may not be in $\mathscr{C}([0,1])$.

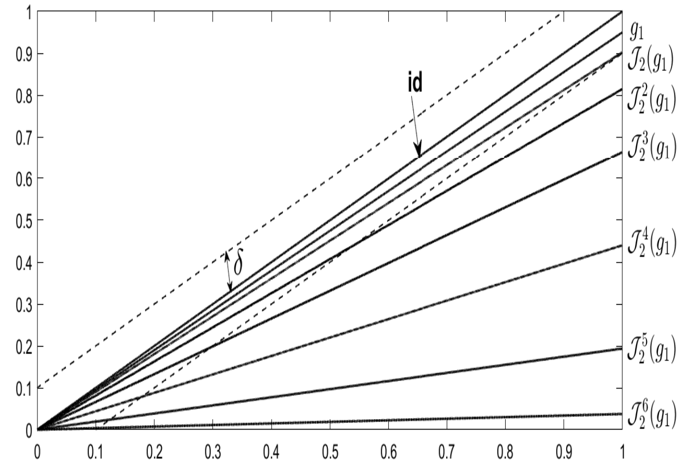

Figure 2.1 Iterates of g_1 under $\mathscr{J}_2$

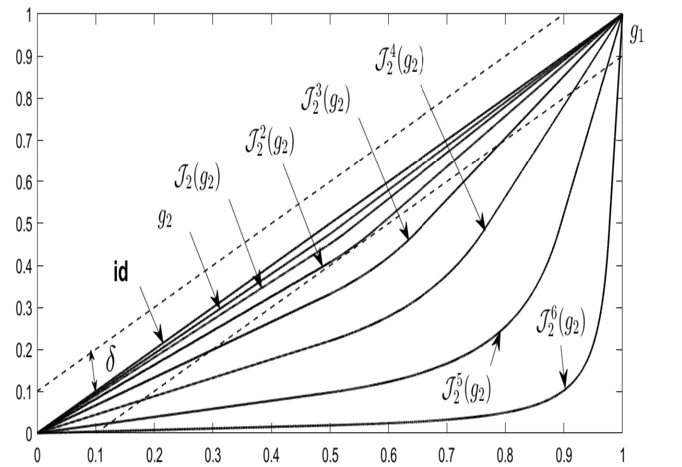

Figure 2.2 Iterates of g_2 under $\mathscr{J}_2$

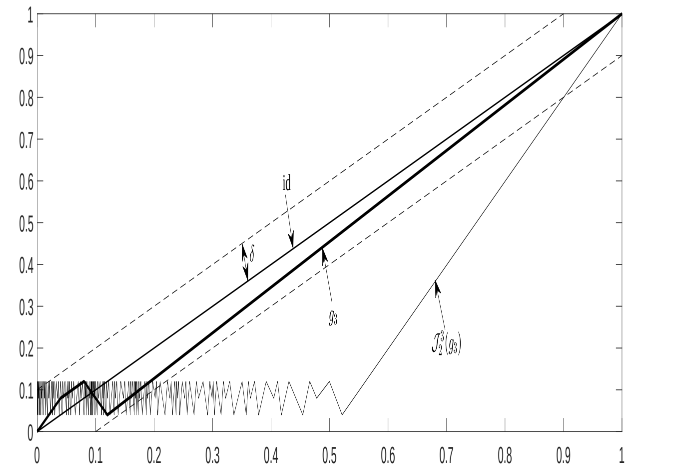

Figure 2.3 Third iterate of g_3 under $\mathscr{J}_2$

CHAPTER 3

TOPOLOGICAL CONJUGACY AND KNEADING THEORY

"The important thing is not to stop questioning.
Curiosity has its own reason for existing."
- Albert Einstein

In this chapter, we study the iterates of an important class of non-monotone maps, the continuous piecewise monotone self-maps of compact intervals.

3.1 NON-INVARIANCE OF KNEADING MATRIX

Let $I = [a,b]$ be a compact interval in $\mathbb{R}$ such that $a < b$. As defined in section 1.1.2, a point $c \in (a,b)$ is said to be a turning point of an $f \in \mathscr{M}(I)$ if f is strictly monotone in no neighbourhood of c. It follows that each $c \in T(f)$ is either a local minimum or local maximum point of f. More concretely, we have the following.

Proposition 3.1.1. (Lemma 2.1 of Zhang (1997)) *Let $f \in \mathscr{M}(I)$. Then $c \in T(f)$ if and only if for every $\varepsilon > 0$ there exist $x, y \in I$ with $x \neq y$, $|x-c| < \varepsilon$ and $|y-c| < \varepsilon$ such that $f(x) = f(y)$.*

One of the main results in kneading theory is the following.

Theorem 3.1.2. (Milnor and Thurston (1988)) *Let $f \in \mathscr{M}(I)$ and $g \in \mathscr{M}(J)$, where $J = [c,d]$ is a compact interval in $\mathbb{R}$ with $c < d$. If f is h-conjugate to g, given that h is orientation-preserving, then $N(f;t) = N(g;t)$ and $D(f;t) = D(g;t)$.*

In this section, we consider if the above theorem is valid when h is orientation-reversing. Henceforth, for the entirety of this section and the one that follows, unless stated otherwise, let $f \in \mathscr{M}(I)$ and $g \in \mathscr{M}(J)$ such that $T(f) = \{c_1, c_2, \dots, c_m\}$,

$T(g) = \{d_1, d_2, \ldots, d_n\}$, $L(f) = \{I_1, I_2, \ldots, I_{m+1}\}$ and $L(g) = \{J_1, J_2, \ldots, J_{n+1}\}$, where $I_j = [c_{j-1}, c_j]$ for $1 \le j \le m+1$ and $J_i = [d_{i-1}, d_i]$ for $1 \le i \le n+1$ with $c_0 = a$, $c_{m+1} = b$, $d_0 = c$ and $d_{n+1} = d$. To prove our main result, we require the following three technical lemmas, the first of which is an immediate consequence of the conjugacy, but we include its proof for completeness.

Lemma 3.1.3. *If f is h-conjugate to g, then $\#T(f) = \#T(g)$.*

Proof. To prove $n = m$, it suffices to show that

$$T(g) = \{h(c_1), h(c_2), \ldots, h(c_m)\}. \tag{3.1.1}$$

Let $1 \le i \le m$ and $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $(c_i - \delta, c_i + \delta) \subseteq h^{-1}(U)$, where $U := (h(c_i) - \varepsilon, h(c_i) + \varepsilon)$. Since $c_i \in T(f)$, by Proposition 3.1.1, there exist $x, y \in I$ with $c_i - \delta < x < c_i < y < c_i + \delta$ such that $f(x) = f(y)$, implying that

$$\begin{aligned} g\big(h(x)\big) = (g \circ h)(x) = (h \circ f)(x) &= h\big(f(x)\big) \\ &= h\big(f(y)\big) = (h \circ f)(y) = (g \circ h)(y) = g\big(h(y)\big). \end{aligned}$$

Since $x, y \in (c_i - \delta, c_i + \delta) \subseteq h^{-1}(U)$, clearly $h(x), h(y) \in U$. Further, either $h(x) < h(c_i) < h(y)$ or $h(y) < h(c_i) < h(x)$ according as h preserves or reverses orientation, respectively. Hence, by Proposition 3.1.1, we have $h(c_i) \in T(g)$.

For the reverse inclusion, consider an arbitrary $w \in T(g)$ and suppose that $w \ne h(c_i)$ for all $1 \le i \le m$. Then $h^{-1}(w) \notin T(f)$, implying by Proposition 3.1.1 that f is strictly monotone on $V := \big(h^{-1}(w) - \delta, h^{-1}(w) + \delta\big)$ for some $\delta > 0$. Let $f \nearrow V$. The proof in the case that $f \searrow V$ is similar.

Claim: g is strictly monotone on the neighbourhood $W := h(V)$ of d.

Since h^{-1} is continuous, clearly W is a neighbourhood of d in J. Consider arbitrary $u, v \in W$ such that $u < v$. Then $u = h(x)$ and $v = h(y)$ for some $x, y \in V$. If h preserves orientation, then $x < y$, implying that $f(x) < f(y)$. Now,

$$\begin{aligned} g\big(h(x)\big) = (g \circ h)(x) = (h \circ f)(x) &= h\big(f(x)\big) \\ &< h\big(f(y)\big) = (h \circ f)(y) = (g \circ h)(y) = g\big(h(y)\big), \end{aligned}$$

and therefore $g(u) < g(v)$. We arrive at the same conclusion if h is orientation-reversing. So, $g \nearrow W$, and the claim is proved.

Hence, by the claimed fact, there exists $\delta_1 > 0$ such that g is strictly monotone on $(w - \delta_1, w + \delta_1)$, a contradiction to our assumption that $w \in T(g)$. So, $w = h(c_i)$ for some $1 \le i \le m$. Therefore the assertion (3.1.1) is proved and the result follows. $\square$

Lemma 3.1.4. *Let f be h-conjugate to g. Then the following statements are true.*

(i) *If h is orientation-preserving, then $d_i = h(c_i)$ for $1 \le i \le m$ and $J_j = h(I_j)$ for $1 \le j \le m+1$.*

(ii) *If h is orientation-reversing, then $d_i = h(c_{m+1-i})$ for $1 \le i \le m$ and $J_j = h(I_{m+2-j})$ for $1 \le j \le m+1$.*

Proof. By Lemma 3.1.3, we have $n = m$ and $T(g) = \{h(c_1), h(c_2), \dots, h(c_m)\}$. If h preserves orientation, then $h(c_1) < h(c_2) < \cdots < h(c_m)$, implying that $d_i = h(c_i)$ for $1 \le i \le m$. Further, by intermediate value theorem, we obtain $J_j = h(I_j)$ for $1 \le j \le m+1$. This proves result **(i)**. The proof of result **(ii)** is similar. □

Lemma 3.1.5. *Let f be h-conjugate to g. Then the following statements are true.*

(i) *If h is orientation-reversing, then $\varepsilon(J_j) = \varepsilon(I_{m+2-j})$ for $1 \le j \le m+1$.*

(ii) *If h is orientation-preserving, then $\varepsilon(J_j) = \varepsilon(I_j)$ for $1 \le j \le m+1$.*

(iii) *For $1 \le i \le m$ and $k \ge 0$,*

$$\varepsilon(A((h \circ f^k)(c_i+))) = \varepsilon_k(c_i+, f) \tag{3.1.2}$$

and

$$\varepsilon(A((h \circ f^k)(c_i-))) = \varepsilon_k(c_i-, f). \tag{3.1.3}$$

Proof. By Lemma 3.1.3, we have $n = m$ and g satisfies (3.1.1). Consider an arbitrary $j \in \{1, 2, \dots, m+1\}$. In the case that $\varepsilon(I_{m+2-j}) = +1$, we have to show that $g \nearrow J_j$. So, let $x, y \in J_j$ such that $x < y$. Since $h : I \to J$ is an injective map, by result **(ii)** of Lemma 3.1.4, it follows that $h : I_{m+2-j} \to J_j$ is a bijective map. Let $u, v \in I_{m+2-j}$ be such that $h(u) = x$ and $h(v) = y$. Since $h \searrow I_{m+2-j}$, we have $v < u$, and therefore $f(v) < f(u)$, implying that $(h \circ f)(u) < (h \circ f)(v)$. Hence

$$\begin{aligned} g(x) = (h \circ f)(h^{-1}(x)) &= (h \circ f)(u) \\ &< (h \circ f)(v) = (h \circ f)(h^{-1}(y)) = g(y), \end{aligned}$$

which proves that $g \nearrow J_j$. The proofs of the equality $\varepsilon(J_j) = \varepsilon(I_{m+2-j})$ in the case that $\varepsilon(I_{m+2-j}) = -1$, and that of result **(ii)** are similar.

Next, we prove result **(iii)**. Consider the case that h reverses orientation. Let $1 \le i \le m$ and $k \ge 0$ be arbitrary.

Claim: $A((h\circ f^k)(c_i+)) = J_j$ whenever $j \in \{1,2,\dots,m+1\}$ such that $A_k(c_i+,f) = I_{m+2-j}$.

Let $A_k(c_i+,f) = I_{m+2-j}$, where $j \in \{1,2,\dots,m+1\}$. Choose $\delta > 0$ such that $A(f^k(x)) = I_{m+2-j}$ for $c_i < x < c_i+\delta$. Then $(h\circ f^k)(x) \in h(I_{m+2-j})$ for $c_i < x < c_i+\delta$. By result **(ii)** of Lemma 3.1.4, we have $h(I_{m+2-j}) = J_j$. So, $A((h\circ f^k)(x)) = J_j$ for $c_i < x < c_i+\delta$, implying that $A((h\circ f^k)(c_i+)) = J_j$. This proves the claim. By result **(i)**, we have $\varepsilon(J_j) = \varepsilon(I_{m+2-j})$. Hence, by the claimed fact, it follows that $\varepsilon(A((h\circ f^k)(c_i+))) = \varepsilon(A_k(c_i+,f))$, proving (3.1.2). The proofs of (3.1.3), and those of (3.1.2) and (3.1.3) in the case that h is orientation-preserving are similar. □

Corollary 3.1.6. *Let m be an even positive integer. If $f \nearrow I_1$ and $g \searrow J_1$, then f is not conjugate to g.*

Proof. Since $f \nearrow I_1$ and $g \searrow J_1$, by definition $\varepsilon(I_1) = +1$ and $\varepsilon(J_1) = -1$. Suppose that there exists a conjugacy h of f and g. If h preserves orientation, then by result **(ii)** of Lemma 3.1.5, we have $\varepsilon(J_1) = \varepsilon(I_1) = +1$, which is a contradiction. If h reverses orientation, then by result **(i)** of Lemma 3.1.5, $\varepsilon(J_{m+1}) = \varepsilon(I_1) = +1$. This implies that $\varepsilon(J_j) = +1$ for every odd $j \in \{1,2,\dots,m+1\}$, because $m+1$ is odd. So, in particular $\varepsilon(J_1) = +1$, again a contradiction. Hence f is not conjugate to g. □

Having Lemmas 3.1.3, 3.1.4 and 3.1.5, we are ready to prove our main result.

Theorem 3.1.7. *Let f be h-conjugate to g, given that h is orientation-reversing. Then*

$$N(g;t) = -S_m N(f;t) S_{m+1}, \tag{3.1.4}$$

where $S_m := [s_{ij}]_{m\times m}$ such that

$$s_{ij} = \begin{cases} 1 & \text{if } i+j = m+1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. By Lemma 3.1.3, we have $n = m$ and g satisfies (3.1.1). Consider an arbitrary $i \in \{1,2,\dots,m\}$ and $l \in \mathbb{N}$. Then

$$A((g^l\circ h)(c_{m+1-i}-)) = \lim_{y\uparrow c_{m+1-i}} A((g^l\circ h)(y)) = \lim_{y\uparrow c_{m+1-i}} A(g^l(h(y))). \tag{3.1.5}$$

Since h is an orientation-reversing homeomorphism, it follows that $y \uparrow c_{m+1-i}$ if and only if $h(y) \downarrow h(c_{m+1-i})$. Therefore

$$\lim_{y\uparrow c_{m+1-i}} A((g^l(h(y))) = \lim_{h(y)\downarrow h(c_{m+1-i})} A((g^l(h(y))) = A(g^l(h(c_{m+1-i})+)),$$

implying by (3.1.5) that

$$A((g^l \circ h)(c_{m+1-i}-)) = A(g^l(h(c_{m+1-i})+)).$$

Since h reverses orientation, by result **(ii)** of Lemma 3.1.4, we have $d_i = h(c_{m+1-i})$, and therefore

$$A_l(d_i+, g) = A((g^l \circ h)(c_{m+1-i}-)). \tag{3.1.6}$$

Since f is h-conjugate to g, we have $g^l \circ h = h \circ f^l$, implying that

$$A((g^l \circ h)(c_{m+1-i}-)) = A((h \circ f^l)(c_{m+1-i}-)). \tag{3.1.7}$$

Using (3.1.7) in (3.1.6), we have

$$A_l(d_i+, g) = A((h \circ f^l)(c_{m+1-i}-)), \tag{3.1.8}$$

and hence

$$\varepsilon_l(d_i+, g) = \varepsilon(A((h \circ f^l)(c_{m+1-i}-))). \tag{3.1.9}$$

Now

$$\begin{aligned} A((g^l \circ h)(c_{m+1-i}+)) &= \lim_{y \downarrow c_{m+1-i}} A((g^l(h(y)))) \\ &= \lim_{h(y) \uparrow h(c_{m+1-i})} A((g^l(h(y)))) \\ &= A(g^l(h(c_{m+1-i})-)) = A_l(d_i-, g). \end{aligned} \tag{3.1.10}$$

Also, by using (1.1.3) for g, we have $A_l(d_i-, g) = A_l(d_i+, g)$. Therefore, (3.1.10) implies that

$$\begin{aligned} A\big((g^l \circ h)(c_{m+1-i}+) &= A_l(d_i+, g) \\ &= A((g^l \circ h)(c_{m+1-i}-)) \text{ (by using (3.1.6)).} \end{aligned}$$

Thus

$$A((h \circ f^l)(c_{m+1-i}-)) = A((h \circ f^l)(c_{m+1-i}+)), \tag{3.1.11}$$

and hence

$$\varepsilon(A((h \circ f^l)(c_{m+1-i}-))) = \varepsilon(A((h \circ f^l)(c_{m+1-i}+))). \tag{3.1.12}$$

Moreover, by (3.1.2) we have

$$\varepsilon(A((h \circ f^l)(c_{m+1-i}+))) = \varepsilon(A(f^l(c_{m+1-i}+))), \tag{3.1.13}$$

and

$$\varepsilon(A(h(c_{m+1-i}-))) = \varepsilon(A(c_{m+1-i}-)) = -\varepsilon(A(c_{m+1-i}+)), \qquad (3.1.14)$$

where the last equality in (3.1.14) is true, because $A(c_{m+1-i}-)$ and $A(c_{m+1-i}+)$ are two consecutive laps of f. Thus from (3.1.9) and (3.1.8), we obtain

$$\begin{aligned}
\theta_k(d_i+,g) &= \left(\prod_{l=0}^{k-1}\varepsilon_l(d_i+,g)\right)A_k(d_i+,g)\\
&= \left(\prod_{l=0}^{k-1}\varepsilon(A((h\circ f^l)(c_{m+1-i}-)))\right)A((h\circ f^k)(c_{m+1-i}-))\\
&= -\varepsilon(A(c_{m+1-i}+))\left(\prod_{l=1}^{k-1}\varepsilon(A((h\circ f^l)(c_{m+1-i}+)))\right)\\
&\qquad A((h\circ f^k)(c_{m+1-i}+), \qquad (3.1.15)
\end{aligned}$$

for each $k\in\mathbb{N}$, where the equality in (3.1.15) follows from (3.1.14), (3.1.12) and (3.1.11). Using (3.1.13) in this equation, we have

$$\begin{aligned}
\theta_k(d_i+,g) &= -\varepsilon(A(c_{m+1-i}+))\left(\prod_{l=1}^{k-1}\varepsilon(A(f^l(c_{m+1-i}+)))\right)A(h(f^k(c_{m+1-i}+)))\\
&= -\left(\prod_{l=0}^{k-1}\varepsilon(A(f^l(c_{m+1-i}+)))\right)A(h(f^k(c_{m+1-i}+)))\\
&= -\left(\prod_{l=0}^{k-1}\varepsilon_l(c_{m+1-i}+,f)\right)A(h(f^k(c_{m+1-i}+))) \qquad (3.1.16)
\end{aligned}$$

for every $k\in\mathbb{N}$. Now, for a fixed $k\in\mathbb{N}$, $A(h(f^k(c_{m+1-i}+))) = J_{m+2-j}$ whenever $A(f^k(c_{m+1-i}+)) = I_j$ for some $j\in\{1,2,\ldots,m+1\}$ and conversely. Thus, from (3.1.16), it follows that the coefficient of J_j in $\theta_k(d_i+,g)$ is equal to $-$ve of the coefficient of I_{m+2-j} in $\theta_k(c_{m+1-i}+,f)$. This holds for every $k\in\mathbb{N}$. By a similar argument, it follows that

$$\theta_k(d_i-,g) = -\left(\prod_{l=0}^{k-1}\varepsilon_l(c_{m+1-i}-,f)\right)A(h(f^k(c_{m+1-i}-))),$$

for $1\le i\le m$ and $k\in\mathbb{N}$, and the coefficient of J_j in $\theta_k(d_i-,g)$ is equal to $-$ve of the coefficient of I_{m+2-j} in $\theta_k(c_{m+1-i}-,f)$. Therefore

$$\begin{aligned}
M(g;t) = [M_{ij}(g;t)] &= -[M_{m+1-i,m+2-j}(f;t)]\\
&= -S_m[M_{ij}(f;t)]S_{m+1} = -S_mM(f;t)S_{m+1}. \qquad (3.1.17)
\end{aligned}$$

Also, $N_0(g;t) = -S_m N_0(f;t) S_{m+1}$. Thus, from (3.1.17), we have

$$N(g;t) = N_0(g;t) + M(g;t) = -S_m N_0(f;t) S_{m+1} - S_m [M_{ij}(f;t)] S_{m+1} = -S_m N(f;t) S_{m+1}.$$

This proves (3.1.4) and the proof is completed. $\square$

Lemma 3.1.8. *If $N(g;t) = -S_m N(f;t) S_{m+1}$ for some $m \in \mathbb{N}$, then $D(g;t) = D(f;t)$.*

Proof. Without loss of generality, we assume that $f \nearrow I_1$. Then

$$\begin{aligned} D(f;t) &= (-1)^{1+1}\big(1-\varepsilon(I_1)t\big)^{-1} \det\big(N^{(1)}(f;t)\big) \\ &= (1-t)^{-1} \det\big(N^{(1)}(f;t)\big), \end{aligned} \tag{3.1.18}$$

and

$$\begin{aligned} D(g;t) &= (-1)^{(m+1)+1}\big(1-\varepsilon(J_{m+1})t\big)^{-1} \det\big(N^{(m+1)}(g;t)\big) \\ &= (-1)^{m+2}\big(1-\varepsilon(J_{m+1})t\big)^{-1} \det\big(N^{(m+1)}(g;t)\big). \end{aligned} \tag{3.1.19}$$

Since $N(g;t) = -S_m N(f;t) S_{m+1}$, we have $N^{(m+1)}(g;t) = -S_m N^{(1)}(f;t) S_m$. Therefore

$$\det\big(N^{(m+1)}(g;t)\big) = (-1)^m (\det S_m)^2 \det\big(N^{(1)}(f;t)\big) = (-1)^m \det\big(N^{(1)}(f;t)\big),$$

because $\det S_m = (-1)^{\lfloor \frac{m}{2} \rfloor}$. Hence from (3.1.18) and (3.1.19), we obtain

$$D(g;t) = \big(1-\varepsilon(J_{m+1})t\big)^{-1}(1-t)D(f;t). \tag{3.1.20}$$

By result **(i)** of Lemma 3.1.5, we have $\varepsilon(J_{m+1}) = \varepsilon(I_1)$, and therefore $\varepsilon(J_{m+1}) = 1$, since $\varepsilon(I_1) = 1$. Then (3.1.20) implies that $D(g;t) = (1-t)^{-1}(1-t)D(f;t) = D(f;t)$. $\square$

Theorem 3.1.7 shows that the kneading matrices $N(f;t)$ and $N(g;t)$ are not equal whenever f and g are h-conjugates, given that h is orientation-reversing. However, since the matrices S_m and S_{m+1} are invertible, Theorems 3.1.2 and 3.1.7 together imply that $N(f;t)$ and $N(g;t)$ are equivalent whenever f and g are topologically conjugates. Further, Theorem 3.1.7 and Lemma 3.1.8 together prove the following.

Corollary 3.1.9. *Let f be h-conjugate to g, given that h is orientation-reversing. Then $D(g;t) = D(f;t)$.*

Although that of Milnor and Thurston motivated our work, we can indeed deduce Theorem 3.1.7 from Theorem 3.1.2 as follows. Let f be h-conjugate to g, given that h

is orientation-preserving. Define $f_1 : I \to I$ by

$$f_1(x) = a + b - f(a + b - x), \quad \forall x \in I.$$

Then $f_1 \in \mathscr{M}(I)$, f is h_1-conjugate to f_1, and f_1 is $h \circ h_1$-conjugate to g, where $h_1 : I \to I$ is defined by $h_1(x) = a+b-x, \forall x \in I$. Since h_1 and $h \circ h_1$ are orientation-reversing, by Theorem 3.1.7, we have $N(f_1;t) = -S_m N(f;t) S_{m+1}$ and $N(g;t) = -S_m N(f_1;t) S_{m+1}$ such that $m = \#T(f) (= \#T(f_1) = \#T(g))$. Therefore

$$N(g;t) = -S_m(-S_m N(f;t) S_{m+1}) S_{m+1} = S_m^2 N(f;t) S_{m+1}^2 = N(f;t),$$

proving Theorem 3.1.2. On the other hand, we cannot use the above approach to deduce Theorem 3.1.2 from Theorem 3.1.7 because an orientation-reversing homeomorphism can never be equal to the composition of two or more orientation-preserving homeomorphisms.

Further, converses of Theorems 3.1.7 and Corollary 3.1.9 are not valid in general. For example, consider the maps f, $g : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2}, \\ 1-x & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

and $g(x) = 1 - 2x(1-x)$, $\forall x \in [0,1]$. Then $f, g \in \mathscr{M}([0,1])$ such that $T(f) = \{c_1\}$, $L(f) = \{I_1, I_2\}$, $T(g) = \{d_1\}$ and $L(g) = \{J_1, J_2\}$, where $c_1 = d_1 = \frac{1}{2}$, $I_1 = J_1 = [0, \frac{1}{2}]$ and $I_2 = J_2 = [\frac{1}{2}, 1]$. Since $f(I) \subseteq I_1$, we have $A_0(c_1+, f) = I_2$ and $A_k(c_1+, f) = I_1$ for $k \ge 1$. Also, $\varepsilon_0(c_1+, f) = -1$ and $\varepsilon_k(c_1+, f) = 1$ for $k \ge 1$. Therefore $\theta_0(c_1+, f) = I_2$ and $\theta_k(c_1+, f) = -I_1$ for $k \ge 1$, implying that

$$\theta(c_1+, f; t) = I_2 - I_1 t - I_1 t^2 - \cdots = (-t - t^2 - \cdots) I_1 + I_2.$$

Further, $A_0(c_1-, f) = I_1$, and since $A_k(c_1-, f) = A_k(c_1+, f)$, we get that $A_1(c_1-, f) = I_1$ for $k \ge 1$. Moreover, $\varepsilon_k(c_1-, f) = 1$ for $k \ge 0$. Therefore $\theta_0(c_1-, f) = I_1$ for $k \ge 0$, implying that

$$\theta(c_1-, f; t) = I_1 + I_1 t + I_1 t^2 + \cdots = (1 + t + t^2 + \cdots) I_1.$$

Thus

$$\nu(c_1, f; t) = (I_2 - I_1) - 2I_1 t - 2I_1 t^2 - \cdots = (-1 - 2t - 2t^2 - \cdots) I_1 + I_2,$$

and hence

$$N(f;t) = \begin{bmatrix} -1-2t-2t^2-\cdots, & 1 \end{bmatrix}_{1\times 2}.$$

Since $g(J) \subseteq J_2$, by a similar argument as above, we obtain

$$\nu(d_1,g;t) = (J_2-J_1)+2J_2t+2J_2t^2+\cdots = -J_1+(1+2t+2t^2+\cdots)J_2.$$

Therefore

$$N(g;t) = \begin{bmatrix} -1, & 1+2t+2t^2+\cdots \end{bmatrix}_{1\times 2}.$$

Clearly, $N(g;t) = -S_1N(f;t)S_2$ and $D(g;t) = D(f;t)$. However, $[0,\frac{1}{2}]$ and $\{\frac{1}{2},1\}$ are precisely the set of fixed points of f and g, respectively. Therefore f and g are not topologically conjugates, because g has only two fixed points whereas f has uncountably many.

As defined in Holmgren (1994), we say that f is *topologically h-semiconjugate* to g if there exists a continuous onto map $h : I \to J$ such that $h \circ f = g \circ h$. As proved in Remark 3.16 of Block et al. (2012), the map $f \in \mathscr{M}([0,1])$ defined by

$$f(x) = \begin{cases} \frac{8}{3}x & \text{if } \ 0 \le x \le \frac{3}{8}, \\ 2-\frac{8}{3}x & \text{if } \ \frac{3}{8} \le x \le \frac{3}{4}, \\ 2x-\frac{3}{2} & \text{if } \ \frac{3}{4} \le x \le 1 \end{cases}$$

with $T(f) = \{\frac{3}{8},\frac{3}{4}\}$ is h-semiconjugate to the tent map T defined in Example 1.1.4, where a semiconjugacy h is given in Figure 4 of Block et al. (2012). However, f and T do not have the same number of turning points. Further, $h(\frac{3}{8}) \neq \frac{1}{2}$ although $\frac{3}{8}$ is a turning point of f, and $h(\frac{9}{32}) = \frac{1}{2}$ although $\frac{9}{32}$ is not a turning point of f. Therefore h does not map a turning point of f to the turning point of T and maps a point that is not a turning point of f to the turning point of T. Hence the results of Lemmas 3.1.3, 3.1.4 and 3.1.5, and Theorems 3.1.7 and 3.1.9 are not true in general whenever f is h-semiconjugate to g.

Example 3.1.10. Let $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \mathsf{T}^2(x), \quad \forall x \in [0,1],$$

where T is the Tent map defined as in Example 1.1.4. Then $f \in \mathscr{M}([0,1])$ such that $T(f) = \{c_1,c_2,c_3\}$ and $L(f) = \{I_1,I_2,I_3,I_4\}$, where $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{3}{4}$, $I_1 = \left[0,\frac{1}{4}\right]$,

$I_2=\left[\frac{1}{4},\frac{1}{2}\right]$, $I_3=\left[\frac{1}{2},\frac{3}{4}\right]$ and $I_4=\left[\frac{3}{4},1\right]$.

We have $f(c_1)=f(c_3)=1$, $f(c_2)=0$, $\varepsilon(I_1)=\varepsilon(I_3)=+1$ and $\varepsilon(I_2)=\varepsilon(I_4)=-1$. Since $f(0)=f(1)=0$,

$$f^k(c_i)=\begin{cases} 0 & \text{if} \quad i=1,3 \text{ and } k\geq 2,\\ 0 & \text{if} \quad i=2 \text{ and } k\geq 1.\end{cases}$$

Now, for each $i=1,3$, we have $A_0(c_i+,f)=I_{i+1}$, $A_1(c_i+,f)=I_4$, and $A_k(c_i+,f)=I_1$ for $k\geq 2$. Therefore $\varepsilon_0(c_i+,f)=\varepsilon_1(c_i+,f)=-1$, and $\varepsilon_k(c_i+,f)=1$ for $k\geq 2$. Hence $\theta_0(c_i+,f)=I_{i+1}$, $\theta_1(c_i+,f)=-I_4$, and $\theta_k(c_i+,f)=I_1$ for $k\geq 2$. This implies that

$$\theta(c_i+,f;t)=I_{i+1}-I_4t+I_1t^2+I_1t^3+\cdots=(t^2+t^3+\cdots)I_1+I_{i+1}-tI_4$$

for $i=1,3$. Also, $A_0(c_i-,f)=I_i$. Further, since $A_k(c_i-,f)=A_k(c_i+,f)$, we have $A_1(c_i-,f)=I_4$ and $A_k(c_i-,f)=I_1$ for $k\geq 2$. Therefore $\varepsilon_0(c_i-,f)=1$, $\varepsilon_1(c_i-,f)=-1$, and $\varepsilon_k(c_i-,f)=1$ for $k\geq 2$. Hence $\theta_0(c_i-,f)=I_i$, $\theta_1(c_i-,f)=I_4$, and $\theta_k(c_i-,f)=-I_1$ for $k\geq 2$. This implies that

$$\theta(c_i-,f;t)=I_i+I_4t-I_1t^2-I_1t^3-\cdots=(-t^2-t^3-\cdots)I_1+I_i+tI_4,$$

and therefore

$$\nu(c_i,f;t)=(I_{i+1}-I_i)-2I_4t+2I_1t^2+2I_1t^3\cdots=(2t^2+2t^3+\cdots)I_1-I_i+I_{i+1}-2tI_4$$

for $i=1,3$. By a similar argument as above, we obtain

$$\nu(c_2,f;t)=(2t+2t^2+\cdots)I_1-I_i+I_{i+1}.$$

Thus

$$N(f;t)=\begin{bmatrix}-1+2t^2+2t^3+\cdots & 1 & 0 & -2t\\ 2t+2t^2+\cdots & -1 & 1 & 0\\ 2t^2+2t^3+\cdots & 0 & -1 & 1-2t\end{bmatrix}_{3\times 4},$$

and hence $D(f;t)=(-1)^{1+1}(1-t)^{-1}(1-4t)=1-3t-3t^2-\cdots$.

Let $g\in\mathscr{M}([0,1])$ be defined as in Example 1.1.5. Then f and g are topologically conjugates, where a conjugacy h is given by

$$h(x)=\frac{2}{\pi}\arcsin(\sqrt{1-x}),\quad \forall x\in[0,1].$$

Also, as seen in Example 1.1.5, $N(g;t)$ is given by (1.1.4) and $D(g;t) = 1-3t-3t^2-\cdots$. Clearly $D(f;t) = D(g;t)$, and an easy computation shows that $N(g;t) = -S_3 N(f;t) S_4$.

3.2 SOME CONSEQUENCES

In this section, we discuss two immediate consequences of Theorem 3.1.7.

3.2.1 Reduction of Computational Complexity

The kneading matrix of a map f in $\mathscr{M}(I)$ captures crucial dynamical information of all the iterates of f. Moreover, by (1.1.2), it follows that f satisfies the ascending relation (1.2.3). Therefore the 'complexity' of the behaviour of the iterates of f increases with the increase in the order of iteration. So, in general, the process of finding the kneading matrix of a piecewise monotone map involves tedious computations. However, Theorem 3.1.7 is very effective in reducing this computational complexity to a reasonable extent. More precisely, if

$$\mathscr{N}_{\nearrow} = \{N(f;t) : \#T(f) \text{ is odd and } f \nearrow I_1\}$$

and

$$\mathscr{N}_{\searrow} = \{N(g;t) : \#T(g) \text{ is odd and } g \searrow J_1\},$$

then from Theorem 3.1.7 and the following corollary it follows that any one of the two sets $\mathscr{N}_{\nearrow}$ and $\mathscr{N}_{\searrow}$ completely determines the other.

Corollary 3.2.1. *If f satisfy that $\#T(f)$ is odd and $f \nearrow I_1$, then there exists g such that $g \searrow J_1$ and g is topologically conjugate to f.*

Proof. Let $\#T(f) = m$ such that m is a positive odd integer. Let $g := h \circ f \circ h^{-1}$, where $h : I \to J$ is the orientation-reversing homeomorphism defined by

$$h(x) = \frac{c-d}{b-a}x + \frac{bd-ac}{b-a}, \quad \forall x \in I.$$

Then $g \in \mathscr{M}(J)$ and g is h-conjugate to f. Since $f \nearrow I_1$, we have $\varepsilon(I_1) = +1$, implying by result **(i)** of Lemma 3.1.5 that $\varepsilon(J_{m+1}) = +1$. Then $\varepsilon(J_j) = +1$ for every even $j \in \{1,2,\ldots,m+1\}$, because $m+1$ is even. This implies that $\varepsilon(J_1) = -1$, and hence $g \searrow J_1$. □

3.2.2 Classification up to Topological Conjugacy

As seen in section 3.1, Theorems 3.1.2 and 3.1.7 together imply that any $f \in \mathscr{M}(I)$ and $g \in \mathscr{M}(J)$ such that $\#T(f) = \#T(g) = m$ are not topologically conjugates whenever neither of the relations $N(f;t) = N(g;t)$ and $N(f;t) = -S_m N(g;t) S_{m+1}$ are satisfied. This provides a combinatorial approach to prove the nonexistence of topological conjugacy and thereby helps to classify the dynamical systems up to topological conjugacy. As an illustration, we have the following.

Example 3.2.2. Consider the maps $f, g : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 2.8x + 0.3 & \text{if} \quad 0 \le x \le 0.25, \\ -2.8x + 1.7 & \text{if} \quad 0.25 \le x \le 0.5, \\ 7x - 3.2 & \text{if} \quad 0.5 \le x \le 0.6, \\ -1.125x + 1.675 & \text{if} \quad 0.6 \le x \le 1, \end{cases}$$

and

$$g(x) = \begin{cases} -6.5x + 0.65 & \text{if} \quad 0 \le x \le 0.1, \\ 1.5x - 0.15 & \text{if} \quad 0.1 \le x \le 0.6, \\ -2.5x + 2.25 & \text{if} \quad 0.6 \le x \le 0.9, \\ 4x - 3.6 & \text{if} \quad 0.9 \le x \le 1. \end{cases}$$

Then $f, g \in \mathscr{M}([0,1])$ (see Figure 3.1) such that

$$T(f) = \{c_1 = 0.25,\ c_2 = 0.5,\ c_3 = 0.6\}, \ \ T(g) = \{d_1 = 0.1,\ d_2 = 0.6,\ d_3 = 0.9\},$$

$$L(f) = \{I_1 = [0, 0.25],\ I_2 = [0.25, 0.5],\ I_3 = [0.5, 0.6],\ I_4 = [0.6, 1]\}$$

and

$$L(g) = \{J_1 = [0, 0.1],\ J_2 = [0.1, 0.6],\ J_3 = [0.6, 0.9],\ J_4 = [0.9, 1]\}.$$

Claim: $N(g;t) \neq N(f;t)$ and $N(g;t) \neq -S_3 N(f;t) S_4$.

For an indirect proof, assume that $N(g;t) = N(f;t)$ or $N(g;t) = -S_3 N(f;t) S_4$. Then an easy computation shows that

$$N^2_{24}(f;t) = N^2_{24}(g;t) \quad \text{or} \quad N^2_{24}(f;t) = -N^2_{21}(g;t). \tag{3.2.1}$$

Since $f^2(c_2) = f^2(0.5) = 0.86 \in I_4 \setminus T(f)$, we have $A_2(c_2+, f) = A_2(c_2-, f) = I_4$, implying that $N^2_{24}(f;t) \neq 0$. Similarly, since $g^2(d_2) = g^2(0.6) = 0.375 \in J_2 \setminus T(g)$, we get that $A_2(d_2+, g) = A_2(d_2-, g) = J_2$. This implies $N^2_{22}(g;t) \neq 0$, and hence $N^2_{21}(g;t) = N^2_{24}(g;t) = 0$, because $N^2_{2j}(g;t) \neq 0$ for one and only one $j \in \{1,2,3,4\}$. Thus, $N^2_{24}(f;t)$

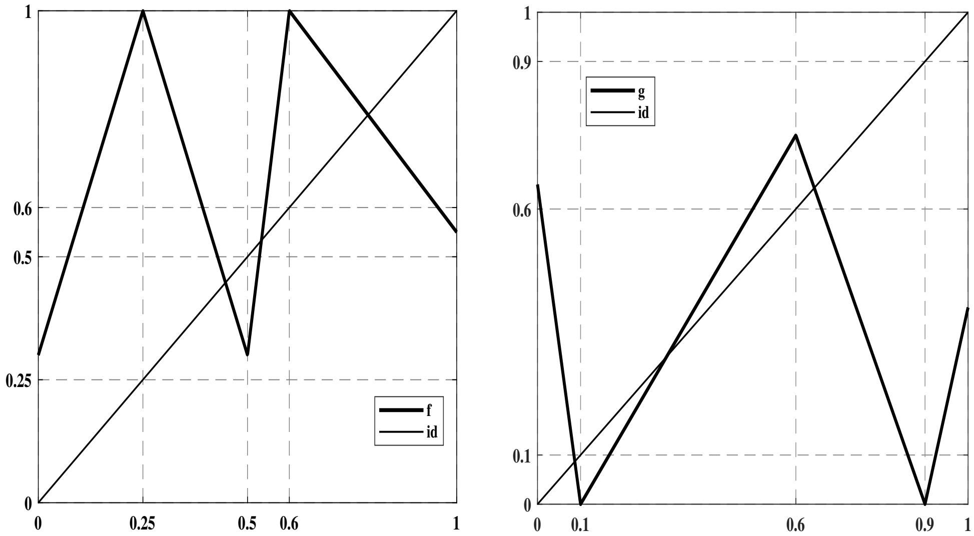

Figure 3.1 Maps f and g

$\neq N^2_{24}(g;t)$ and $N^2_{24}(f;t) \neq -N^2_{21}(g;t)$, a contradiction to (3.2.1). Hence, the claim is proved, which implies by Theorems 3.1.2 and 3.1.7 that f and g are not conjugates.

The above results are for general $f \in \mathscr{M}(I)$. If we consider f to be in a particular subclass of $\mathscr{M}(I)$, we may obtain more details on the dynamics of f, as seen in the following sections.

3.3 THE COMBINATORICS OF TENT-LIKE MAPS

For each $f \in \mathscr{C}(I)$, let $\mathscr{I}_f := \{f^l \mid l \geq 0\}$, the set of iterates of f. As seen in section 1.1.2, the kneading matrix $N(f;t)$ of an $f \in \mathscr{M}(I)$ contains important combinatorial information concerning all the elements of $\mathscr{I}_f$ and hence that of $\mathscr{I}_{f^k}$ for each $k \in \mathbb{N}$, because $\mathscr{I}_{f^k} \subseteq \mathscr{I}_f$. Motivated by this observation, we expect that $N(f^k;t)$ and $N(f;t)$ are related for every $k \in \mathbb{N}$. However, the problem of finding a matrix equation that relates these two matrices is highly non-trivial because the order of these matrices is different, and the problem of computing the kneading matrix of a map is very hard. In this section, we derive matrix equations that relate the kneading matrices of the map and its iterates for the family of *tent-like maps*, i.e., the set

$$\mathscr{M}_0(I) = \left\{f \in \mathscr{M}(I) : f\big(T(f) \cup \{a,b\}\big) \subseteq \{a,b\}\right\},$$

of all chaotic maps in $\mathscr{M}(I)$ each of which is onto on its every lap.

Proposition 3.3.1. **(i)** *If* $f, g \in \mathscr{M}_0(I)$*, then* $f \circ g \in \mathscr{M}_0(I)$*.* **(ii)** *If* $f \in \mathscr{M}_0(I)$*, then* $f^k \in \mathscr{M}_0(I)$ *for each* $k \in \mathbb{N}$*.*

Proof. Given $f, g \in \mathscr{M}_0(I)$, clearly $f \circ g \in \mathscr{M}(I)$ and $(f \circ g)(\{a,b\}) \subseteq \{a,b\}$. Also, if $c \in T(f \circ g)$, then by (1.1.1) we have $c \in T(g)$ or $g(c) \in T(f)$, implying that $(f \circ g)(c) \in \{a,b\}$. Therefore $f \circ g \in \mathscr{M}_0(I)$, proving result **(i)**. Result **(ii)** follows from **(i)** by induction. □

For each $k \in \mathbb{N}$ and $n_1, n_2, \ldots, n_k \in \mathbb{N} \cup \{0\}$, let

$$S(n_1, n_2, \ldots, n_k) := \sum_{j=1}^{k} S_j(n_1, n_2, \ldots, n_k),$$

where

$$S_j(n_1, n_2, \ldots, n_k) := \sum_{1 \le i_1 < i_2 < \cdots < i_j \le k} n_{i_1} n_{i_2} \cdots n_{i_j} \text{ for } 1 \le j \le k.$$

Proposition 3.3.2. **(i)** *If* $f_1, f_2, \ldots, f_k \in \mathscr{M}_0(I)$*, then*

$$\#T(f_1 \circ f_2 \circ \cdots \circ f_k) = S(\#T(f_1), \#T(f_2), \ldots, \#T(f_k)). \tag{3.3.1}$$

(ii) *If* $f \in \mathscr{M}_0(I)$ *such that* $\#T(f) = m$*, then* $\#T(f^k) = (m+1)^k - 1$ *for each* $k \in \mathbb{N}$*.*

(iii) $\#T(f^k) \equiv \#T(f) (\mathrm{mod}\ 2)$ *for each* $f \in \mathscr{M}_0(I)$ *and* $k \in \mathbb{N}$*.*

Proof. We prove result **(i)** by induction on k. For each $f_1 \in \mathscr{M}_0(I)$, we have $S(\#T(f_1)) = S_1(\#T(f_1)) = \#T(f_1)$, and therefore (3.3.1) is true for $k = 1$.

To prove it for $k = 2$, consider arbitrary $f_1, f_2 \in \mathscr{M}_0(I)$ such that $\#T(f_1) = m_1$ and $\#T(f_2) = m_2$. If $m_1 = m_2 = 0$, then $f_1 \circ f_2$ is strictly monotone on I, implying that

$$\#T(f_1 \circ f_2) = 0 = S(0,0) = S(m_1, m_2).$$

If $m_1 = 0$ and $m_2 \neq 0$, then by using (1.1.1) for f_1 and f_2, we have $T(f_1 \circ f_2) = T(f_2)$, and hence $\#T(f_1 \circ f_2) = m_2 = S(0, m_2) = S(m_1, m_2)$. If $m_1 \neq 0$ and $m_2 = 0$, then again by using (1.1.1) for f_1 and f_2, we obtain

$$T(f_1 \circ f_2) = f_2^{-1}(T(f_1)) \cap (a,b).$$

Also, since f_2 is strictly monotone on I, we get that

$$\#\{c :\ c \in f_2^{-1}(T(f_1)) \cap (a,b)\} = \#T(f_1).$$

Therefore $\#T(f_1 \circ f_2) = \#T(f_1) = m_1 = S(m_1, 0) = S(m_1, m_2)$.

Now, let both m_1 and m_2 be non-zero. Let $T(f_1) = \{c_1, c_2, \ldots, c_{m_1}\}$, $T(f_2) = \{d_1, d_2, \ldots, d_{m_2}\}$, $L(f_1) = \{I_1, I_2, \ldots, I_{m_1+1}\}$ and $L(f_2) = \{J_1, J_2, \ldots, J_{m_2+1}\}$, where $a = c_0 < c_1 < \cdots < c_{m_1} < c_{m_1+1} = b$, $I_j = [c_{j-1}, c_j]$ for $1 \le j \le m_1 + 1$, $a = d_0 < d_1 < d_2 < \cdots < d_{m_2} < d_{m_2+1} = b$ and $J_i = [d_{i-1}, d_i]$ for $1 \le i \le m_2 + 1$. Since $f_2(T(f_2)) \subseteq \{a, b\}$, by using (1.1.1) for f_1 and f_2, we have

$$T(f_1 \circ f_2) = T(f_2) \bigsqcup \left(\bigsqcup_{j=0}^{m_2} (f_2^{-1}(T(f_1)) \cap (d_j, d_{j+1})) \right), \tag{3.3.2}$$

where $\sqcup$ indicates that the union is disjoint. Also, since f_2 is strictly monotone on (d_j, d_{j+1}), there exists unique $p_i \in (d_j, d_{j+1})$ such that $f_2(p_i) = c_i$ for each $0 \le j \le m_2$ and $1 \le i \le m_1$,. That is, $f_2^{-1}(c_i) \cap (d_j, d_{j+1})$ is a singleton set for all $1 \le i \le m_1$ and $0 \le j \le m_2$. Therefore, from (3.3.2) we have

$$\begin{aligned} \#T(f_1 \circ f_2) &= \#T(f_2) + \sum_{j=0}^{m_2} \#\{d : d \in f_2^{-1}(T(f_1)) \cap (d_j, d_{j+1})\} \\ &= m_2 + \sum_{j=0}^{m_2} \sum_{i=1}^{m_1} \#\{d : d \in f_2^{-1}(c_i) \cap (d_j, d_{j+1})\} \\ &= m_2 + \sum_{j=0}^{m_2} \sum_{i=1}^{m_1} 1 = m_2 + m_1(m_2 + 1) = S(m_1, m_2). \end{aligned}$$

proving (3.3.1) for $k = 2$. Next, suppose that (3.3.1) true for certain $k \ge 2$ and consider arbitrary $f_1, f_2, \ldots, f_{k+1}$ in $\mathscr{M}_0(I)$ such that $\#T(f_j) = m_j$ for $1 \le j \le k + 1$. Let $g := f_1 \circ f_2 \circ \cdots \circ f_k$. Then, by using (3.3.1) for the case $k = 2$, we have

$$\begin{aligned} \#T(g \circ f_{k+1}) = S(\#T(g), m_{k+1}) &= S_1(\#T(g), m_{k+1}) + S_2(\#T(g), m_{k+1}) \\ &= \#T(g) + m_{k+1} + \#T(g) \cdot m_{k+1}. \end{aligned} \tag{3.3.3}$$

By induction hypothesis, $\#T(g) = S(m_1, m_2, \ldots, m_k)$. Therefore, by (3.3.3) we have

$$\#T(g \circ f_{k+1}) = S(m_1, m_2, \ldots, m_k) + m_{k+1} + S(m_1, m_2, \ldots, m_k) m_{k+1}. \tag{3.3.4}$$

Now,

$$S_1(m_1, m_2, \ldots, m_{k+1}) = S_1(m_1, m_2, \ldots, m_k) + m_{k+1}, \tag{3.3.5}$$

$$S_{k+1}(m_1, m_2, \ldots, m_{k+1}) = S_k(m_1, m_2, \ldots, m_k) m_{k+1}, \tag{3.3.6}$$

$$S_j(m_1, m_2, \ldots, m_{k+1}) = S_j(m_1, m_2, \ldots, m_k) + S_{j-1}(m_1, m_2, \ldots, m_k) m_{k+1} \tag{3.3.7}$$

for $2 \le j \le k$. Therefore by adding the equations (3.3.5), (3.3.6) and (3.3.7), on simplification, we obtain

$$\begin{aligned} S(m_1, m_2, \ldots, m_{k+1}) &= S(m_1, m_2, \ldots, m_k) + m_{k+1} + S(m_1, m_2, \ldots, m_k) m_{k+1} \\ &= \#T(g \circ f_{k+1}) \qquad \text{(by using (3.3.4))} \\ &= \#T(f_1 \circ f_2 \circ \cdots \circ f_{k+1}). \end{aligned}$$

Thus, (3.3.1) is true for $k+1$, and therefore by induction it is true for every $k \in \mathbb{N}$. This proves result **(i)**.

In order to prove result **(ii)**, consider an arbitrary $f \in \mathscr{M}_0(I)$ such that $\#T(f) = m$ and let $k \in \mathbb{N}$. Set $m_j = m$ for $1 \le j \le k$. Then

$$S_j(m_1, m_2, \ldots, m_k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le k} m^j = \binom{k}{j} m^j$$

for $1 \le j \le k$, implying by result **(i)** that

$$\#T(f^k) = S(m_1, m_2, \ldots, m_k) = \sum_{j=1}^{k} \binom{k}{j} m^j = (m+1)^k - 1.$$

Result **(iii)** follows from **(ii)** by observing that $(m+1)^k - 1 \equiv m (\text{mod } 2)$. □

We now introduce some particular subsets of $\mathscr{M}_0(I)$. Let

$$\begin{aligned} \mathscr{M}_{\nearrow}(I) &:= \{f \in \mathscr{M}_0(I) : T(f) = \emptyset, f(a) = a \text{ and } f(b) = b\}, \\ \mathscr{M}_{\searrow}(I) &:= \{f \in \mathscr{M}_0(I) : T(f) = \emptyset, f(a) = b \text{ and } f(b) = a\}, \\ \mathscr{M}_{\wedge}(I) &:= \{f \in \mathscr{M}_0(I) : f \text{ is unimodal and } f(a) = f(b) = a\}, \\ \mathscr{M}_{\vee}(I) &:= \{f \in \mathscr{M}_0(I) : f \text{ is unimodal and } f(a) = f(b) = b\}, \\ \mathscr{M}_{\mathrm{N}}(I) &:= \{f \in \mathscr{M}_0(I) : T(f) \neq \emptyset, f(a) = a \text{ and } f(b) = b\}, \\ \mathscr{M}_{\text{И}}(I) &:= \{f \in \mathscr{M}_0(I) : T(f) \neq \emptyset, f(a) = b \text{ and } f(b) = a\}, \\ \mathscr{M}_{\mathrm{M}}(I) &:= \{f \in \mathscr{M}_0(I) : T(f) \neq \emptyset \text{ and } f(a) = f(b) = a\}, \\ \mathscr{M}_{\mathrm{W}}(I) &:= \{f \in \mathscr{M}_0(I) : T(f) \neq \emptyset \text{ and } f(a) = f(b) = b\}. \end{aligned}$$

Then $\mathscr{M}_0(I)$ is indeed the disjoint union of $\mathscr{M}_{\nearrow}(I)$, $\mathscr{M}_{\searrow}(I)$, $\mathscr{M}_{\mathrm{N}}(I)$, $\mathscr{M}_{\text{И}}(I)$, $\mathscr{M}_{\mathrm{M}}(I)$ and $\mathscr{M}_{\mathrm{W}}(I)$.

Proposition 3.3.3. **(i)** *If $f, g \in \mathscr{M}_N(I)$, then $f \circ g \in \mathscr{M}_N(I)$. This is also true when $\mathscr{M}_N(I)$ is replaced by $\mathscr{M}_{\nearrow}(I)$, $\mathscr{M}_M(I)$ and $\mathscr{M}_W(I)$.*

(ii) *If $f^k \in \mathscr{M}_0(I)$ for some $k \in \mathbb{N}$, then $f \in \mathscr{M}_0(I)$. This is also true when $\mathscr{M}_0(I)$ is replaced by $\mathscr{M}_M(I)$ and $\mathscr{M}_W(I)$.*

Proof. Consider arbitrary $f, g \in \mathscr{M}_N(I)$. Then by result **(i)** of Proposition 3.3.1, we have $f \circ g \in \mathscr{M}_0(I)$. Also, $f(a) = g(a) = a$ and $f(b) = g(b) = b$, implying that $(f \circ g)(a) = a$ and $(f \circ g)(b) = b$. Therefore $f \circ g \in \mathscr{M}_N(I)$, proving result **(i)** for $\mathscr{M}_N(I)$. The proofs for $\mathscr{M}_{\nearrow}(I), \mathscr{M}_M(I)$, and $\mathscr{M}_W(I)$ are similar.

In order to prove result **(ii)**, consider an arbitrary $f \in \mathscr{C}(I)$ such that $f^k \in \mathscr{M}_0(I)$ for some $k \geq 2$. Then, clearly $f \in \mathscr{M}(I)$. We discuss in the two cases.

Case (a): Suppose that $\#T(f^k) = 0$. Then f is strictly monotone on I. Also, since f^k is onto on I, so is f. Therefore $f(\{a,b\}) \subseteq \{a,b\}$, and hence $f \in \mathscr{M}_0(I)$.

Case (b): Suppose that $\#T(f^k) \neq 0$. Since f^{k-1} is onto, there exists $u \in I$ such that $f^{k-1}(u) = a$. If $u = a$, then $f(a) = f(f^{k-1}(a)) = f^k(a) \in \{a,b\}$. If $u = b$, then $f(a) = f(f^{k-1}(b)) = f^k(b) \in \{a,b\}$. If $u \in (a,b)$, then $u \in T(f^{k-1}) \subseteq T(f^k)$, implying that $f(a) = f(f^{k-1}(u)) = f^k(u) \in \{a,b\}$. Thus $f(a) \in \{a,b\}$. By a similar argument, it follows that $f(b) \in \{a,b\}$.

Next, consider an arbitrary $w \in T(f)$. Since f^{k-1} is onto, there exists $v \in I$ such that $f^{k-1}(v) = w$, implying that $v \in f^{-(k-1)}(w) \subseteq f^{-(k-1)}(T(f)) \subseteq T(f^k)$. Then $f(w) = f(f^{k-1}(v)) = f^k(v) \in \{a,b\}$. Therefore $f \in \mathscr{M}_0(I)$. □

For each $m \in \mathbb{N} \cup \{0\}$, let $\mathscr{M}_{M,m}(I) := \{f \in \mathscr{M}_M(I) \ : \ \#T(f) = m\}$ and $\mathscr{M}_{W,m}(I)$, $\mathscr{M}_{N,m}(I)$, $\mathscr{M}_{\text{И},m}(I)$ be defined similarly.

Lemma 3.3.4. *The kneading matrix $N(f;t)$ is independent of the choice of f in $\mathscr{M}_{M,m}(I)$ for each $m \in \mathbb{N}$. This is also true if $\mathscr{M}_{M,m}(I)$ is replaced by $\mathscr{M}_{W,m}(I)$, $\mathscr{M}_{N,m}(I)$ and $\mathscr{M}_{\text{И},m}(I)$.*

Proof. Let $m \in \mathbb{N}$ and $f \in \mathscr{M}_{M,m}(I)$. Then

$$f(c_i) = \begin{cases} b & \text{if} \quad i \in \{1,3,\ldots,m\}, \\ a & \text{if} \quad i \in \{2,4,\ldots,m-1\}, \end{cases}$$

and

$$\varepsilon(I_j) = \begin{cases} +1 & \text{for} \quad j \in \{1,3,5,\ldots,m\}, \\ -1 & \text{for} \quad j \in \{2,4,6,\ldots,m+1\}. \end{cases} \tag{3.3.8}$$

Also, since $f(a) = f(b) = a$, we have

$$f^k(c_i) = \begin{cases} a & \text{if} \quad i \in \{1,3,\ldots,m\} \ \text{ and } \ k \geq 2, \\ a & \text{if} \quad i \in \{2,4,\ldots,m-1\} \ \text{ and } \ k \geq 1. \end{cases} \tag{3.3.9}$$

Let $i \in \{2,4,6,\ldots,m-1\}$ be arbitrary. Since $A_0(c_i+,f) = I_{i+1}$ and $A_k(c_i+,f) = I_1$ for $k \geq 1$, by (3.3.8) we have $\varepsilon_k(c_i+,f) = 1$ for $k \geq 0$, implying that $\theta_0(c_i+,f) = I_{i+1}$ and $\theta_k(c_i+,f) = I_1$ for $k \geq 1$. Therefore

$$\theta(c_i+,f;t) = I_{i+1} + I_1 t + I_1 t^2 + I_1 t^3 + \cdots = (t + t^2 + t^3 + \cdots) I_1 + I_{i+1}.$$

Also, $A_0(c_i-,f) = I_i$, and since $A_k(c_i-,f) = A_k(c_i+,f)$, we have $A_k(c_i-,f) = I_1$ for $k \geq 1$. Therefore, by (3.3.8) we get $\varepsilon_0(c_i-,f) = -1$ and $\varepsilon_k(c_i-,f) = 1$ for $k \geq 1$, which implies that $\theta_0(c_i-,f) = I_i$ and $\theta_k(c_i-,f) = -I_1$ for $k \geq 1$. Hence

$$\theta(c_i-,f;t) = I_i - I_1 t - I_1 t^2 - I_1 t^3 - \cdots = (-t - t^2 - t^3 - \cdots) I_1 + I_i,$$

and thus

$$\begin{aligned}
\nu(c_i,f;t) &= \theta(c_i+,f;t) - \theta(c_i-,f;t) \\
&= (I_{i+1} + I_1 t + I_1 t^2 + \cdots) - (I_i - I_1 t - I_1 t^2 - \cdots) \\
&= (I_{i+1} - I_i) + 2I_1 t + 2I_1 t^2 + \cdots \\
&= (2t + 2t^2 + \cdots) I_1 - I_i + I_{i+1}.
\end{aligned}$$

By a similar argument as above, we obtain $\nu(c_i,f;t) = (2t^2 + 2t^3 + \cdots) I_1 - I_i + I_{i+1} - 2t I_{m+1}$ for each $i \in \{1,3,5,\ldots,m\}$. Therefore the kneading matrix of f is

$$N(f;t) = \begin{bmatrix}
-1 + 2t^2 + 2t^3 + \cdots & & -2t \\
2t + 2t^2 + \cdots & & 0 \\
2t^2 + 2t^3 + \cdots & & -2t \\
2t + 2t^2 + \cdots & & 0 \\
\vdots & M_m & \vdots \\
2t + 2t^2 + \cdots & & 0 \\
2t^2 + 2t^3 + \cdots & & 1 - 2t
\end{bmatrix}_{m \times (m+1)}, \tag{3.3.10}$$

where M_m is the $m \times (m-1)$ matrix obtained from $N_0(f;t)$ by deleting its first and last columns. Since $f \in \mathscr{M}_{M,m}(I)$ was arbitrary, (3.3.10) is true for every $f \in \mathscr{M}_{M,m}(I)$. Therefore $N(f;t)$ is independent of the choice of f in $\mathscr{M}_{M,m}(I)$.

Next, to prove the result for $\mathscr{M}_{W,m}(I)$, consider an arbitrary $f \in \mathscr{M}_{W,m}(I)$. Let $g := h \circ f \circ h^{-1}$, where $h : I \to I$ is the orientation-reversing homeomorphism defined by $h(x) = a + b - x$. Then $g \in \mathscr{M}(I)$ and g is topologically h-conjugate to f, implying by Lemma 3.1.3 that $\#T(g) = m$. Also, since $f(a) = f(b) = a$, $h(a) = h^{-1}(a) = b$ and $h(b) = h^{-1}(b) = a$, we have $g(\{a,b\}) \subseteq \{a,b\}$ such that $g(a) = g(b) = a$. Further, as seen in the proof of Lemma 3.1.3, we have $T(g) = h(T(f))$, and therefore $g(T(g)) \subseteq \{a,b\}$. Hence $g \in \mathscr{M}_{M,m}(I)$, and thus by the result for $\mathscr{M}_{M,m}(I)$, it follows that $N(g;t)$ is equal to the matrix on the right hand side of (3.3.10). Since f is h-conjugate to g, by Theorem 3.1.7, we have $N(g;t) = -S_m N(f;t) S_{m+1}$. Therefore the kneading matrix of f is

$$\begin{aligned} N(f;t) &= -S_m N(g;t) S_{m+1} \\ &= \begin{bmatrix} -1+2t & & -2t^2-2t^3-\cdots \\ 0 & & -2t-2t^2-\cdots \\ 2t & & -2t^2-2t^3-\cdots \\ 0 & & -2t-2t^2-\cdots \\ \vdots & M_m & \vdots \\ 0 & & -2t-2t^2-\cdots \\ 2t & & 1-2t^2-2t^3-\cdots \end{bmatrix}_{m\times(m+1)} . \end{aligned} \tag{3.3.11}$$

Since $f \in \mathscr{M}_{W,m}(I)$ was arbitrary, (3.3.11) is true for every $f \in \mathscr{M}_{W,m}(I)$. The proofs for $\mathscr{M}_{N,m}(I)$ and $\mathscr{M}_{\text{И},m}(I)$ are similar to that for $\mathscr{M}_{M,m}(I)$. In fact, it follows that

$$N(f;t) = \begin{bmatrix} -1 & & -2t-2t^2-2t^3-\cdots \\ 2t+2t^2+\cdots & & 0 \\ 0 & & -2t-2t^2-2t^3-\cdots \\ 2t+2t^2+\cdots & & 0 \\ \vdots & M_m & \vdots \\ 0 & & -2t-2t^2-2t^3-\cdots \\ 2t+2t^2+\cdots & & 1 \end{bmatrix}_{m\times(m+1)} \tag{3.3.12}$$

whenever $f \in \mathscr{M}_{N,m}(I)$ and

$$N(f;t) = \begin{bmatrix} -1+2t+2t^3+\cdots & & -2t^2-2t^4-\cdots \\ 2t^2+2t^4+\cdots & & -2t-2t^3-\cdots \\ 2t+2t^3+\cdots & & -2t^2-2t^4-\cdots \\ 2t^2+2t^4+\cdots & & -2t-2t^3-\cdots \\ \vdots & M_m & \vdots \\ 2t+2t^3+\cdots & & -2t^2-2t^4-\cdots \\ 2t^2+2t^4+\cdots & & 1-2t-2t^3-\cdots \end{bmatrix}_{m\times(m+1)} \tag{3.3.13}$$

whenever $f \in \mathscr{M}_{\text{И},m}(I)$. □

Parity	$f \in$	$g \in$	$f \circ g \in$	$g \circ f \in$	$N(f \circ g;t)$	$N(g \circ f;t)$	Conclusion
	$\mathscr{M}_{M,m_1}(I)$	$\mathscr{M}_{N,m_2}(I)$	$\mathscr{M}_{M,m}(I)$	$\mathscr{M}_{M,m}(I)$	$N_{M,m}(t)$	$N_{M,m}(t)$	$(*)$
m_1 odd	$\mathscr{M}_{M,m_1}(I)$	$\mathscr{M}_{\text{И},m_2}(I)$	$\mathscr{M}_{M,m}(I)$	$\mathscr{M}_{W,m}(I)$	$N_{M,m}(t)$	$N_{W,m}(t)$	$(**)$
m_2 even	$\mathscr{M}_{W,m_1}(I)$	$\mathscr{M}_{N,m_2}(I)$	$\mathscr{M}_{W,m}(I)$	$\mathscr{M}_{W,m}(I)$	$N_{W,m}(t)$	$N_{W,m}(t)$	$(*)$
	$\mathscr{M}_{W,m_1}(I)$	$\mathscr{M}_{\text{И},m_2}(I)$	$\mathscr{M}_{W,m}(I)$	$\mathscr{M}_{M,m}(I)$	$N_{W,m}(t)$	$N_{M,m}(t)$	$(**)$
	$\mathscr{M}_{N,m_1}(I)$	$\mathscr{M}_{N,m_2}(I)$	$\mathscr{M}_{N,m}(I)$	$\mathscr{M}_{N,m}(I)$	$N_{N,m}(t)$	$N_{N,m}(t)$	$(*)$
m_1 even	$\mathscr{M}_{N,m_1}(I)$	$\mathscr{M}_{\text{И},m_2}(I)$	$\mathscr{M}_{\text{И},m}(I)$	$\mathscr{M}_{\text{И},m}(I)$	$N_{\text{И},m}(t)$	$N_{\text{И},m}(t)$	$(*)$
m_2 even	$\mathscr{M}_{\text{И},m_1}(I)$	$\mathscr{M}_{\text{И},m_2}(I)$	$\mathscr{M}_{N,m}(I)$	$\mathscr{M}_{N,m}(I)$	$N_{N,m}(t)$	$N_{N,m}(t)$	$(*)$

Table 3.1 Comparison of $N(f \circ g;t)$ and $N(g \circ f;t)$

An easy computation shows that $N(f;t) = -S_m N(g;t) S_{m+1}$ whenever $f \in \mathscr{M}_{M,m}(I)$ and $g \in \mathscr{M}_{W,m}(I)$. However, this relation is not true when $\mathscr{M}_{M,m}(I)$ and $\mathscr{M}_{W,m}(I)$ are replaced by $\mathscr{M}_{N,m}(I)$ and $\mathscr{M}_{\text{И},m}(I)$, respectively. In fact, it follows from Corollary 3.1.6 that f is not conjugate to g whenever $f \in \mathscr{M}_{N,m}(I)$ and $g \in \mathscr{M}_{\text{И},m}(I)$.

For each $m \in \mathbb{N}$, let $N_{M,m}(t) := N(f;t)$ for some $f \in \mathscr{M}_{M,m}(I)$. The matrices $N_{W,m}(t)$, $N_{N,m}(t)$ and $N_{\text{И},m}(t)$ are defined similarly. Although any two arbitrary $f, g \in \mathscr{M}_0(I)$ do not commute in general, the kneading matrices of composite maps $f \circ g$ and $g \circ f$ are related as described in the following.

Theorem 3.3.5. *If $f, g \in \mathscr{M}_0(I)$, then $N(f \circ g;t) = N(g \circ f;t)$ or $N(f \circ g;t) = -S_m N(g \circ f;t) S_{m+1}$ for some $m \in \mathbb{N}$.*

Proof. Consider arbitrary $f, g \in \mathscr{M}_0(I)$. Without loss of generality, we assume that either $T(f) \neq \emptyset$ or $T(g) \neq \emptyset$. Let $\#T(f) = m_1$ and $\#T(g) = m_2$ such that $m_1, m_2 \geq 0$, but not both zero. Since $S(m_1, m_2) = S(m_2, m_1)$, by result **(i)** of Proposition 3.3.2, we have $\#T(f \circ g) = \#T(g \circ f)$. Let this common number be m.

Suppose that both m_1 and m_2 are odd. Then it suffices to consider the following cases.

Case (a): If $f \in \mathscr{M}_{M,m_1}(I)$ and $g \in \mathscr{M}_{M,m_2}(I)$, then $f \circ g,\ g \circ f \in \mathscr{M}_{M,m}(I)$, and hence by Lemma 3.3.4, $N(f \circ g;t) = N_{M,m}(t) = N(g \circ f;t)$.

Case (b): If $f \in \mathscr{M}_{M,m_1}(I)$ and $g \in \mathscr{M}_{W,m_2}(I)$, then $f \circ g \in \mathscr{M}_{M,m}(I)$ and $g \circ f \in \mathscr{M}_{W,m}(I)$. Therefore, by Lemma 3.3.4, $N(f \circ g;t) = N_{M,m}(t)$ and $N(g \circ f;t) = N_{W,m}(t)$. This implies $N(f \circ g;t) = N_{M,m}(t) = -S_m N_{W,m}(t) S_{m+1} = -S_m N(g \circ f;t) S_{m+1}$.

Case (c): If $f \in \mathscr{M}_{W,m_1}(I)$ and $g \in \mathscr{M}_{W,m_2}(I)$, then $f \circ g,\ g \circ f \in \mathscr{M}_{W,m}(I)$. So, again by Lemma 3.3.4, $N(f \circ g;t) = N_{W,m}(t) = N(g \circ f;t)$.

Remaining instances for the parities of m_1 and m_2, and the corresponding cases can be discussed similarly. A summary of the premises and the corresponding conclusions is given in Table 3.1, where $(*)$ and $(**)$ denote the equations $N(f \circ g;t) = N(g \circ f;t)$ and $N(f \circ g;t) = -S_m N(g \circ f;t) S_{m+1}$, respectively. □

Corollary 3.3.6. *$D(f \circ g;t) = D(g \circ f;t)$ for every $f, g \in \mathscr{M}_0(I)$.*

Proof. Since $f, g \in \mathscr{M}_0(I)$, by Theorems 3.3.5, we have $N(f \circ g;t) = N(g \circ f;t)$ or $N(f \circ g;t) = -S_m N(g \circ f;t) S_{m+1}$ for some $m \in \mathbb{N}$. In the first case, the equality $D(f \circ g;t) = D(g \circ f;t)$ follows from the definition of kneading determinant, while in the second, it follows from Lemma 3.1.8. □

3.4 RELATION BETWEEN $N(f^k;t)$ AND $N(f;t)$

Although we aim to describe a relation between $N(f^k;t)$ and $N(f;t)$, in view of the relation $N(f;t) = N_0(f;t) + M(f;t)$, where $N_0(f;t)$ is independent of choice of f, it suffices to describe a relation between $M(f^k;t)$ and $M(f;t)$. So in what follows, we prove results for $M(f;t)$ instead of $N(f;t)$.

For $k, l \geq 1$, let e_k denote the matrix $[0, 0, \ldots, 0, 1]_{1\times k}$, $\mathbb{I}_k$ the identity matrix of order k, $\mathbb{O}_{k\times l}$ the zero matrix of order $k \times l$, and $R_{k\times l}$ the $k \times l$ matrix $[r_{ij}]$ defined by

$$r_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \quad \text{or} \quad i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

For each positive even integer k, let $\mathfrak{I}_k$ denote the transpose of $[\mathbb{I}_2 \ \mathbb{I}_2 \ \cdots \mathbb{I}_2]_{2\times k}$. As defined in Preston (1988), an $f \in \mathscr{M}(I)$ is said to be *uniformly piecewise linear* if it is linear on each of its laps with slope $\pm\alpha$ for some positive real α. For $k \geq 1$, let $f_{N,k}$, $f_{M,k}$, $f_{W,k}$ and $f_{\text{И},k}$ be the uniformly piecewise linear maps in $\mathscr{M}_{N,k}(I)$, $\mathscr{M}_{M,k}(I)$, $\mathscr{M}_{W,k}(I)$ and $\mathscr{M}_{\text{И},k}(I)$, respectively. The following theorem describe a relation between kneading matrices of elements of $\mathscr{M}_0(I)$ with that of bimodal or trimodal uniformly piecewise linear maps, whose dynamical properties are relatively easy to investigate.

Theorem 3.4.1. **(i)** *If $f \in \mathscr{M}_{M,m}(I)$, then*

$$M(f;t) = \begin{bmatrix} \mathfrak{I}_{m-1} & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} M(f_{M,3};t) R_{4\times(m+1)}. \tag{3.4.1}$$

(ii) *If $f \in \mathscr{M}_{W,m}(I)$, then*

$$M(f;t) = \begin{bmatrix} \mathfrak{I}_{m-1} & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} M(f_{W,3};t) R_{4\times(m+1)}. \tag{3.4.2}$$

(iii) *If* $f \in \mathscr{M}_{N,m}(I)$*, then* $M(f;t) = \mathfrak{I}_m M(f_{N,2};t) R_{3\times(m+1)}$.

(iv) *If* $f \in \mathscr{M}_{\mathsf{V\!\!\Lambda},m}(I)$*, then* $M(f;t) = \mathfrak{I}_m M(f_{\mathsf{V\!\!\Lambda},2};t) R_{3\times(m+1)}$.

Proof. Consider an arbitrary $f \in \mathscr{M}_{M,m}(I)$. Since $f_{M,3} \in \mathscr{M}_{M,3}(I)$, by (3.3.10) we have

$$M(f_{M,3};t) = \begin{bmatrix} 2t^2+2t^3+\cdots & 0 & 0 & -2t \\ 2t+2t^2+\cdots & 0 & 0 & 0 \\ 2t^2+2t^3+\cdots & 0 & 0 & -2t \end{bmatrix}_{3\times 4}.$$

Let

$$A = \begin{bmatrix} 2t^2+2t^3+\cdots \\ 2t+2t^2+\cdots \end{bmatrix}_{2\times 1} \quad \text{and} \quad B = \begin{bmatrix} -2t \\ 0 \end{bmatrix}_{2\times 1}.$$

Then

$$M(f_{M,3};t) = \begin{bmatrix} A & \mathbb{O}_{2\times 1} & \mathbb{O}_{2\times 1} & B \\ 2t^2+2t^3+\cdots & 0 & 0 & -2t \end{bmatrix}_{3\times 4}.$$

Therefore, by (3.3.10) we have

$$\begin{aligned} M(f;t) &= \begin{bmatrix} A & & B \\ A & & B \\ \vdots & \mathbb{O}_{m\times(m-1)} & \vdots \\ A & & B \\ 2t^2+2t^3+\cdots & & -2t \end{bmatrix}_{m\times(m+1)} \\ &= \begin{bmatrix} \mathbb{I}_2 & \\ \mathbb{I}_2 & \\ \vdots & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{I}_2 & \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix}_{m\times 3} \begin{bmatrix} A & \mathbb{O}_{2\times 1} & \mathbb{O}_{2\times 1} & B \\ 2t^2+2t^3+\cdots & 0 & 0 & -2t \end{bmatrix} R_{4\times(m+1)} \\ &= \begin{bmatrix} \mathfrak{I}_{m-1} & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} M(f_{M,3};t) R_{4\times(m+1)}, \end{aligned}$$

proving (3.4.1).

Next, consider an arbitrary $f \in \mathscr{M}_{W,m}(I)$. Let g and h be the maps defined as in Lemma 3.3.4. Then $g \in \mathscr{M}_{M,m}(I)$ and g is h-conjugate to f. Therefore, by result **(i)**, we have

$$M(g;t) = \begin{bmatrix} \mathfrak{I}_{m-1} & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} M(f_{M,3};t) R_{4\times(m+1)}. \tag{3.4.3}$$

Also, since h is orientation-reversing, by Theorem 3.1.7 we get

$$M(g;t) = -S_m M(f;t) S_{m+1}. \tag{3.4.4}$$

Further, as $f_{M,3}$ is h-conjugate to $f_{W,3}$, again by Theorem 3.1.7 we have

$$M(f_{M,3};t) = -S_3 M(f_{W,3};t) S_4. \tag{3.4.5}$$

Now, by using (3.4.4) and (3.4.5) in (3.4.3), we have

$$S_m M(f;t) S_{m+1} = \begin{bmatrix} \mathfrak{I}_{m-1} & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} S_3 M(f_{W,3};t) S_4 R_{4\times(m+1)},$$

implying that

$$M(f;t) = \begin{bmatrix} 1 & \mathbb{O}_{1\times 2} \\ \mathbb{O}_{(m-1)\times 1} & \mathfrak{I}_{m-1} \end{bmatrix} M(f_{W,3};t) R_{4\times(m+1)}. \tag{3.4.6}$$

But

$$\begin{bmatrix} 1 & \mathbb{O}_{1\times 2} \\ \mathbb{O}_{(m-1)\times 1} & \mathfrak{I}_{m-1} \end{bmatrix} M(f_{W,3};t) = \begin{bmatrix} \mathfrak{I}_{m-1} & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} M(f_{W,3};t).$$

Therefore (3.4.2) follows from (3.4.6). This proves results **(ii)**. The proofs of results **(iii)** and **(iv)** are similar to that of **(i)**. □

We now derive a relation between $M(f^k;t)$ and $M(f;t)$ for maps in $\mathscr{M}_0(I)$.

Theorem 3.4.2. **(i)** *If $f \in \mathscr{M}_{N,m}(I)$, then*

$$M(f^k;t) = [\mathfrak{I}_l \quad \mathbb{O}_{l\times(m-2)}] M(f;t) R_{(m+1)\times(l+1)}, \quad \forall k \geq 1,$$

where $l = (m+1)^k - 1$. This is also true when $\mathscr{M}_{N,m}(I)$ is replaced by $\mathscr{M}_{\Lambda,m}(I)$ and k is a positive odd integer.

(ii) *If $f \in \mathscr{M}_{M,m}(I)$, then*

$$M(f^k;t) = \begin{bmatrix} \mathfrak{I}_{l-1} & \mathbb{O}_{(l-1)\times(m-2)} \\ \mathbb{O}_{1\times 2} & e_{m-2} \end{bmatrix} M(f;t) R_{(m+1)\times(l+1)}, \quad \forall k \geq 1,$$

where $l = (m+1)^k - 2$. This is also true when $\mathscr{M}_{M,m}(I)$ is replaced by $\mathscr{M}_{W,m}(I)$.

Proof. Let $f \in \mathscr{M}_{N,m}(I)$ and $k \in \mathbb{N}$ be arbitrary. Then, by result **(i)** of Proposition 3.3.3, we have $f^k \in \mathscr{M}_{N,m}(I)$. Also, from result **(ii)** of Proposition 3.3.2, we get $\#T(f^k) =$

$(m+1)^k-1$. Therefore $f \in \mathscr{M}_{N,(m+1)^k-1}(I)$. Hence, by Lemma 3.3.4, we have

$$M(f;t) = \begin{bmatrix} A & & B \\ \vdots & \mathbb{O}_{l\times(l-1)} & \vdots \\ A & & B \end{bmatrix}_{l\times(l+1)},$$

where $l = (m+1)^k - 1$,

$$A = \begin{bmatrix} 0 \\ 2t+2t^2+\cdots \end{bmatrix}_{2\times 1} \quad \text{and} \quad B = \begin{bmatrix} -2t-2t^2-2t^3-\cdots \\ 0 \end{bmatrix}_{2\times 1}.$$

Thus

$$\begin{aligned} M(f;t) &= \begin{bmatrix} \mathbb{I}_2 & \\ \vdots & \mathbb{O}_{l\times(m-2)} \\ \mathbb{I}_2 & \end{bmatrix}_{l\times m} \begin{bmatrix} A & & B \\ \vdots & \mathbb{O}_{m\times(m-1)} & \vdots \\ A & & B \end{bmatrix}_{m\times(m+1)} R_{(m+1)\times(l+1)} \\ &= [\mathfrak{I}_l \quad \mathbb{O}_{l\times(m-2)}] M(f;t) R_{(m+1)\times(l+1)}, \end{aligned}$$

proving the first part of result **(i)**. The proofs of the second part and that of result **(ii)** are similar. In fact, by a similar argument as in Theorem 3.4.1, we can indeed prove result **(ii)** for $\mathscr{M}_{W,m}(I)$ from that for $\mathscr{M}_{M,m}(I)$. □

3.5 RELATION BETWEEN $D(f^k;t)$ AND $D(f;t)$

In this section, we derive a relation between $D(f^k;t)$ and $D(f;t)$ for maps in $\mathscr{M}_0(I)$.

Lemma 3.5.1. *Let $m \in \mathbb{N}$. Then*

$$\det(N^{(1)}_{M,m}(t)) = \det(N^{(1)}_{W,m}(t)) = 1-(m+1)t \tag{3.5.1}$$

and

$$\det(N^{(m+1)}_{N,m}(t)) = \det(N^{(m+1)}_{\Lambda,m}(t)) = \frac{1-(m+1)t}{1-t}.$$

Proof. First, by induction on m, we prove the following result **(R)** on determinants. **(R):** *If A_m is a square matrix of order $m \geq 2$ defined by*

$$A_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & a_{1m} \\ -1 & 1 & 0 & \cdots & 0 & 0 & a_{2m} \\ 0 & -1 & 1 & \cdots & 0 & 0 & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & a_{m-1m} \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1+a_{mm} \end{bmatrix}_{m\times m},$$

then $\det A_m = 1 + a_{1m} + a_{2m} + \cdots + a_{mm}$.

We have

$$A_2 = \begin{bmatrix} 1 & a_{12} \\ -1 & 1+a_{22} \end{bmatrix},$$

so that $\det A_2 = 1 + a_{12} + a_{22}$, and therefore the result is true for $m = 2$. Now, let $m > 2$ and suppose that the result is true for $m-1$. Then, by expanding the determinant about first row, we have

$$\det A_m = \det B_{m-1} + (-1)^{m+1} a_{1m} \det U_{m-1}, \tag{3.5.2}$$

where

$$B_{m-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & a_{2m} \\ -1 & 1 & \cdots & 0 & 0 & a_{3m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 & a_{m-1m} \\ 0 & 0 & \cdots & 0 & -1 & 1+a_{mm} \end{bmatrix}_{(m-1)\times(m-1)}$$

and

$$U_{m-1} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}_{(m-1)\times(m-1)}.$$

Note that $\det U_{m-1} = (-1)^{m-1}$. Also, by induction hypothesis, we have $\det B_{m-1} = 1 + a_{2m} + a_{3m} + \cdots + a_{mm}$. Therefore, by (3.5.2), we get

$$
\begin{aligned}
\det A_m &= 1 + a_{2m} + a_{3m} + \cdots + a_{mm} + (-1)^{m+1} a_{1m} (-1)^{m-1} \\
&= 1 + a_{1m} + a_{2m} + \cdots + a_{mm},
\end{aligned}
$$

proving the result for m. Hence, by induction, the result **(R)** is valid for each $m \geq 2$.

We now prove the identity

$$
\det(N_{M,m}^{(1)}(t)) = 1 - (m+1)t, \quad m \geq 1. \tag{3.5.3}
$$

By (3.3.10), we have $N_{M,1}^{(1)}(t) = [1-2t]$, implying that $\det(N_{M,1}^{(1)}(t)) = 1 - 2t$. If $m > 1$, then by (3.3.10) and result **(R)**, we obtain

$$
\begin{aligned}
\det(N_{M,m}^{(1)}(t)) &= 1 + \big((-2t+0) + \cdots ((m-1)/2 \text{ times}) \cdots (-2t+0)\big) - 2t \\
&= 1 - 2t\left(\frac{m-1}{2}\right) - 2t = 1 - (m+1)t.
\end{aligned}
$$

Hence, (3.5.3) is proved. The proofs of other identities are similar. □

Theorem 3.5.2. *Let $m \in \mathbb{N}$. If $f \in \mathscr{M}_{M,m}(I) \cup \mathscr{M}_{W,m}(I) \cup \mathscr{M}_{N,m}(I)$, then*

$$
D(f^k;t) = \frac{1-(m+1)^k t}{1-(m+1)t} D(f;t), \quad \forall k \in \mathbb{N}. \tag{3.5.4}
$$

This is also true when $\mathscr{M}_{N,m}(I)$ is replaced by $\mathscr{M}_{\mathrm{V\!\!\Lambda},m}(I)$ and k is any positive odd integer.

Proof. Let $f \in \mathscr{M}_{M,m}(I)$ and let $k \in \mathbb{N}$ be arbitrary. By definition, we have

$$
D(f;t) = (1-t)^{-1} \det(N^{(1)}(f;t)).
$$

Since $N(f;t) = N_{\mathscr{M},m}(t)$, by (3.5.1) we get that $\det(N^{(1)}(f;t)) = 1 - (m+1)t$. Therefore

$$
D(f;t) = (1-t)^{-1}(1-(m+1)t). \tag{3.5.5}
$$

By Propositions 3.3.3 and 3.3.2, we have $f^k \in \mathscr{M}_{M,(m+1)^k-1}(I)$, implying that $N(f^k;t) = N_{M,(m+1)^k-1}(t)$ and $\varepsilon(I_1') = 1$, where I_1' is the first lap of f^k. Therefore by (3.5.1) we get $\det(N^{(1)}(f^k;t)) = 1 - (m+1)^k t$, and thus

$$
D(f^k;t) = (1-\varepsilon(I_1')t)^{-1} \det(N^{(1)}(f^k;t)) = (1-t)^{-1}(1-(m+1)^k t). \tag{3.5.6}
$$

Then (3.5.4) follows from (3.5.5) and (3.5.6), proving the result for $\mathscr{M}_{M,m}(I)$. The proofs for $\mathscr{M}_{W,m}(I)$, $\mathscr{M}_{N,m}(I)$ and $\mathscr{M}_{\mathrm{V\!\!\Lambda},m}(I)$ are similar. □

3.6 MODIFIED KNEADING MATRIX

As seen in section 1.1.2, the kneading matrix of an $f \in \mathscr{M}(I)$ is defined using only the kneading increments corresponding to the turning points of f. In what follows, we use the 'kneading data' associated with endpoints a and b of I, with suitable one-sided limits, to define a new kneading matrix for f.

Let $\nu(c_0, f;t) := \theta(c_0+, f;t)$ and $\nu(c_{m+1}, f;t) := -\theta(c_{m+1}-, f;t)$ Then the *modified kneading matrix* of f, denoted by $N'(f;t)$, is defined by

$$N'(f;t) = \begin{bmatrix} N'_{01}(f;t) & N'_{02}(f;t) & \cdots & N'_{0,m+1}(f;t) \\ & & N(f;t) & \\ N'_{m+1,1}(f;t) & N'_{m+1,2}(f;t) & \cdots & N'_{m+1,m+1}(f;t) \end{bmatrix}_{(m+2)\times(m+1)},$$

where the entries $N'_{ij}(f;t)$ for $i = 0, m+1$ and $j = 1, 2, \ldots m+1$ are obtained by setting

$$\nu(c_0, f;t) = N'_{01}(f;t)I_1 + N'_{02}(f;t)I_2 + \cdots + N'_{0,m+1}(f;t)I_{m+1}$$

and

$$\nu(c_{m+1}, f;t) = N'_{m+1,1}(f;t)I_1 + N'_{m+1,2}(f;t)I_2 + \cdots + N'_{m+1,m+1}(f;t)I_{m+1}.$$

For $1 \le i \le m+2$, let $N'_{(i)}(f;t)$ denote the $(m+1)\times(m+1)$ matrix obtained by deleting the i^{th} row of $N'(f;t)$.

Theorem 3.6.1. **(i)** *If $f \in \mathscr{M}_{M,m}(I) \cup \mathscr{M}_{W,m}(I)$, then*

$$D(f;t) = \det N'_{(i)}(f;t), \quad i = 1, m+2. \tag{3.6.1}$$

(ii) *If $f \in \mathscr{M}_{N,m}(I) \cup \mathscr{M}_{\mathrm{V\!\!\Lambda},m}(I)$, then*

$$D(f;t) = (-1)^i \det N'_{(i)}(f;t), \quad i = 1, m+2.$$

Proof. Let $f \in \mathscr{M}_{M,m}(I)$. Then $f^k(c_0) = f^k(c_{m+1}) = a$ for each $k \in \mathbb{N}$. Since $A_k(c_0+, f)$ $= I_1$ and $\varepsilon_k(c_i+, f) = 1$, we have $\theta_k(c_0+, f) = I_1$ for all $k \ge 0$. Therefore

$$\begin{aligned} \nu(c_0, f;t) = \theta(c_0+, f;t) &= I_1 + I_1 t + I_1 t^2 + I_1 t^3 + \cdots \\ &= (1 + t + t^2 + t^3 + \cdots)I_1. \end{aligned}$$

Also, since $A_0(c_{m+1}-, f) = I_{m+1}$ and $A_k(c_{m+1}-, f) = I_1$, we get $\varepsilon_0(c_{m+1}-, f) = -1$ and $\varepsilon_k(c_{m+1}-, f) = 1$ for all $k \ge 1$, respectively. Therefore $\theta_0(c_{m+1}-, f) = I_{m+1}$ and

$\theta_k(c_{m+1}-,f) = -I_1$ for $k \geq 1$. Hence

$$\begin{aligned}\nu(c_{m+1},f;t) = -\theta(c_{m+1}-,f;t) &= -I_{m+1} + I_1 t + I_1 t^2 + I_1 t^3 + \cdots \\ &= (t + t^2 + t^3 + \cdots) I_1 - I_{m+1}.\end{aligned}$$

Further, since $f \in \mathscr{M}_{M,m}(I)$, we have $N(f;t) = N_{M,m}(t)$. Therefore

$$N'(f;t) = \begin{bmatrix} 1+t+t^2+\cdots & 0 & 0 & \cdots & 0 & 0 \\ & & & N_{M,m}(f;t) & & \\ t+t^2+t^3+\cdots & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}_{(m+2)\times(m+1)},$$

and thus

$$\begin{aligned}\det N'_{(m+2)}(f;t) &= (1+t+t^2+\cdots)\det N^{(1)}(f;t) \\ &= (1-t)^{-1}\det N^{(1)}(f;t) \\ &= (-1)^{1+1}(1-\varepsilon(I_1)t)^{-1}\det N^{(1)}(f;t) = D(f;t).\end{aligned}$$

Moreover, sine m is odd, we have

$$\det N^{(1)}(f;t) = -(1-t)(1+t)^{-1}\det N^{(m+1)}(f;t).$$

Therefore

$$\begin{aligned}\det N'_{(1)}(f;t) &= (-1)^{(m+1)+1}(t+t^2+\cdots)\det N^{(1)}(f;t) \\ &\quad +(-1)^{(m+1)+(m+1)}(-1)\det N^{(m+1)}(f;t) \\ &= (t+t^2+\cdots)(1-t)(1+t)^{-1}\det N^{(m+1)}(f;t) - \det N^{(m+1)}(f;t) \\ &= -(1+t)^{-1}\det N^{(m+1)}(f;t) \\ &= (-1)^{(m+1)+1}(1-\varepsilon(I_{m+1})t)^{-1}\det N^{(m+1)}(f;t) \\ &= D(f;t).\end{aligned}$$

This proves (3.6.1) for $f \in \mathscr{M}_{M,m}(I)$. The proofs of (3.6.1) for $f \in \mathscr{M}_{W,m}(I)$ and that of result **(ii)** are similar. □

CHAPTER 4

AN ITERATIVE EQUATION WITH MULTIPLICATION

"A problem well put is half solved."
- John Dewey

In this chapter, we give results on existence, uniqueness, stability and construction of solutions of (1.2.4) on $\mathbb{R}_+$ and $\mathbb{R}_-$.

4.1 SOME TECHNICAL LEMMAS

Let $\mathscr{C}_b(\mathbb{R}_+)$ consist of all bounded continuous self-maps of $\mathbb{R}_+$, and let $\mathscr{C}_b(\mathbb{R})$ be defined as in section 1.1.3. Consider g on $\mathbb{R}_+$. We can use the exponential map $\psi(x) = e^x$ to conjugate g to a map on the whole $\mathbb{R}$, i.e., let $f(x) := \log g(e^x)$, a map from $\mathbb{R}$ into $\mathbb{R}$ (one-to-one if g is one-to-one), and reduces equation (1.2.4) to the polynomial-like one

$$\alpha_1 f(x) + \alpha_2 f^2(x) + \cdots + \alpha_n f^n(x) = F(x), \qquad x \in \mathbb{R}, \tag{4.1.1}$$

where $F(x) := \log G(e^x)$.

Proposition 4.1.1. *The map g is a solution (resp. unique solution) of* (1.2.4) *in $\mathscr{X} \subseteq \mathscr{C}_b(\mathbb{R}_+)$ if and only if $f(x) := \psi^{-1}(g(\psi(x)))$ is a solution (resp. unique solution) of* (4.1.1) *in $\mathscr{Y} \subseteq \mathscr{C}_b(\mathbb{R})$, where $\psi(x) = e^x$ and $\mathscr{Y} = \{\psi^{-1} \circ g \circ \psi : g \in \mathscr{X}\}$.*

Proof. Let g be a solution of (1.2.4) in $\mathscr{X}$. Since ψ is a homeomorphism of $\mathbb{R}$ onto $\mathbb{R}_+$, clearly $\mathscr{Y} \subseteq \mathscr{C}_b(\mathbb{R})$ and $f \in \mathscr{Y}$. Also, for each $x \in \mathbb{R}$, we have

$$\sum_{k=1}^{n} \alpha_k f^k(x) = \sum_{k=1}^{n} \alpha_k \log g^k(e^x) = \log\left(\prod_{k=1}^{n} (g^k(e^x))^{\alpha_k}\right) = \log G(e^x) = F(x),$$

implying that f is a solution of (4.1.1) on $\mathbb{R}$. The converse follows similarly. Now, in order to prove the uniqueness, assume that (1.2.4) has a unique solution in $\mathscr{X}$ and suppose that f_1, f_2 are any two solutions of (4.1.1) in $\mathscr{Y}$. Then, by "if" part of what we have proved above, there exist solutions g_1 and g_2 of (1.2.4) in $\mathscr{X}$ such that $f_1 = \psi^{-1} \circ g_1 \circ \psi$ and $f_2 = \psi^{-1} \circ g_2 \circ \psi$. By our assumption, we have $g_1 = g_2$ and therefore $f_1 = f_2$. The proof of converse is similar. $\square$

Considering g on $\mathbb{R}_-$, we have the following.

Proposition 4.1.2. *Let $\alpha_k \in \mathbb{Z}$ for $1 \le k \le n$ such that $\sum_{k=1}^{n} \alpha_k$ is odd. Then the map g is a solution (resp. unique solution) of (1.2.4) in $\mathscr{X} \subseteq \mathscr{C}_b(\mathbb{R}_-)$ if and only if $h(x) := \psi^{-1}(g(\psi(x)))$ is a solution (resp. unique solution) of the equation*

$$(h(x))^{\alpha_1} (h^2(x))^{\alpha_2} \cdots (h^n(x))^{\alpha_n} = H(x) \tag{4.1.2}$$

in $\mathscr{Y} \subseteq \mathscr{C}_b(\mathbb{R}_+)$, where $\psi(x) = -x$, $H(x) = \psi^{-1}(G(\psi(x)))$ and $\mathscr{Y} = \{\psi^{-1} \circ g \circ \psi : g \in \mathscr{X}\}$.

Proof. Let g be a solution of (1.2.4) in $\mathscr{X}$. Since ψ is a homeomorphism of $\mathbb{R}_+$ onto $\mathbb{R}_-$, clearly $\mathscr{Y} \subseteq \mathscr{C}_b(\mathbb{R}_+)$ and $h \in \mathscr{Y}$. Also, for each $x \in \mathbb{R}_+$ and $k \in \{1, 2, \ldots, n\}$, we have $H(x) = -G(-x)$ and $h^k(x) = -g^k(-x)$. Therefore,

$$\begin{aligned} \prod_{k=1}^{n} (h^k(x))^{\alpha_k} &= \prod_{k=1}^{n} (-g^k(-x))^{\alpha_k} \\ &= (-1)^{\sum_{k=1}^{n} \alpha_k} \prod_{k=1}^{n} (g^k(-x))^{\alpha_k} \\ &= -\prod_{k=1}^{n} (g^k(-x))^{\alpha_k} \\ &= -G(-x) = H(x) \end{aligned}$$

since $\sum_{k=1}^{n} \alpha_k$ is odd, implying that h is a solution of (4.1.2) on $\mathbb{R}_+$. The converse follows similarly. Further, the proof of uniqueness is similar to that of Proposition 4.1.1. $\square$

By Proposition 4.1.1, it suffices to prove existence for (4.1.1) on the whole $\mathbb{R}$ in order to prove the existence of solution for (1.2.4) on $\mathbb{R}_+$. Further, in order to extend the solutions from $\mathbb{R}_+$ to its closure, we require the continuity of g and G at 0, i.e., we require the necessary conditions $\lim_{x \to -\infty} F(x) = \log G(0)$, $\lim_{x \to -\infty} f^k(x) = \log g^k(0)$

for all $1 \leq k \leq n$ and

$$\sum_{k=1}^{n} \alpha_k \log g^k(0) = \log G(0).$$

These conditions can indeed be satisfied if $\sum_{k=1}^{n} \alpha_k = 1$, $G|_{\mathscr{R}(G)} = \mathrm{id}$, $0 \notin \mathscr{R}(G)$, and $g \equiv G$ on $[0, +\infty)$.

Let $I = [a,b]$ and $J = [c,d]$ be compact intervals in $\mathbb{R}$ and $\mathbb{R}_+$ respectively with non-empty interiors. Let $\mathscr{C}(\mathbb{R}, I)$ (resp. $\mathscr{C}(I,I)$) be the set of all continuous maps of $\mathbb{R}$ (resp. I) into I. Similarly we define $\mathscr{C}(\mathbb{R}_+, J)$ and $\mathscr{C}(J,J)$. For any f in $\mathscr{C}(\mathbb{R}, I)$ or $\mathscr{C}(I,I)$, let

$$\|f\|_I := \sup\{|f(x)| : x \in I\},$$

and for any g in $\mathscr{C}(\mathbb{R}_+, J)$ or $\mathscr{C}(J,J)$, let

$$\|g\|_J := \sup\{|g(x)| : x \in J\}.$$

For any map f, let $\mathscr{R}(f)$ denote the range of f. For $M, \delta \geq 0$, let

$$\begin{aligned} \mathscr{F}_I(\delta, M) &:= \{f \in \mathscr{C}_b(\mathbb{R}) : \mathscr{R}(f) = I, f(a) = a, f(b) = b \text{ and} \\ &\qquad \delta(x-y) \leq f(x) - f(y) \leq M(x-y), \forall x, y \in I \text{ with } x \geq y\}, \\ \mathscr{G}_J(\delta, M) &:= \Big\{g \in \mathscr{C}_b(\mathbb{R}_+) : \mathscr{R}(g) = J, g(c) = c, g(d) = d \text{ and} \\ &\qquad \left(\frac{x}{y}\right)^{\delta} \leq \frac{g(x)}{g(y)} \leq \left(\frac{x}{y}\right)^{M}, \forall x, y \in J \text{ with } x \geq y \Big\}. \end{aligned}$$

Then it can be observed that $\mathscr{F}_I(\delta, M) \subseteq \mathscr{F}_I(\delta_1, M_1)$ and $\mathscr{G}_J(\delta, M) \subseteq \mathscr{G}_J(\delta_1, M_1)$ whenever $\delta \geq \delta_1 \geq 0$ and $M_1 \geq M \geq 0$.

Proposition 4.1.3. *Let $M, \delta \geq 0$. Then $g \in \mathscr{G}_J(\delta, M)$ if and only if $\psi^{-1} \circ g \circ \psi \in \mathscr{F}_I(\delta, M)$, where $\psi(x) = e^x$ and $I = \log(J) := \{\log x : x \in J\}$.*

Proof. Let $g \in \mathscr{G}_J(\delta, M)$ and $I := \log(J) = [a,b]$. Then $a = \log c$ and $b = \log d$. Clearly, $f := \psi^{-1} \circ g \circ \psi \in \mathscr{C}_b(\mathbb{R})$. Also, we have $f(a) = \log g(e^a) = \log g(c) = \log c = a$, and similarly $f(b) = b$. So, $I \subseteq \mathscr{R}(f)$. The reverse inclusion follows by definitions of f and I, because $\mathscr{R}(g) = J$. Therefore $\mathscr{R}(f) = I$.

Next, let $x, y \in I$ with $x \geq y$. Then there exist $u, v \in J$ with $u \geq v$ such that $x = \log u$ and $y = \log v$. So, from the assumption on g, we have

$$\left(\frac{u}{v}\right)^{\delta} \leq \frac{g(u)}{g(v)} \leq \left(\frac{u}{v}\right)^{M},$$

implying that

$$\delta \log\left(\frac{e^x}{e^y}\right) \leq \log\left(\frac{g(e^x)}{g(e^y)}\right) \leq M \log\left(\frac{e^x}{e^y}\right).$$

i.e., $\delta(x-y) \leq f(x) - f(y) \leq M(x-y)$. Therefore $f \in \mathscr{F}_I(\delta, M)$. The converse follows similarly. $\square$

Proposition 4.1.4. *If $M < 1$ or $\delta > 1$, then $\mathscr{G}_J(\delta, M) = \emptyset$. If $M = 1$ or $\delta = 1$, then $\mathscr{G}_J(\delta, M) := \{g \in \mathscr{C}_b(\mathbb{R}_+) : g|_J = \mathrm{id}\}$.*

Proof. Let $g \in \mathscr{G}_J(\delta, M)$. Then by Proposition 4.1.3, $f := \psi^{-1} \circ g \circ \psi \in \mathscr{F}_I(\delta, M)$, where $\psi(x) = e^x$ and $I = \log(J)$. So, for any $x, y \in I$ such that $x \geq y$, we have

$$\delta(x-y) \leq f(x) - f(y) \leq M(x-y). \tag{4.1.3}$$

If $M < 1$, then by setting $y = a$ in (4.1.3), we get that $f(x) < x, \forall x \in I$ with $x > a$. This is a contradiction to the fact that $f(b) = b$, because $b > a$. So, $\mathscr{F}_I(\delta, M) = \emptyset$ and hence $\mathscr{G}_J(\delta, M) = \emptyset$, whenever $M < 1$. A similar argument holds when $\delta > 1$.

If $M = 1$, then from (4.1.3), we have

$$f(x) - f(y) \leq (x-y), \quad \forall x, y \in I \text{ with } x \geq y. \tag{4.1.4}$$

Now for $x = b$, (4.1.4) implies that $f(y) \geq y, \forall y \in I$ with $y < b$. Moreover, setting $y = a$ in (4.1.4), we have $f(x) \leq x, \forall x \in I$ with $x > a$. Thus $f(x) = x, \forall x \in I$, and therefore $f|_I = \mathrm{id}$. This implies that $g|_J = \mathrm{id}$. The reverse inclusion is trivial. So, $\mathscr{G}_J(\delta, M) := \{g \in \mathscr{C}_b(\mathbb{R}_+) : g|_J = \mathrm{id}\}$. A similar argument holds when $\delta = 1$. $\square$

Proposition 4.1.5. *The set $\mathscr{F}_I(\delta, M)$ is a complete metric space under the metric induced by $\|\cdot\|$.*

Proof. It can be easily seen that $\mathscr{F}_I(\delta, M)$ is a closed subset of $\mathscr{C}_b(\mathbb{R})$. So, since $\mathscr{C}_b(\mathbb{R})$ is complete with respect to the metric induced by $\|\cdot\|$, it follows that $\mathscr{F}_I(\delta, M)$ is also complete. $\square$

In view of Proposition 4.1.4, we cannot seek solutions of (1.2.4) without imposing conditions on M and δ. So, henceforth we assume that $0 < \delta \leq 1 \leq M$. We need the following six technical lemmas, the last three of which look similar to some of the results given in Murugan and Subrahmanyam (2009). However, we have to rewrite their proofs carefully because of the following difference: It is assumed in Murugan and Subrahmanyam (2009) that $f \in \mathscr{C}(I, I)$ and f is a homeomorphism of I onto itself, implying that f^{-1} is well defined on the whole domain I of f, but this thesis deals with $f \in \mathscr{C}(\mathbb{R}, I)$ satisfying that $f|_I$ is a homeomorphism of I onto itself. So, f is not a homeomorphism on $\mathbb{R}$, that is, f^{-1} is not defined on the whole $\mathbb{R}$. In this case, we can consider only the inverse of $f|_I$. For a specific instance, the conclusion $L_f \in \mathscr{F}_I(K_0, K_1)$, made in Lemma 3.2 of Murugan and Subrahmanyam (2009), is not true here, simply because we have defined L_f only on I. However, even if we define it on the whole of $\mathbb{R}$, it does not belong to $\mathscr{F}_I(K_0, K_1)$, because in that case $\mathscr{R}(L_f) \neq I$. So, in view of this, we include their proofs here in order to avoid ambiguities.

Lemma 4.1.6. (Zhang (1990)) *Let $f, g \in \mathscr{C}(I, I)$ satisfy $|f(x) - f(y)| \leq M|x - y|$ and $|g(x) - g(y)| \leq M|x - y|$ for all $x, y \in I$, where $M \geq 1$. Then*

$$\|f^k - g^k\|_I \leq \left(\sum_{j=0}^{k-1} M^j\right) \|f - g\|_I \text{ for } k = 1, 2, \ldots. \tag{4.1.5}$$

Lemma 4.1.7. (Zhang and Baker (2000)) *Let $f \in \mathscr{C}(I, I)$ satisfy $f(a) = a$, $f(b) = b$ and $\delta(x - y) \leq f(x) - f(y) \leq M(x - y)$ for all $x, y \in I$ with $x \geq y$, where $0 < \delta \leq 1 \leq M$. Then f is a homeomorphism of I onto itself and*

$$\frac{1}{M}(x - y) \leq f^{-1}(x) - f^{-1}(y) \leq \frac{1}{\delta}(x - y), \tag{4.1.6}$$

for each $x, y \in I$ with $x \geq y$.

Lemma 4.1.8. (Zhang and Baker (2000)) *Let $f, g : I \to I$ be homeomorphisms such that $\delta(x - y) \leq f(x) - f(y) \leq M(x - y)$ and $\delta(x - y) \leq g(x) - g(y) \leq M(x - y)$ for all $x, y \in I$ with $x \geq y$, where $0 < \delta \leq 1 \leq M$. Then*

$$\delta\|f^{-1} - g^{-1}\|_I \leq \|f - g\|_I \leq M\|f^{-1} - g^{-1}\|_I.$$

For $\alpha_k \geq 0$ ($1 \leq k \leq n$) with $\sum_{k=1}^n \alpha_k = 1$ and $f \in \mathscr{F}_I(\delta, M)$, define $L_f : I \to I$ by

$$L_f(x) = \alpha_1 x + \alpha_2 f(x) + \cdots + \alpha_n f^{n-1}(x), \quad \forall x \in I.$$

Lemma 4.1.9. *Let $f \in \mathscr{F}_I(\delta, M)$, where $0 < \delta \leq 1 \leq M$. Then $L_f(a) = a$, $L_f(b) = b$, $\mathscr{R}(L_f) = I$ and*

$$K_0(x - y) \leq L_f(x) - L_f(y) \leq K_1(x - y), \tag{4.1.7}$$

for each $x, y \in I$ with $x \geq y$, where $K_0 := \sum_{k=1}^n \alpha_k \delta^{k-1}$ and $K_1 := \sum_{k=1}^n \alpha_k M^{k-1}$.

Proof. It can be easily seen that $L_f(a) = a$, $L_f(b) = b$ and $\mathscr{R}(L_f) = I$. Also, for any $x, y \in I$ with $x \geq y$, we have

$$\begin{aligned} L_f(x) - L_f(y) &= \sum_{k=1}^{n} \alpha_k f^{k-1}(x) - \sum_{k=1}^{n} \alpha_k f^{k-1}(y) \\ &= \sum_{k=1}^{n} \alpha_k (f^{k-1}(x) - f^{k-1}(y)) \\ &\leq \left(\sum_{k=1}^{n} \alpha_k M^{k-1} \right) (x - y) = K_1(x - y) \end{aligned}$$

and

$$\begin{aligned} L_f(x) - L_f(y) &= \sum_{k=1}^{n} \alpha_k f^{k-1}(x) - \sum_{k=1}^{n} \alpha_k f^{k-1}(y) \\ &= \sum_{k=1}^{n} \alpha_k (f^{k-1}(x) - f^{k-1}(y)) \\ &\geq \left(\sum_{k=1}^{n} \alpha_k \delta^{k-1} \right) (x - y) = K_0(x - y). \end{aligned}$$

This completes the proof. □

Lemma 4.1.10. *Let $0 < \delta \leq 1 \leq M$ and $f \in \mathscr{F}_I(\delta, M)$. Then*

$$\frac{1}{K_1}(x - y) \leq L_f^{-1}(x) - L_f^{-1}(y) \leq \frac{1}{K_0}(x - y), \tag{4.1.8}$$

for each $x, y \in I$ with $x \geq y$, where K_0, K_1 are as in Lemma 4.1.9.

Proof. Follows from the proof of Lemma 4.1.7, by noting from Lemma 4.1.9 that $L_f(a) = a$, $L_f(b) = b$, $\mathscr{R}(L_f) = I$ and L_f satisfies (4.1.7) with $0 < K_0 \leq 1 \leq K_1$. □

Lemma 4.1.11. *Let $0 < \delta \leq 1 \leq M$ and $f_1, f_2 \in \mathscr{F}_I(\delta, M)$. Then*

$$\|L_{f_1} - L_{f_2}\|_I \leq K_2 \|f_1 - f_2\|_I \quad \textit{and} \quad \|L_{f_1}^{-1} - L_{f_2}^{-1}\|_I \leq \frac{K_2}{K_0} \|f_1 - f_2\|_I,$$

where K_0, K_1 are as in Lemma 4.1.9 and $K_2 := \sum_{k=2}^{n} \alpha_k (\sum_{j=0}^{k-2} M^j)$.

Proof. Let $f_1, f_2 \in \mathscr{F}_I(\delta, M)$. Then for each $x \in I$, we have

$$\begin{aligned} |L_{f_1}(x) - L_{f_2}(x)| &= \left| \sum_{k=1}^{n} \alpha_k f_1^{k-1}(x) - \sum_{k=1}^{n} \alpha_k f_2^{k-1}(x) \right| \\ &\leq \sum_{k=2}^{n} \alpha_k |f_1^{k-1}(x) - f_2^{k-1}(x)| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=2}^{n} \alpha_k \|f_1^{k-1} - f_2^{k-1}\|_I \\
&\leq \left(\sum_{k=2}^{n} \alpha_k \left(\sum_{j=0}^{k-2} M^j\right)\right) \|f_1 - f_2\|_I \quad \text{(using Lemma 4.1.6)} \\
&= K_2 \|f_1 - f_2\|_I,
\end{aligned}$$

implying that

$$\|L_{f_1} - L_{f_2}\|_I \leq K_2 \|f_1 - f_2\|_I. \tag{4.1.9}$$

Moreover, since $L_{f_1}, L_{f_2} : I \to I$ are homeomorphisms satisfying (4.1.7) with $0 < K_0 \leq 1 \leq K_1$, by Lemma 4.1.8 we have

$$\|L_{f_1}^{-1} - L_{f_2}^{-1}\|_I \leq \frac{1}{K_0} \|L_{f_1} - L_{f_2}\|_I. \tag{4.1.10}$$

Therefore (4.1.9) and (4.1.10) together implies that

$$\|L_{f_1}^{-1} - L_{f_2}^{-1}\|_I \leq \frac{K_2}{K_0} \|f_1 - f_2\|_I. \tag{4.1.11}$$

This completes the proof. □

4.2 EXISTENCE, UNIQUENESS AND STABILITY

In this section we give results on existence, uniqueness and stability of solutions of (1.2.4).

Theorem 4.2.1. *Let $0 < \alpha_1 < 1$, $\alpha_k \geq 0$ for $2 \leq k \leq n$ such that $\sum_{k=1}^{n} \alpha_k = 1$ and $G \in \mathscr{G}_J(K_1\delta, K_0 M)$, where $J = [c,d]$, $c < d$ and $0 < \delta \leq 1 \leq M$. Let K_0, K_1 and K_2 be as defined in Lemmas 4.1.9 and 4.1.11, respectively. If $K_2 < K_0$, then* (1.2.4) *has a unique solution in $\mathscr{G}_J(\delta, M)$.*

Proof. Let $G \in \mathscr{G}_J(K_1\delta, K_0 M)$, $a := \log c$ and $b := \log d$. Then we obtain the interval $I = [a,b]$ with $a < b$, which satisfies $I = \log J$. By Proposition 4.1.3, we have $F := \psi^{-1} \circ G \circ \psi \in \mathscr{F}_I(K_1\delta, K_0 M)$, where $\psi(x) = e^x$.

Define $T : \mathscr{F}_I(\delta, M) \to \mathscr{C}_b(\mathbb{R})$ by

$$Tf(x) = L_f^{-1}(F(x)), \quad x \in \mathbb{R}.$$

By definitions of F and L_f, we have $Tf(a) = a$ and $Tf(b) = b$. This implies that $I \subseteq \mathscr{R}(Tf)$. Also, since $L_f^{-1} : I \to I$, we have $\mathscr{R}(L_f^{-1}) \subseteq I$, and therefore $\mathscr{R}(Tf) \subseteq I$.

So, $\mathscr{R}(Tf) = I$. Further, for any $x, y \in I$ with $x \geq y$, as $F \in \mathscr{F}_I(K_1\delta, K_0M)$, we have

$$\begin{aligned} Tf(x) - Tf(y) &= L_f^{-1}(F(x)) - L_f^{-1}(F(y)) \\ &\leq \frac{1}{K_0}(F(x) - F(y)) \quad \text{(by using Lemma 4.1.10)} \\ &\leq \frac{1}{K_0}K_0M(x-y) \\ &= M(x-y) \end{aligned}$$

and

$$\begin{aligned} Tf(x) - Tf(y) &= L_f^{-1}(F(x)) - L_f^{-1}(F(y)) \\ &\geq \frac{1}{K_1}(F(x) - F(y)) \quad \text{(again by using Lemma 4.1.10)} \\ &\geq \frac{1}{K_1}K_1\delta(x-y) \\ &= \delta(x-y). \end{aligned}$$

Hence $Tf \in \mathscr{F}_I(\delta, M)$, which proves that T is a self-map on $\mathscr{F}_I(\delta, M)$.

We now prove that T is a contraction. For $f_1, f_2 \in \mathscr{F}_I(\delta, M)$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} |Tf_1(x) - Tf_2(x)| &= |L_{f_1}^{-1}(F(x)) - L_{f_2}^{-1}(F(x))| \\ &\leq \|L_{f_1}^{-1} - L_{f_2}^{-1}\|_I \quad \text{(since } F(x) \in I\text{)} \\ &\leq \frac{K_2}{K_0}\|f_1 - f_2\|_I \quad \text{(by using Lemma 4.1.11)} \\ &\leq \frac{K_2}{K_0}\|f_1 - f_2\|, \end{aligned} \tag{4.2.1}$$

implying that $\|Tf_1 - Tf_2\| \leq \frac{K_2}{K_0}\|f_1 - f_2\|$. Since $0 < K_2 < K_0$, it follows that T is a contraction. By Proposition 4.1.5, $\mathscr{F}_I(\delta, M)$ is complete, and hence by Banach's contraction principle, T has a unique fixed point in $\mathscr{F}_I(\delta, M)$. That is, there exists unique $f \in \mathscr{F}_I(\delta, M)$ such that $L_f^{-1}(F(x)) = f(x), \forall x \in \mathbb{R}$, which proves that f is the unique solution of (4.1.1) in $\mathscr{F}_I(\delta, M)$. This implies by Propositions 4.1.1 and 4.1.3 that $g := \psi \circ f \circ \psi^{-1}$ is the unique solution of (1.2.4) in $\mathscr{G}_J(\delta, M)$. The proof is completed. $\square$

Although Lemmas 4.1.9, 4.1.10 and 4.1.11 are true for $\alpha_1 \in [0,1]$, in Theorem 4.2.1 we have assumed that $\alpha_1 \in (0,1)$ for the following reason: If $\alpha_1 = 1$, then $g = G$ is the unique solution of (1.2.4) so that the problem is trivial. On the other hand, if $\alpha_1 = 0$, then we have $K_0 = \sum_{k=2}^{n} \alpha_k \delta^{k-1}$. So, the condition (in Theorem 4.2.1) $K_2 < K_0$ is not satisfied, because

$$K_2 = \sum_{k=2}^{n} \alpha_k \left(\sum_{j=0}^{k-2} M^j \right) > \sum_{k=2}^{n} \alpha_k \geq \sum_{k=2}^{n} \alpha_k \delta^{k-1} = K_0.$$

Thus this theorem is not true for $\alpha_1 = 0$. In particular, one cannot solve the iterative root problem $g^n = G$ on $\mathbb{R}_+$ using this theorem.

Corollary 4.2.2. *In addition to the assumptions of Theorem 4.2.1, suppose that* $G|_J =$ id. *If* $K_2 < K_0$*, then* G *is the unique solution of* (1.2.4) *in* $\mathscr{G}_J(\delta, M)$.

Proof. Follows by Theorem 4.2.1, because G is a solution of (1.2.4) in $\mathscr{G}_J(\delta, M)$. □

Given $G \in \mathscr{Y} \subseteq \mathscr{C}_b(\mathbb{R}_+)$, as in Agarwal et al. (2003), we say that a solution g of (1.2.4) in $\mathscr{X} \subseteq \mathscr{C}_b(\mathbb{R}_+)$ *depends continuously on* G if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|g - g_1\| < \varepsilon$ whenever $G_1 \in \mathscr{Y}$ with $\|G - G_1\| < \delta$, and $g_1 \in \mathscr{X}$ satisfy that

$$\prod_{k=1}^{n} (g_1^k(x))^{\alpha_k} = G_1(x), \quad \forall x \in \mathbb{R}_+. \tag{4.2.2}$$

Under the assumptions of Theorem 4.2.1, we will show that the solution obtained depends continuously on the function G. More precisely, we have the following.

Theorem 4.2.3. *In addition to assumptions of Theorem 4.2.1, suppose that* $G_1 \in \mathscr{G}_J(K_1\delta, K_0M)$ *and* $g_1 \in \mathscr{G}_J(\delta, M)$ *satisfies* (4.2.2). *Then*

$$\|g - g_1\| \leq \frac{d}{c(K_0 - K_2)} \|G - G_1\|. \tag{4.2.3}$$

Proof. Given G, G_1, g and g_1 as above, let $F(x) = \log G(e^x), F_1(x) = \log G_1(e^x), f(x) = \log g(e^x)$ and $f_1(x) = \log g_1(e^x), \forall x \in \mathbb{R}$. Since $G, G_1 \in \mathscr{G}_J(K_1\delta, K_0M)$, by Proposition 4.1.3, we have $F, F_1 \in \mathscr{F}_I(K_1\delta, K_0M)$, where $I = [a, b]$ such that $a = \log c$ and $b = \log d$. Using a similar argument, we see that $f, f_1 \in \mathscr{F}_I(\delta, M)$. Moreover, f and f_1 satisfy equation (4.1.1) and the equation

$$\sum_{k=1}^{n} \alpha_k f^k(x) = F_1(x), \quad x \in \mathbb{R},$$

respectively, implying that $L_f^{-1}(F(x)) = f(x)$ and $L_{f_1}^{-1}(F_1(x)) = f_1(x), \forall x \in \mathbb{R}$. Therefore, for each $x \in \mathbb{R}$,

$$|f(x) - f_1(x)| = |L_f^{-1}(F(x)) - L_{f_1}^{-1}(F_1(x))|$$

$$
\begin{aligned}
&\le |L_f^{-1}(F(x)) - L_{f_1}^{-1}(F(x))| + |L_{f_1}^{-1}(F(x)) - L_{f_1}^{-1}(F_1(x))| \\
&\le \|L_f^{-1} - L_{f_1}^{-1}\|_I + \frac{1}{K_0}|F(x) - F_1(x)| \ \text{ (using (4.1.8))} \\
&\le \frac{K_2}{K_0}\|f - f_1\|_I + \frac{1}{K_0}\|F - F_1\|_I \ \text{ (using (4.1.11))} \\
&\le \frac{K_2}{K_0}\|f - f_1\| + \frac{1}{K_0}\|F - F_1\|,
\end{aligned}
$$

and hence

$$
\|f - f_1\| \le \frac{K_2}{K_0}\|f - f_1\| + \frac{1}{K_0}\|F - F_1\|.
$$

Since $K_2 < K_0$, the above inequality shows

$$
\|f - f_1\| \le \frac{1}{K_0 - K_2}\|F - F_1\|. \tag{4.2.4}
$$

Since the map $x \mapsto e^x$ is continuously differentiable on I with bounded derivative, it is a Lipschitz map on I. In fact, $|e^x - e^y| < e^b|x - y|$ for all $x, y \in I$. So, for each $x \in \mathbb{R}_+$, we have

$$
\begin{aligned}
|g(x) - g_1(x)| &= |e^{f(\log x)} - e^{f_1(\log x)}| \\
&< e^b|f(\log x) - f_1(\log x)| \le d\|f - f_1\|,
\end{aligned}
$$

implying that

$$
\begin{aligned}
\|g - g_1\| &\le d\|f - f_1\| \\
&\le \frac{d}{K_0 - K_2}\|F - F_1\| \ \text{ (using (4.2.4))}.
\end{aligned} \tag{4.2.5}
$$

Since the map $x \mapsto \log x$ is continuously differentiable on J with bounded derivative, it is a Lipschitz map on J. In fact, $|\log x - \log y| < \frac{1}{c}|x - y|$ for all $x, y \in J$. Therefore, for each $x \in \mathbb{R}$, we have

$$
\begin{aligned}
|F(x) - F_1(x)| &= |\log G(e^x) - \log G_1(e^x)| \\
&\le \frac{1}{c}|G(e^x) - G_1(e^x)| \le \frac{1}{c}\|G - G_1\|,
\end{aligned}
$$

implying that

$$
\|F - F_1\| \le \frac{1}{c}\|G - G_1\|. \tag{4.2.6}
$$

Then (4.2.3) follows from (4.2.5) using (4.2.6). □

The assumptions that $0 < \alpha_1 < 1$ and $\sum_{k=1}^n \alpha_k = 1$ made in Theorem 4.2.1 are not strong. In fact, if $\alpha_1 > 1$ or $\sum_{k=1}^n \alpha_k > 1$, then we can divide all the exponents α_ks in (1.2.4) by $\sum_{k=1}^n \alpha_k$ to get the normalized equation, but the assumptions on G have to be changed suitably. Moreover, by using the above observation, Theorem 4.2.1 and Proposition 4.1.2, we can indeed extend the solutions of (1.2.4) on $\mathbb{R}_+$ to $\mathbb{R}_-$ whenever $\alpha_k \in \mathbb{Z}$ for $1 \le k \le n$ such that $\sum_{k=1}^n \alpha_k$ is odd.

4.3 CONSTRUCTION OF SOLUTIONS

The method used in section 4.2 is an application of Banach's fixed point theorem, which gives an recursive algorithm to approach the unique solution. Unlike section 4.2, in this section we can use another method, sewing piece by piece, to find more continuous solutions of (1.2.4) on the whole $\mathbb{R}_+$.

Consider (1.2.4) with real α_ks, $1 \le k \le n$, and without loss of generality assume that $\alpha_n \neq 0$. Then (1.2.4) and its modified equation (4.1.1) can be represented equivalently as

$$g^n(x) = \prod_{k=1}^{n-1} (g^k(x))^{\lambda_k} G(x) \tag{4.3.1}$$

and (1.1.10), respectively, where λ_k is real for $1 \le k \le n-1$. Let $\lambda := \sum_{k=1}^{n-1} \lambda_k$. We will discuss for $\lambda \ge 0$ and $\lambda < 0$ separately.

First, we consider the case that $\lambda \ge 0$. In 2007, Xu and Zhang (2007a) proved the existence of continuous solution of (1.1.10) on the compact interval I with the assumption that $\lambda \in [0,1)$. In what follows, solving (1.1.10) with $\lambda \in [0,1)$ on the whole $\mathbb{R}$, we obtain solutions of (4.3.1) on $\mathbb{R}_+$.

Let $\mathbb{I}$ denote the interval $|a,b|$ defined as in section 1.1.3, and let $\mathbb{J} = |c,d|$ be an interval in $\mathbb{R}_+$, where c and d may be 0 and ∞ respectively. For $\zeta \in \bar{\mathbb{I}}$, the closure of $\mathbb{I}$, $\eta \in \bar{\mathbb{J}}$ and $\lambda \in [0,1)$, let

$$S_{\eta,\lambda}[\mathbb{R}_+;\mathbb{J}] := \{g \in \mathscr{C}_b(\mathbb{R}_+) : g|_{\mathbb{J}} \text{ is strictly increasing and satisfies } \textbf{(B1)} \text{ and } \textbf{(B2)}\}$$

and $R_{\zeta,\lambda}[\mathbb{R};\mathbb{I}]$ be defined as in section 1.1.3, where

(B1) $(g(x) - x^{1-\lambda})(\eta - x) > 0$ for $x \neq \eta$,

(B2) $(g(x) - \eta^{1-\lambda})(\eta - x) < 0$ for $x \neq \eta$.

Proposition 4.3.1. *Let $\lambda \in [0,1)$. Then a map $g \in S_{\eta,\lambda}[\mathbb{R}_+;\mathbb{J}]$ for $\eta \in \bar{\mathbb{J}}$ if and only if $f = \psi^{-1} \circ g \circ \psi \in R_{\zeta,\lambda}[\mathbb{R};\mathbb{I}]$, where $\psi(x) = e^x$, $\zeta = \log \eta$ and $\mathbb{I} = \log(\mathbb{J})$.*

Proof. Let $g \in S_{\eta,\lambda}[\mathbb{R}_+;\mathbb{J}]$, where $\eta \in \bar{\mathbb{J}}$. Since $f|_{\mathbb{I}} = \psi^{-1} \circ (g|_{\mathbb{J}}) \circ \psi$ and $g|_{\mathbb{J}}$ is strictly increasing, clearly $f|_{\mathbb{I}}$ is also strictly increasing. In order to prove that f satisfies condition **(A1)**, consider any $x \in \mathbb{I}$ such that $x \neq \zeta$. Let $y \in \mathbb{J}$ be such that $x = \log y$. Then either $g(y) < y^{1-\lambda}$ or $g(y) > y^{1-\lambda}$ according as $y > \eta$ or $y < \eta$, respectively. This implies that either $f(x) < (1-\lambda)x$ or $f(x) > (1-\lambda)x$ according as ether $x > \zeta$ or $x < \zeta$, respectively. In any case, we have $(f(x) - (1-\lambda)x)(\zeta - x) > 0$. By a similar argument, using condition **(B2)** for g, we can prove that f satisfies condition **(A2)**. Hence $f \in R_{\zeta,\lambda}[\mathbb{R};\mathbb{I}]$. The converse follows similarly. □

Theorem 4.3.2. *Let $\lambda \in [0,1)$ and $G \in S_{c,\lambda}[\mathbb{R}_+;\mathbb{J}]$ such that $\mathscr{R}(G) = \mathscr{R}(G|_{\mathbb{J}})$, where $\mathbb{J} = |c,d]$. Then* (4.3.1) *has solutions in $S_{c,0}[\mathbb{R}_+;\mathbb{J}]$. Moreover, each solution depends on $n-1$ arbitrarily chosen orientation-preserving homeomorphisms $f_j : [x_j, x_{j-1}] \to [x_{j+1}, x_j]$, $j = 1,2,\ldots,n-1$, where $x_0 = b$ and $x_1, x_2, \ldots, x_n$ are given as in Lemma 1.1.12.*

Proof. Given $G \in S_{c,\lambda}[\mathbb{R}_+;\mathbb{J}]$, by Proposition 4.3.1, we have $F = \psi^{-1} \circ G \circ \psi \in R_{a,\lambda}[\mathbb{R};\mathbb{I}]$, where $\psi(x) = e^x$ and $\mathbb{I} = \log(\mathbb{J})$. Also, since $\mathscr{R}(G) = \mathscr{R}(G|_{\mathbb{J}})$, it follows that $\mathscr{R}(F) = \mathscr{R}(F|_{\mathbb{I}})$. So, $F_1 := F|_{\mathbb{I}} \in R_{a,\lambda}[\mathbb{I};\mathbb{I}]$. Therefore by Lemma 1.1.12, (1.1.10) has a solution ϕ_1 in $R_{a,0}[\mathbb{I};\mathbb{I}]$. Let f be the extension of map ϕ_1 to $\mathbb{R}$ defined by

$$f(x) = \phi_1 \circ F_1^{-1} \circ F(x), \quad x \in \mathbb{R}. \tag{4.3.2}$$

We assert that f is a solution of (1.1.10) in $R_{a,0}[\mathbb{R};\mathbb{I}]$. Being a strictly increasing continuous map, $F_1 : \mathbb{I} \to \mathscr{R}(F_1)$ has the inverse F_1^{-1}, which is also strictly increasing and continuous on $\mathscr{R}(F_1)$. Therefore, as $\mathscr{R}(F) = \mathscr{R}(F_1)$, clearly f is a well-defined map on $\mathbb{R}$. Also, f is continuous on $\mathbb{R}$, being the composition of continuous maps ϕ_1, F_1^{-1} and F. Further, as $\phi_1 \in R_{a,0}[\mathbb{I};\mathbb{I}]$, it follows that f is strictly increasing on $\mathbb{I}$, and satisfies the conditions **(A1)** and **(A2)**. Therefore $f \in R_{a,0}[\mathbb{R};\mathbb{I}]$. Moreover, for each $x \in \mathbb{R}$,

$$\begin{aligned}
f^n(x) - \sum_{k=1}^{n-1} \lambda_k f^k(x) &= f^{n-1}(f(x)) - \sum_{k=1}^{n-1} \lambda_k f^{k-1}(f(x)) \\
&= f^{n-1}|_I(f(x)) - \sum_{k=1}^{n-1} \lambda_k f^{k-1}|_I(f(x)) \\
&\qquad \text{(since } f(x) \in \mathscr{R}(f) = \mathscr{R}(f|_{\mathbb{I}}) \subseteq \mathbb{I}) \\
&= (f|_I)^{n-1}(f(x)) - \sum_{k=1}^{n-1} \lambda_k (f|_I)^{k-1}(f(x))
\end{aligned}$$

$$
\begin{aligned}
&= \phi_1^{n-1}(\phi_1 \circ F_1^{-1} \circ F(x)) - \sum_{k=1}^{n-1} \lambda_k \phi_1^{k-1}(\phi_1 \circ F_1^{-1} \circ F(x)) \\
&= \phi_1^{n}(F_1^{-1} \circ F(x)) - \sum_{k=1}^{n-1} \lambda_k \phi_1^{k}(F_1^{-1} \circ F(x)) \\
&= F_1(F_1^{-1} \circ F(x)) \quad (\text{since } F_1^{-1} \circ F(x) \in \mathbb{I}) \\
&= F(x).
\end{aligned}
$$

Therefore f is a solution of (1.1.10) in $R_{a,0}[\mathbb{R};\mathbb{I}]$. Hence by Propositions 4.1.1 and 4.3.1, $g = \psi \circ f \circ \psi^{-1}$ is a solution of (4.3.1) in $S_{c,0}[\mathbb{R}_+;\mathbb{J}]$. Further, by Lemma 1.1.12, ϕ_1 and hence g depends on $n-1$ arbitrarily chosen orientation-preserving homeomorphisms $f_j : [x_j, x_{j-1}] \to [x_{j+1}, x_j]$, $j = 1, 2, \ldots, n-1$, where $x_0 = b$. □

For the other class $S_{d,\lambda}[\mathbb{R}_+;\mathbb{J}]$, we can similarly prove the following result using Lemma 1.1.13.

Theorem 4.3.3. *Let $\lambda \in [0,1)$ and $G \in S_{d,\lambda}[\mathbb{R}_+;\mathbb{J}]$ such that $\mathscr{R}(G) = \mathscr{R}(G|_{\mathbb{J}})$, where $\mathbb{J} = [c,d|$. Then* (4.3.1) *has solutions in $S_{d,0}[\mathbb{R}_+;\mathbb{J}]$. Moreover, each solution depends on $n-1$ arbitrarily chosen orientation-preserving homeomorphisms $f_j : [x_{j-1}, x_j] \to [x_j, x_{j+1}]$, $j = 1, 2, \ldots, n-1$, where $x_0 = a$ and $x_1, x_2, \ldots, x_n$ are given as in Lemma 1.1.13.*

In the special case that $\lambda = 0$, (4.3.1) reduces to the equation

$$
g^n(x) = G(x), \tag{4.3.3}
$$

i.e., the problem of iterative roots for a given function G. We have following results for solutions of (4.3.3) on $\mathbb{R}_+$.

Corollary 4.3.4. *Let G be a continuous function on $\mathbb{R}_+$ such that G is strictly increasing on J, $G(c) = c, G(d) < d, \mathscr{R}(G) = [c, G(d)]$ and $G(x) < x$ for $x \in (c,d)$, where $J = [c,d]$. Then* (4.3.3) *has solutions on $\mathbb{R}_+$. Moreover, each solution depends on $n-1$ arbitrarily chosen orientation-preserving homeomorphisms $f_j : [x_j, x_{j-1}] \to [x_{j+1}, x_j]$, $j = 1, 2, \ldots, n-1$, where $x_0 = b$ and $x_1, x_2, \ldots, x_n$ are given as in Lemma 1.1.12.*

Proof. Follows from Theorem 4.3.2, because $G \in S_{c,0}[\mathbb{R}_+;J]$ with $J = [c,d]$ such that $\mathscr{R}(F) = \mathscr{R}(F|_J)$. □

We have the following analogous result for the case $G(x) > x$, whose proof is similar.

Corollary 4.3.5. *Let G be a continuous function on $\mathbb{R}_+$ such that G is strictly increasing on $\mathbb{J}$, $G(c) > c, G(d) = d, \mathscr{R}(G) = [G(c), d]$ and $G(x) > x$ for $x \in (c,d)$, where $J =$*

$[c,d]$. *Then* (4.3.3) *has solutions on* $\mathbb{R}_+$. *Moreover, each solution depends on* $n-1$ *arbitrarily chosen orientation-preserving homeomorphisms* $f_j : [x_{j-1}, x_j] \to [x_j, x_{j+1}]$, $j = 1, 2, \ldots, n-1$, *where* $x_0 = a$ *and* $x_1, x_2, \ldots, x_n$ *are given as in Lemma 1.1.13.*

Theorems 4.3.2 and 4.3.3 each give infinitely many solutions of (4.3.1) on $\mathbb{R}_+$ since infinitely many choices can be made for the initial function $f_1, f_2, \ldots, f_{n-1}$ in Lemmas 1.1.12 and 1.1.13. Similar conclusions hold for Corollaries 4.3.4 and 4.3.5.

Next, we consider the case that $\lambda \le 0$. In 2013, assuming that $\lambda \le 0$, Zhang et al. (2013) proved the existence of continuous solutions for (1.1.10) on the compact interval I. In what follows, solving (1.1.10) with $\lambda \le 0$ on the whole $\mathbb{R}$, we obtain solutions of (4.3.1) on $\mathbb{R}_+$. For compact intervals $I = [a,b]$ and $J = [c,d]$ of $\mathbb{R}$ and $\mathbb{R}_+$ respectively, and for $\lambda \in \mathbb{R}$, let

$$\begin{aligned} \mathscr{A}_\lambda[\mathbb{R};I] &:= \{f \in \mathscr{C}_b(\mathbb{R}) : f|_I \text{ is strictly increasing}, f(a) = \lambda a \text{ and } f(b) = \lambda b\}, \\ \mathscr{B}_\lambda[\mathbb{R}_+;J] &:= \{g \in \mathscr{C}_b(\mathbb{R}_+) : g|_J \text{ is strictly increasing}, g(c) = c^\lambda \text{ and } g(d) = d^\lambda\}. \end{aligned}$$

Proposition 4.3.6. *Let* $\lambda \in \mathbb{R}$. *Then* $g \in \mathscr{B}_\lambda[\mathbb{R}_+;J]$ *if and only if* $f = \psi^{-1} \circ g \circ \psi \in \mathscr{A}_\lambda[\mathbb{R};I]$, *where* $\psi(x) = e^x$ *and* $I = \log(J)$.

Proof. Let $g \in \mathscr{B}_\lambda[\mathbb{R}_+;J]$, where $\lambda \in \mathbb{R}$. Since $f|_I = \psi^{-1} \circ (g|_J) \circ \psi$ and $g|_J$ is strictly increasing, clearly $f|_I$ is also strictly increasing. Also, $f(a) = \log g(e^a) = \log(g(c)) = \log(c^\lambda) = \lambda \log c = \lambda a$ and similarly $f(b) = \lambda b$. Hence $f \in \mathscr{A}_\lambda[\mathbb{R};I]$. The converse follows similarly. □

Lemma 4.3.7. (Corollary 1 of Zhang et al. (2013)) *Let* $\lambda \le 0$ *and* $F \in \mathscr{A}_{1-\lambda}[I;I]$, *where* $I = [a,b]$. *Then* (1.1.10) *has infinitely many solutions in* $\mathscr{A}_1[I;I]$.

Remark 4.3.8. The proof of the above lemma, seen in pp. 82-89 of Zhang et al. (2013), shows steps to obtain those solutions:

Step 1: For each $\zeta, \xi \in (a,b)$ and $\lambda \le 0$, let

$$\begin{aligned} \mathscr{A}_\lambda^\zeta[I] &:= \{f \in \mathscr{C}(I, \lambda I) : f \text{ is strictly increasing on } I, f(a) = \lambda a, f(b) = \lambda b, \\ &\qquad f(x) > \lambda x \text{ for } x \in (a,b) \text{ and } f \text{ is linear on } [\zeta, b]\}, \\ \mathscr{B}_\lambda^\xi[I] &:= \{f \in \mathscr{C}(I, \lambda I) : f \text{ is strictly increasing on } I, f(a) = \lambda a, f(b) = \lambda b, \\ &\qquad f(x) < \lambda x \text{ for } x \in (a,b) \text{ and } f \text{ is linear on } [a, \xi]\}. \end{aligned}$$

In this step, we construct solutions of (1.1.10) for $F \in \mathscr{A}_{1-\lambda}^\zeta[I] \cup \mathscr{B}_{1-\lambda}^\xi[I]$ (see Theorem 1 in Zhang et al. (2013)). This enables us to construct a sequence (F_m)

in $\mathscr{A}^{\zeta}_{1-\lambda}[I]\cup\mathscr{B}^{\xi}_{1-\lambda}[I]$ which converges to a given function F of more general form and find the corresponding solutions f_m for $m=1,2,\ldots$.

Step 2: Using the sequential compactness of (f_m) and verifying that its limit f is a solution of (1.1.10), we arrive at the existence of solution of (1.1.10) for $F\in\mathscr{A}_{1-\lambda}[I]\cup\mathscr{B}_{1-\lambda}[I]$, where

$$\begin{aligned}\mathscr{A}_{\lambda}[I] &:= \{f\in\mathscr{C}(I,\lambda I): f \text{ is strictly increasing on } I, f(a)=\lambda a,\\ &\qquad f(b)=\lambda b \text{ and } f(x)>\lambda x \text{ for } x\in(a,b)\},\\ \mathscr{B}_{\lambda}[I] &:= \{f\in\mathscr{C}(I,\lambda I): f \text{ is strictly increasing on } I, f(a)=\lambda a,\\ &\qquad f(b)=\lambda b \text{ and } f(x)<\lambda x \text{ for } x\in(a,b)\}\end{aligned}$$

for $\lambda\le 0$ (see Theorem 2 in Zhang et al. (2013)).

Step 3: Dropping the assumption that location of F is below or above the line $y=(1-\lambda)x$ made in $\mathscr{A}_{1-\lambda}[I]$ and $\mathscr{B}_{1-\lambda}[I]$, we obtain solutions of (1.1.10) for $F\in\mathscr{A}_{1-\lambda}[I;I]$ (see Corollary 1 in Zhang et al. (2013)). In fact, given any $F\in\mathscr{A}_{1-\lambda}[I;I]$, let

$$\Gamma:=\{x\in I: F(x)=(1-\lambda)x\}.$$

Then $I=\Gamma\cup(\cup_j I_j)$ and I_j's are disjoint open intervals, denoted by (a_j,b_j)'s, $a_j,b_j\in\Gamma$, such that $F(x)\neq(1-\lambda)x$ for $x\in(a_j,b_j)$. Then either $F_j\in\mathscr{B}_{1-\lambda}[I]$ or $F_j\in\mathscr{A}_{1-\lambda}[I]$, where $F_j:=F|_{I_j}$ for $j=1,2,\ldots$. By step 2, for each j the equation

$$f^n(x)=\sum_{k=1}^{n-1}\lambda_k f^k(x)+F_j(x)$$

has a solution $f_j\in\mathscr{A}_1[I_j;I_j]$, which depends on the choice of a sequence $(F_{j,m})$ in $\mathscr{A}^{\zeta}_{1-\lambda}[I]\cup\mathscr{B}^{\xi}_{1-\lambda}[I]$. Then it follows that the function $f\in\mathscr{A}_1[I;I]$ defined by

$$f(x)=\begin{cases} f_j(x) & \text{if } x\in I_j,\\ x & \text{if } x\in\Gamma\end{cases}$$

is a solution of (1.1.10) on I.

Since infinitely many choices can be made for every sequence $(F_{j,m})$, Lemma 4.3.7 indeed gives infinitely many solutions of (1.1.10) for $F\in\mathscr{A}_{1-\lambda}[I;I]$.

Theorem 4.3.9. *Let $\lambda\le 0$ and $G\in\mathscr{B}_{1-\lambda}[\mathbb{R}_+;J]$ such that $\mathscr{R}(G)=J^{1-\lambda}:=\{x^{1-\lambda}:x\in J\}$, where $J=[c,d]$. Then* (4.3.1) *has infinitely many solutions in $\mathscr{B}_1[\mathbb{R}_+;J]$. Moreover,* *each solution depends on suitably chosen sequences* $(F_{j,m})$*'s for* $j = 1,2,\ldots$ *as indicated in the above Remark 4.3.8.*

Proof. Given $G \in \mathscr{B}_{1-\lambda}[\mathbb{R}_+;J]$, by Proposition 4.3.6, we have $F = \psi^{-1}\circ G\circ\psi \in \mathscr{A}_{1-\lambda}[\mathbb{R};I]$, where $\psi(x) = e^x$ and $I = \log(J)$. Also, since $\mathscr{R}(G) = J^{1-\lambda}$, we have $\mathscr{R}(F) = (1-\lambda)I$. So $F_1 := F|_I \in \mathscr{A}_{1-\lambda}[I;I]$, and therefore by Lemma 4.3.7, (1.1.10) has a solution ϕ_1 in $\mathscr{A}_1[I;I]$. Let f be the extension of ϕ_1 to $\mathbb{R}$ as defined in (4.3.2). We prove that f is a solution of (1.1.10) in $\mathscr{A}_1[\mathbb{R}_+;I]$. Being a strictly increasing continuous map, $F_1 : I \to (1-\lambda)I$ has the inverse F_1^{-1}, which is also strictly increasing and continuous on $(1-\lambda)I$. Therefore, as by assumption $\mathscr{R}(F) = \mathscr{R}(F_1)$, clearly f is a well-defined map on $\mathbb{R}$. Also, f is continuous on $\mathbb{R}$, being the composition of continuous maps ϕ_1, F_1^{-1} and F. Further, as $\phi_1 \in \mathscr{A}_1[I;I]$, it follows that $f|_I$ is strictly increasing, $f(a) = \lambda a$ and $f(b) = \lambda b$. Therefore $f \in \mathscr{A}_1[\mathbb{R};I]$. Moreover, by a similar argument as in the proof of Theorem 4.3.2, it can be shown that f is a solution of (1.1.10) in $\mathscr{A}_1[\mathbb{R};I]$. Hence by Propositions 4.1.1 and 4.3.6, $g = \psi\circ f\circ\psi^{-1}$ is a solution of (4.3.1) in $\mathscr{B}_1[\mathbb{R}_+;J]$. Further, as indicated in Remark 4.3.8, construction of ϕ_1 and hence that of g depends on the choice of sequences $(F_{j,m})$'s for $j = 1,2,\ldots$. □

In the special case that $\lambda = 0$, we have the following result for solutions of iterative root problem (4.3.3) on $\mathbb{R}_+$.

Corollary 4.3.10. *Let* G *be a continuous function on* $\mathbb{R}_+$ *such that* G *is strictly increasing on* J*,* $G(c) = c$*,* $G(d) = d$ *and* $\mathscr{R}(G) = J$*, where* $J = [c,d]$*. Then* (4.3.3) *has infinitely many solutions on* $\mathbb{R}_+$*. Moreover, each solution depends on the suitably chosen sequences* $(F_{j,m})$*'s for* $j = 1,2,\ldots$ *as indicated in Remark 4.3.8.*

Proof. Follows from Theorem 4.3.9, because $G \in \mathscr{B}_1[\mathbb{R}_+;J]$ such that $\mathscr{R}(G) = J$. □

By comparing the coefficients of g^k, $1 \le k \le n$, in equations (4.3.1) and (1.2.4), we have $\alpha_k = -\lambda_k$ for $1 \le k \le n-1$ and $\alpha_n = 1$. Further, if $\lambda_k \in \mathbb{Z}$ for $1 \le k \le n-1$, then the assumption that $\sum_{k=1}^{n}\alpha_k$ is odd, made in Proposition 4.1.2, demands that $1-\sum_{k=1}^{n-1}\lambda_k$ is odd, i.e., λ is even. Thus, using Proposition 4.1.2, we can indeed extend solutions of (1.2.4) on $\mathbb{R}_+$ to $\mathbb{R}_-$ whenever $\lambda_k \in \mathbb{Z}$ for all $1 \le k \le n-1$ such that λ is even.

4.4 EXAMPLES AND REMARKS

Example 4.4.1. Consider the equation

$$(g(x))^{\frac{3}{4}}(g^2(x))^{\frac{1}{4}} = G(x), \tag{4.4.1}$$

where $G:\mathbb{R}_+\to\mathbb{R}_+$ is defined by

$$G(x)=\begin{cases} 1 & \text{if} \quad x\in(0,1],\\ e^{(1+\log x)\log\sqrt{x}} & \text{if} \quad x\in[1,e],\\ e & \text{if} \quad x\in[e,\infty). \end{cases}$$

Let $f(x):=\log g(e^x)$ and $F(x):=\log G(e^x)$ for $x\in\mathbb{R}$. Then (4.4.1) reduces to the polynomial-like equation

$$\frac{3}{4}f(x)+\frac{1}{4}f^2(x)=F(x),$$

where $F:\mathbb{R}\to\mathbb{R}$ is the map defined by

$$F(x)=\begin{cases} 0 & \text{if} \quad x\le 0,\\ \frac{x^2+x}{2} & \text{if} \quad x\in[0,1],\\ 1 & \text{if} \quad x\ge 1. \end{cases}$$

Note that $F\in\mathscr{F}_I(\frac{1}{2},\frac{3}{2})$, where $I=[0,1]$. Let $\delta=\frac{2}{3}$ and $M=2$. Then $K_1\delta=\frac{1}{2}$, $K_0M=\frac{3}{2}$, and therefore $F\in\mathscr{F}_I(K_1\delta,K_0M)$. This implies by Proposition 4.1.3 that $G\in\mathscr{G}_J(K_1\delta,K_0M)$, where $J=[1,e]$. Also, $K_2=\frac{1}{4}<\frac{11}{12}=K_0$. Thus, all the hypotheses of Theorem 4.2.1 are satisfied. Hence (4.4.1) has a unique solution g in $\mathscr{G}_J(\frac{2}{3},2)$.

Example 4.4.2. Consider the equation

$$\frac{(g^2(x))^3}{(g(x))^2}=G(x), \tag{4.4.2}$$

where $G:\mathbb{R}_+\to\mathbb{R}_+$ is defined by

$$G(x)=\begin{cases} 1 & \text{if}\, x\in(0,1],\\ \sqrt[3]{x} & \text{if}\, x\in[1,e],\\ e^{\frac{1}{3\log x}} & \text{if}\, x\in[e,\infty). \end{cases}$$

Let $f(x):=\log g(e^x)$ and $F(x):=\log G(e^x)$ for $x\in\mathbb{R}$. Then (4.4.2) reduces to

$$-2f(x)+3f^2(x)=F(x),$$

which is equivalent to

$$f^2(x)-\frac{2}{3}f(x)=\frac{1}{3}F(x),$$

where $F : \mathbb{R} \to \mathbb{R}$ is the map defined by

$$F(x) = \begin{cases} 0 & \text{if } \; x \le 0, \\ \frac{x}{3} & \text{if } \; x \in [0,1], \\ \frac{1}{3x} & \text{if } \; x \ge 1. \end{cases}$$

Note that $H := \frac{1}{3}F \in R_{0,\frac{2}{3}}[\mathbb{R};I]$, where $I = [0,1]$. Therefore by Proposition 4.3.1, it follows that the map G_1 defined by $G_1(x) = e^{H(\log x)}$ lies in $\mathscr{S}_{1,\frac{2}{3}}[\mathbb{R}_+;J]$, where $J = [1,e]$. Also, since $\mathscr{R}(H) = [0, \frac{1}{9}] = \mathscr{R}(H|_I)$, we have $\mathscr{R}(G_1) = [1, \sqrt[9]{e}] = \mathscr{R}(G_1|_J)$. Therefore, by Theorem 4.3.2,

$$\frac{(g^2(x))}{(g(x))^{\frac{2}{3}}} = G_1(x),$$

and hence (4.4.2) has a solution g on $\mathbb{R}_+$.

Example 4.4.3. Consider the equation

$$(g^2(x))^3(g(x))^6 = G(x), \tag{4.4.3}$$

where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$G(x) = \begin{cases} 1 & \text{if } x \in (0,1], \\ x^3 & \text{if } x \in [1,2], \\ \frac{7x+2}{x} & \text{if } x \in [2,\infty). \end{cases}$$

Then $G \in \mathscr{B}_3[\mathbb{R}_+;J]$, where $J = [1,2]$. Also, $\mathscr{R}(G) = J^3$. Hence by Theorem 4.3.9, (4.4.3) has a solution in $\mathscr{B}_1[\mathbb{R}_+;J]$.

We make the following observations regarding the two approaches (i.e., using fixed point theorem and constructing solutions piece by piece) considered to solve (1.2.4). First, the solutions g of (1.2.4) obtained in Theorems 4.3.2 and 4.3.3 have exactly one fixed point at an end-point of $\mathscr{R}(g)$, whereas each solution g obtained in Theorems 4.2.1 and 4.3.9 has fixed points at both end-points of $\mathscr{R}(g)$. Second, as noted before, using Theorem 4.2.1, we cannot solve iterative root problem (4.3.3). On the other hand, we can indeed obtain solutions of (4.3.3) using Corollaries 4.3.4, 4.3.5 and 4.3.10.

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

"Our imagination is the only limit to what we can hope to have in the future."
- Charles F. Kettering

In this thesis, we have addressed some problems which are mainly concerned about the iteration of continuous maps. One of the main contributions of our work is to exhibit a subtle interplay between the theory of dynamical systems and iterative equations. Chapter 2 reports an investigation of various dynamical properties such as fixed points, periodic points, stability, and chaoticness concerned with iteration operators on $\mathscr{C}(K)$ for a compact metric space K. The approach used here is to study the dynamic behaviours using solutions of the Babbage equation. Indeed, we have proved that the iteration operator is not chaotic, but most of the fixed points are unstable, thereby exhibiting its complex behaviour. On the other hand, Chapter 4 presents results on continuous solutions of an iterative equation with multiplication, and the strategy employed is to use an exponential map to reduce this equation in conjugation to the popular form of a polynomial-like iterative equation. Another focus of this thesis is on topological conjugacy itself. Chapter 3 proves the relation that $N(g;t) = -S_m N(f;t) S_{m+1}$ whenever f and g are h-conjugates with $\#T(f) = m$ such that h is orientation-reversing. This identity, in contrast with the Milnor-Thurston's result for the orientation-preserving case, proves that the kneading matrix associated with a piecewise monotone map is not an invariant under orientation-reversing conjugacy. To obtain more details on the dynamics, the thesis also considers the study of a particular subfamily of such maps, called the tent-like maps, and describes the relationship between kneading matrices of these maps with that of their iterates. Additionally, some interesting problems have been left for the future. The following ideas or questions could be considered.

Since any non-constant map f in $\mathscr{C}_{\mathrm{id}}(I)$ or $\mathscr{C}_{\mathrm{id}}(S^1)$ has a unique choice for $f|_{\mathscr{R}(f)}$, namely id, we were indeed able to use the definition of stability to prove the instability of such maps in Theorems 2.4.2 and 2.4.3. However, if f is in $\mathscr{C}_{\mathrm{inv}}(I)$ or $\mathscr{C}_{\mathrm{inv}}(S^1)$ (consisting of all continuous self-maps of S^1 which are orientation-reversing involutions on their range), then $f|_{\mathscr{R}(f)}$ has uncountably many choices and therefore the case-wise approach used in the proofs of above two theorems is impractical. Thus the problem of stability of fixed points of $\mathscr{J}_n$, which are in $\mathscr{C}_{\rm inv}(I)$ and $\mathscr{C}_{\rm inv}(S^1)$, is highly nontrivial. Additionally, although $\mathscr{J}_n$ does not have a non-trivial periodic point in $\mathscr{C}(I)$ by Theorem 2.2.4, the problem of stability of periodic points of $\mathscr{J}_n$ in $\mathscr{C}(S^1)$ is also difficult. Further, it could be interesting to consider the study of the dynamics of $\mathscr{J}_n$ on $\mathscr{C}(K)$ whenever K is not compact. Remark that we have already taken the first step towards this problem by proving that $\mathscr{J}_n$ is indeed continuous on $\mathscr{C}(K)$ whenever K is a locally compact Hausdorff space (see Appendix A).

In section 3.4, we have described the relationship between $N(f^k;t)$ and $N(f;t)$ for $f \in \mathscr{M}_0(I)$. It would be natural to consider this problem for a generic $f \in \mathscr{M}(I)$. More generally, it could also be interesting to investigate the relationship between $N(f \circ g;t)$, $N(f;t)$ and $N(g;t)$ for arbitrary $f,g \in \mathscr{M}(I)$. Further, since both the kneading matrix and the iterative root problem concern about iterates of maps, it would be interesting to question, whether we can comment on solutions of the iterative root problem using kneading theory. Besides these, it could be even more challenging, yet exciting to develop a kneading theory for continuous maps with infinitely many turning points similar to piecewise monotone maps.

We remind that in section 4.3 we did not complete our discussion for all $\lambda \in \mathbb{R}$, because we have assumed that $0 \le \lambda < 1$ in Theorems 4.3.2 and 4.3.3. Remark that theses theorems are not necessarily valid for $\lambda \ge 1$, and therefore our current approach cannot be used in this case to solve (1.2.4) on $\mathbb{R}_+$. More precisely, if $\lambda \ge 1$, then the sets $S_{c,\lambda}[\mathbb{R}_+;\mathbb{J}]$ and $S_{d,\lambda}[\mathbb{R}_+;\mathbb{J}]$ are not necessarily nonempty. In fact, if $G \in S_{1,3}[\mathbb{R}_+;[1,2]]$, then by using the conditions **(B1)** and **(B2)** we have $1 < G(2) < 1/4$, which is a contradiction. We arrive at a similar contradiction that $1 < G(1) < 1$ if $G \in S_{2,3}[\mathbb{R}_+;[1.2]]$. So, both the sets $S_{1,3}[\mathbb{R}_+;[1.2]]$ and $S_{2,3}[\mathbb{R}_+;[1.2]]$ are empty. Also, as observed at the end of section 4.2 (resp. 4.3), we can extend the solutions of (1.2.4) (resp. (4.3.1)) on $\mathbb{R}_+$ to $\mathbb{R}_-$ whenever $\alpha_k \in \mathbb{Z}$ for $1 \le k \le n$ such that $\sum_{k=1}^n \alpha_k$ is odd (resp. whenever $\lambda_k \in \mathbb{Z}$ for $1 \le k \le n-1$ such that λ is even). On the other hand, if $\alpha_k \in \mathbb{R} \setminus \mathbb{Z}$ for some $1 \le k \le n$, then for any $G, g \in \mathscr{C}_b(\mathbb{R}_-)$, $x \mapsto \prod_{k=1}^n (g^k(x))^{\alpha_k}$ is a multi-valued complex map, whereas $x \mapsto G(x)$ is a single valued real map. So, in order to obtain the equality in (1.2.4), we have to choose branches of the complex logarithm suitably, which not only depends on x but also on each term of the product $\prod_{k=1}^n (g^k(x))^{\alpha_k}$. Therefore, solving (1.2.4) on $\mathbb{R}_-$ in this case is very difficult. For a similar reason, solving (4.3.1) on $\mathbb{R}_-$ is difficult if $\lambda_k \in \mathbb{R} \setminus \mathbb{Z}$ for some $1 \le k \le n-1$. Moreover, it could be exciting to investigate (1.2.4) for differentiable, equivariant, and multivalued solutions.

Appendix A

In this appendix, we prove that the iteration operator $\mathscr{J}_n$ is indeed continuous on $\mathscr{C}(K)$ whenever K is a locally compact Hausdorff space.

Let X, Y and Z be topological spaces. As seen in Munkres (2000), the collection

$$\{\mathscr{S}(E,U) : E \text{ is a compact subset of } X \text{ and } U \text{ is open in } Y\}$$

such that

$$\mathscr{S}(E,U) = \{f \in \mathscr{C}^0(X,Y) : f(E) \subseteq U\}.$$

forms a subbasis for a topology on $\mathscr{C}^0(X,Y)$, called the *compact-open topology*. Further, given a function $f : X \times Z \to Y$, there is a corresponding function $F : Z \to \mathscr{C}^0(X,Y)$, defined by the equation

$$F(z)(x) = f(x,z), \tag{A.0.1}$$

called the *induced map* of f. Conversely, given $F : Z \to \mathscr{C}^0(X,Y)$, this equation defines a corresponding function f of $X \times Z$ into Y. Moreover, when X is locally compact Hausdorff, we have the following results.

Lemma A.0.1. (Theorem 46.10 in Munkres (2000), p 286) *Let $\mathscr{C}^0(X,Y)$ have the compact-open topology, where X be locally compact Hausdorff. Then the map $\mathscr{E}$: $X \times \mathscr{C}^0(X,Y) \to Y$ defined by $\mathscr{E}(x,f) = f(x)$ is continuous. The map $\mathscr{E}$ is called the evaluation map.*

Lemma A.0.2. (Theorem 46.11 in Munkres (2000), p 287) *Let $\mathscr{C}^0(X,Y)$ be considered in the compact-open topology, where X is locally compact Hausdorff. Then the map $f : X \times Z \to Y$ defined as in* (A.0.1) *is continuous if and only if the corresponding induced map $F : Z \to \mathscr{C}^0(X,Y)$ is continuous.*

Having the above two lemmas, we are ready to prove our desired result.

Theorem A.0.3. *If K is locally compact Hausdorff, then $\mathscr{J}_n$ is continuous on $\mathscr{C}(K)$ for each $n \in \mathbb{N}$.*

Proof. First, we prove by induction that the *iterated evaluation map* $\mathscr{E}_n : K \times \mathscr{C}(K) \to K$ defined by

$$\mathscr{E}_n(x, f) = f^n(x) \tag{A.0.2}$$

is continuous for each $n \in \mathbb{N}$. The case $n = 1$ follows by Lemma A.0.1, because $\mathscr{E}_1 = \mathscr{E}$. Suppose that $\mathscr{E}_n$ is continuous for certain $n \geq 2$. To prove $\mathscr{E}_{n+1}$ is continuous on $K \times \mathscr{C}(K)$, consider the map $H_n : K \times \mathscr{C}(K) \to K \times \mathscr{C}(K)$ defined by $H_n = (\mathscr{E}_n, p)$, where $p : K \times \mathscr{C}(K) \to \mathscr{C}(K)$ is the projection map defined by

$$p(x, f) = f.$$

Since $\mathscr{E}_n$ and p are continuous, so is H_n. Now

$$(\mathscr{E}_1 \circ H_n)(x, f) = \mathscr{E}_1\big(\mathscr{E}_n(x, f), p(x, f)\big) = \mathscr{E}\big(f^n(x), f\big) = f^{n+1}(x) = \mathscr{E}_{n+1}(x, f)$$

for each $(x, f) \in K \times \mathscr{C}(K)$, implying that $\mathscr{E}_{n+1} = \mathscr{E}_1 \circ H_n$. Therefore $\mathscr{E}_{n+1}$ is continuous, being the composition of continuous maps $\mathscr{E}_1$ and H_n. Hence, by induction, $\mathscr{E}_n$ is continuous on $K \times \mathscr{C}(K)$ for each $n \in \mathbb{N}$.

The iteration operator $\mathscr{J}_n : \mathscr{C}(K) \to \mathscr{C}(K)$ is actually the induced map of $\mathscr{E}_n : K \times \mathscr{C}(K) \to K$ for each $n \in \mathbb{N}$. Since K is locally compact Hausdorff, by above discussions, $\mathscr{E}_n$ is continuous for each $n \in \mathbb{N}$. This implies by Theorem A.0.2 that $\mathscr{J}_n$ is continuous on $\mathscr{C}(K)$ for each $n \in \mathbb{N}$. □

Bibliography

Aczél, J. (1966). *Lectures on functional equations and their applications*, volume 19 of *Mathematics in Science and Engineering*. Academic Press, New York-London.

Agarwal, R. P., Xu, B., and Zhang, W. (2003). Stability of functional equations in single variable. *J. Math. Anal. Appl.*, 288(2):852–869.

Alves, J. F. and Sousa Ramos, J. (1999). Kneading theory: a functorial approach. *Comm. Math. Phys.*, 204(1):89–114.

Alves, J. F. and Sousa Ramos, J. (2004). Kneading theory for tree maps. *Ergodic Theory Dynam. Systems*, 24(4):957–985.

Babbage, C. (1815). An essay towards the calculus of functions. *Philoso. Trans. Royal Soc. London.*, 105:389–423.

Baron, K. and Jarczyk, W. (2001). Recent results on functional equations in a single variable, perspectives and open problems. *Aequationes Math.*, 61(1-2):1–48.

Block, L., Keesling, J., and Ledis, D. (2012). Semi-conjugacies and inverse limit spaces. *J. Difference Equ. Appl.*, 18(4):627–645.

Block, L. S. and Coppel, W. A. (1992). *Dynamics in one dimension*, volume 1513 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

Blokh, A. M. (1992). The set of all iterates is nowhere dense in $C([0,1],[0,1])$. *Trans. Amer. Math. Soc.*, 333(2):787–798.

Bödewadt, U. T. (1944). Zur iteration reeller functionen. *Math. Z.*, 49:497–516.

Brin, M. and Stuck, G. (2002). *Introduction to dynamical systems*. Cambridge University Press, Cambridge.

Collet, P. and Eckmann, J.-P. (1980). *Iterated maps on the interval as dynamical systems*. Birkhäuser, Boston.

de Vries, J. (2014). *Topological dynamical systems*, volume 59 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin.

Devaney, R. L. (2003). *An introduction to chaotic dynamical systems*. Westview Press, Boulder, CO.

Fort, Jr., M. K. (1955). The embedding of homeomorphisms in flows. *Proc. Amer. Math. Soc.*, 6:960–967.

Holmgren, R. A. (1994). *A first course in discrete dynamical systems*. Springer-Verlag, New York.

Huang, W. and Ye, X. (2002). Devaney's chaos or 2-scattering implies Li-Yorke's chaos. *Topology Appl.*, 117(3):259–272.

Iannella, N. and Kindermann, L. (2005). Finding iterative roots with a spiking neural network. *Inform. Process. Lett.*, 95(6):545–551.

Jarczyk, W. (1996). On an equation of linear iteration. *Aequationes Math.*, 51(3):303–310.

Jarczyk, W. (2003). Babbage equation on the circle. *Publ. Math. Debrecen*, 63(3):389–400.

Kuczma, M. (1960). On monotonic solutions of a functional equation. I. *Ann. Polon. Math.*, 9:295–297.

Kuczma, M. (1968). *Functional equations in a single variable*. Monografie Matematyczne, Tom 46. Państwowe Wydawnictwo Naukowe, Warsaw.

Kuczma, M., Choczewski, B., and Ger, R. (1990). *Iterative functional equations*, volume 32 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge.

Leśniak, Z. (2002). On fractional iterates of a homeomorphism of the plane. *Ann. Polon. Math.*, 79(2):129–137.

Li, L. (2009). A note on a multivalued iterative equation. *Acta Math. Univ. Comenian. (N.S.)*, 78(1):71–73.

Li, L., Jarczyk, J., Jarczyk, W., and Zhang, W. (2009). Iterative roots of mappings with a unique set-value point. *Publ. Math. Debrecen*, 75(1-2):203–220.

Li, L. and Liu, L. (2019). Decreasing case on characteristic endpoints question for iterative roots of PM functions. *J. Difference Equ. Appl.*, 25(3):396–407.

Li, L. and Zhang, W. (2018). The question on characteristic endpoints for iterative roots of PM functions. *J. Math. Anal. Appl.*, 458(1):265–280.

Li, T. Y. and Yorke, J. A. (1975). Period three implies chaos. *Amer. Math. Monthly*, 82(10):985–992.

Lind, D. and Marcus, B. (1995). *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge.

Mandelbrot, B. B. (1982). *The fractal geometry of nature*. W. H. Freeman and Co., San Francisco, California.

McCarthy, P. J. (1983). The general exact bijective continuous solution of Feigenbaum's functional equation. *Comm. Math. Phys.*, 91(3):431–443.

McShane, N. (1961). On the periodicity of homeomorphisms of the real line. *Amer. Math. Monthly*, 68:562–563.

Mendes, D. A. and Ramos, J. S. (2004). Kneading theory for triangular maps. *Int. J. Pure Appl. Math.*, 10(4):421–450.

Milnor, J. (2006). *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ.

Milnor, J. and Thurston, W. (1977). On iterated maps of the interval. I. the kneading matrix, and II. periodic points. preprint, Princeton University.

Milnor, J. and Thurston, W. (1988). On iterated maps of the interval. In *Dynamical systems*, pages 465–563. Springer.

Munkres, J. R. (2000). *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ.

Murugan, V. and Subrahmanyam, P. V. (2006). Special solutions of a general iterative functional equation. *Aequationes Math.*, 72(3):269–287.

Murugan, V. and Subrahmanyam, P. V. (2009). Existence of continuous solutions for an iterative functional series equation with variable coefficients. *Aequationes Math.*, 78(1-2):167–176.

Nikodem, K. and Zhang, W. (2004). On a multivalued iterative equation. *Publ. Math. Debrecen*, 64(3-4):427–435.

Preston, C. (1983). *Iterates of maps on an interval*, volume 999 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

Preston, C. (1988). *Iterates of piecewise monotone mappings on an interval*, volume 1347 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

Preston, C. (1989). What you need to know to knead. *Adv. Math.*, 78(2):192–252.

Ruette, S. (2017). *Chaos on the interval*, volume 67 of *University Lecture Series*. American Mathematical Society, Providence, RI.

Rugh, H. H. and Lei, T. (2015). Kneading with weights. *J. Fractal Geom.*, 2(4):339–375.

Sarkovskiĭ, O. M. (1965). On cycles and the structure of a continuous mapping. *Ukrain. Mat. Ž.*, 17(3):104–111.

Schröder, E. (1870). Ueber iterirte Functionen. *Math. Ann.*, 3(2):296–322.

Si, J. G. (1995). Continuous solutions of the iterative equation $G(f(x), f^{n_1}(x), \cdots, f^{n_k}(x)) = F(x)$. *J. Math. Res. Exposition*, 15(1):149–150.

Si, J. G. and Zhang, W. N. (1998). C^2 solutions of a functional equations. *Acta Math. Sinica (Chin. Ser.)*, 41(5):1061–1064.

Targoński, G. (1981). *Topics in iteration theory*, volume 6 of *Studia Mathematica: Skript*. Vandenhoeck & Ruprecht, Göttingen.

Targoński, G. (1985). Phantom iterates of continuous functions. In *Iteration theory and its functional equations*, volume 1163 of *Lecture Notes in Math.*, pages 196–202. Springer, Berlin.

Targoński, G. (1995). Progress of iteration theory since 1981. *Aequationes Math.*, 50(1-2):50–72.

Vincze, E. (1959). Über die Charakterisierung der assoziativen Funktionen von mehreren Veränderlichen. *Publ. Math. Debrecen*, 6:241–253.

Wall, C. T. C. (1993). *A geometric introduction to topology*. Dover Publications, Inc., New York.

Wang, X. and Si, J. (2001). Differentiable solutions of an iterative functional equation. *Aequationes Math.*, 61(1-2):79–96.

Xu, B. and Zhang, W. (2007a). Construction of continuous solutions and stability for the polynomial-like iterative equation. *J. Math. Anal. Appl.*, 325(2):1160–1170.

Xu, B. and Zhang, W. (2007b). Decreasing solutions and convex solutions of the polynomial-like iterative equation. *J. Math. Anal. Appl.*, 329(1):483–497.

Zdun, M. C. (2000). On iterative roots of homeomorphisms of the circle. *Bull. Polish Acad. Sci. Math.*, 48(2):203–213.

Zdun, M. C. and Solarz, P. (2014). Recent results on iteration theory: iteration groups and semigroups in the real case. *Aequationes Math.*, 87(3):201–245.

Zdun, M. C. and Zhang, W. (2007). A general class of iterative equations on the unit circle. *Czechoslovak Math. J.*, 57(132)(3):809–829.

Zeidler, E. (1986). *Nonlinear functional analysis and its applications. I.* Springer-Verlag, New York.

Zhang, W. (1997). PM functions, their characteristic intervals and iterative roots. *Ann. Polon. Math*, 65(2):119–128.

Zhang, W. (2000). Solutions of equivariance for a polynomial-like iterative equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 130(5):1153–1163.

Zhang, W. and Baker, J. A. (2000). Continuous solutions of a polynomial-like iterative equation with variable coefficients. *Ann. Polon. Math.*, 73(1):29–36.

Zhang, W., Xu, B., and Zhang, W. (2013). Global solutions for leading coefficient problem of polynomial-like iterative equations. *Results Math.*, 63(1-2):79–93.

Zhang, W. and Zhang, W. (2011). Continuity of iteration and approximation of iterative roots. *J. Comput. Appl. Math.*, 235(5):1232–1244.

Zhang, W. N. (1988). Stability of the solution of the iterated equation $\sum_{i=1}^{n} \lambda_i f(x) = F(x)$. *Acta Math. Sci. (English Ed.)*, 8(4):421–424.

Zhang, W. N. (1989). On the differentiable solutions of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$. *Acta Math. Sinica*, 32(1):98–109.

Zhang, W. N. (1990). Discussion on the differentiable solutions of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$. *Nonlinear Anal.*, 15(4):387–398.

Publications

1. Chaitanya Gopalakrishna and Murugan Veerapazham (2020). Relation between kneading matrices of a map and its iterates, *Commun. Korean Math. Soc.*, 35(2):571-589, `https://doi.org/10.4134/CKMS.c190255`.

2. Murugan Veerapazham, Chaitanya Gopalakrishna and Weinian Zhang (2021), Dynamics of the iteration operator on the space of continuous self-maps, *Proc. Amer. Math. Soc.*, 149(1):217-229, `https://doi.org/10.1090/proc/15178`.

3. Chaitanya Gopalakrishna and Murugan Veerapazham, Invariance of kneading matrix under conjugacy, *J. Korean Math. Soc.*, `https://doi.org/10.4134/JKMS.j190378`

4. Chaitanya Gopalakrishna, Murugan Veerapazham, Suyun Wang and Weinian Zhang, Continuous solutions of an iterative equation with multiplication (communicated).

5. Chaitanya Gopalakrishna, Murugan Veerapazham and Weinian Zhang, Dynamics of iteration operators on self-maps of locally compact Hausdorff spaces (in preparation).

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