WEIGHTED REGULARIZATION METHODS FOR ILL-POSED PROBLEMS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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June, 2020

To my Teachers and Parents

DECLARATION

By the Ph.D. Research Scholar

I hereby **declare** that the research thesis entitled **"WEIGHTED REGULARIZATION METHODS FOR ILL-POSED PROBLEMS"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this research thesis has not been submitted to any University or Institution for the award of any degree.

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Date : 8-06-2020

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CERTIFICATE

This is to **certify** that the research thesis entitled **"WEIGHTED REGULARIZATION METHODS FOR ILL-POSED PROBLEMS"** submitted by **K. Kanagaraj**, (Reg. No 158024MA15F10) as the record of the research work carried out by him, is *accepted as the research thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Prof. Santhosh George
Research Guide

Chairman - DRPC

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ABSTRACT

This thesis is devoted for obtaining a stable approximate solution for ill-posed operator equation $Fx = y$. In the second Chapter we consider a non-linear ill-posed equation $Fx = y$, where F is monotone operator defined on a Hilbert space. Our analysis in Chapter 2 is in the setting of a Hilbert scale.

In the rest of the thesis, we studied weighted or fractional regularization method for linear ill-posed equation. Precisely, in Chapter 3 we studied fractional Tikhonov regularization method and in Chapters 4 and 5 we studied fractional Lavrentiv regularization method for the linear ill-posed equation $Ax = y$, where A is a positive self-adjoint operator. Numerical examples are provided to show the reliability and effectiveness of our methods.

Keywords: Ill-Posed Problem,Regularization parameter, Discrepancy principle, Fractional Tikhonov regularization method, Monotone Operator, Lavrentiev Regularization, Hilbert Scales, Adaptive Parameter Choice Strategy.

Mathematics Subject Classification: 47A52, 47H09, 47J060, 65J10, 65J15, 65J20, 65R10.

Table of Contents

List of Figures

List of Tables

CHAPTER 1

INTRODUCTION

Inverse problems are reverse side of some direct problem. The transformation of known causes into effects that are determined by some model is called direct problem. Inverse problem occurs in almost all fields of sciences, in particular (Kabanikhin (2008)):

"• Physics (quantum mechanics, acoustics, electrodynamics , etc.);

• Geophysics (seismic exploration, logging, magnetotelluric sounding, etc.);

• Medicine (X-ray and NMR tomography, ultrasound testing, etc.);

• Economics (optimal control theory, financial mathematics, etc.)."

For examples, and more details about inverse problems can be found in (Kabanikhin (2008); Groetsch (2015)).

By nature, inverse problems are ill-posed in the sense of Hadamard (Hadamard (1953); Keller (1976)). A problem is not well-posed then it is said to be ill-posed (see section 1.1).

Many inverse problems can be formulated as an operator equation

$$Tx = y, \tag{1.0.1}$$

where $T : X \to Y$ is a linear or non-linear operator between suitable normed spaces.

1.1 PRELIMINARIES

Throughout the thesis we will be using the following notations.

- X and Y are Hilbert spaces.
- Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively stand for inner product and norm.
- $D(T)$, $R(T)$ and $N(T)$ denote the Domain of T, Range of T, Nullspace of T, respectively.
- $\mathcal{B}(X, Y)$ denote the set of all bounded linear operators from X to Y.
- $B(x, r)$ denote the open ball centered at x with radius $r > 0$.
- $\overline{X}$ stands for the closure of X.
- T^* stands for the adjoint of an operator T.

According to Hadamard (1953), (cf. Kabanikhin (2008); Engl et al. (1996)), the problem of solving the operator equation (1.0.1) is said to be **well-posed** if the following conditions hold:

(*i*) for each $y \in Y$ there exists a solution $\hat{x} \in X$ to the equation $Tx = y$ (the existence condition), i.e., $R(T) = Y$;

(*ii*) the solution $\hat{x}$ to the equation $Tx = y$ is unique in X (the uniqueness condition), i.e., there exists an inverse operator $T^{-1} : Y \to X$;

(*iii*) for any $B(\hat{x}, r) \subset X$ of the solution $\hat{x}$ to the equation $Tx = y$, there is a $B(y, \delta) \subset Y$ such that for each $y^\delta \in B(y, \delta)$ the element $T^{-1} y^\delta = x^\delta$ belongs to $B(\hat{x}, r)$, i.e., the operator T^{-1} is continuous (the stability condition)

A typical example of equation (1.0.1) is the Fredholm integral equation of the first kind

$$\int_a^b k(s,t)x(t)dt = y(s), \quad a \leq s \leq b \tag{1.1.2}$$

with non-degenerate kernel $k(s,t)$. Here $X = Y = L^2[a,b]$ and $T : L^2[a,b] \to L^2[a,b]$ defined by $(Tx)(s) = \int_a^b k(s,t)x(t)dt = y(s),\ a \leq s \leq b$ is a compact operator (i.e., $\overline{T(B)}$ is compact in Y for every bounded set B in X). Note that Fredholm integral equations of the first kind appears in many inverse problems of practical importance, for example

Example 1.1.1. *(A Gravitation problem) (cf. Groetsch (2007)) The inverse problem of determining the interior mass distribution f from observation of the force g on the outer ring is formulated as an integral equation of the first kind given by*

$$g(\varphi) = \gamma \int_0^{2\pi} \frac{2 - \cos(\varphi - \theta)}{(5 - 4\cos(\varphi - \theta))^{3/2}} f(\theta) d\theta$$

where γ is the universal gravity constant.

Example 1.1.2. *(Steady State Heat distributions) (cf. Groetsch (2007))*

Consider the problem of determining the temperature flux on the left edge of a semi-infinte strip from observation of the temperature on that face when the temperature in the strip is at steady state. The problem may be stated mathematically as follows: Let

$$\Omega = \{(x,y) : 0 < x,\ 0 < y < \pi\}$$

and suppose $v = v(x,y)$ is a function defined on the closure of Ω and satisfying

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad in \quad \Omega$$

and

$$v(x,0) = v(x,\pi) = 0 \quad for \quad x > 0.$$

Suppose we wish want to find the temperature flux

$$g(y) = \frac{\partial v}{\partial x}(0,y), \quad 0 < y < \pi$$

given the temperature distribution $h(y) = u(0,y)$. Again it is the problem of solving integral equation of first kind given by

$$\begin{aligned} h(y) &= \sum_{n=1}^{\infty} a_n \, sin\, ny \\ &= -\sum_{n=1}^{\infty} \frac{2}{n\pi} \int_0^{\pi} g(\xi)\, sin\, n\xi \, d\xi \, sin\, ny \\ &= \int_0^{\pi} k(y,\xi)\, g(\xi)\, d\xi \end{aligned}$$

where

$$k(y,\xi) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\, sin\, ny\, sin\, n\xi.$$

The operator equation (1.0.1) has a solution if and only if $y \in R(T)$. If $y \notin R(T)$, then we look for an element $x_0 \in X$ such that Tx_0 is close to y. i.e, to find $x_0 \in X$ such that

$$\|Tx_0 - y\| = \inf\ \{\|Tx - y\| : x \in X\}\,.$$

If such x_0 exists, then we call it a least residual norm solution or LRN-solution of (1.0.1).

The following theorems gives the characterization for LRN-solution.

Theorem 1.1.3. *(cf. Nair (2009), Theorem 4.2) Suppose $T : X \to Y$ be a linear operator. Let $P : Y \to Y$ be the orthogonal projection onto $\overline{R(T)}$. For $y \in Y$, the following are equivalent.*

(i) $T\,x = y$ has an LRN-solution.

(ii) $y \in R(T) + R(T)^{\perp}$.

(iii) The equation $Tx = Py$ has a solution.

Theorem 1.1.4. *(cf. Nair (2009), Theorem 4.5) Let $T \in \mathcal{B}(X,Y)$, and $y \in R(T) + R(T)^{\perp}$. Then $x \in X$ is an LRN-solution of (1.0.1) if and only if*

$$T^*\,T\,x = T^*\,y.$$

Note that, if T is not one-to-one then LRN-solution is not unique because, if u is a LRN-solution then so is $u + v$ for any $v \in N(T)$. By Theorem 1.1.4, the set of all LRN-solution of (1.0.1) given by the set

$$S_y = \{u \in X : T^*\,T\,u = T^*\,y\}.$$

By using the continuity and linearity of T and T^*, we can prove that S_y is closed convex set.

It is known that (cf. Groetsch (1977), Theorem 1.1.4) closed convex set in a Hilbert space has unique element of minimal norm. So there exists a unique $\hat{x} \in S_y$ such that

$$\|\hat{x}\| = \inf\{\|u\| : u \in S_y\} \quad \text{and}\ \ T\,\hat{x} = Py.$$

Definition 1.1.5. *(cf. Groetsch (1977), page 115) Let* $T \in \mathcal{B}(X,Y)$. *The operator* $T^\dagger : D(T^\dagger) \subset Y \to X$, *where* $D(T^\dagger) = R(T) + R(T)^\perp$, *defined by* $T^\dagger y = \hat{x}$, *where* $\hat{x}$ *is the LRN-solution of minimal norm of the equation* $Tx = y$, *is called the* ***Generalized inverse*** *of* T.

It is known that (cf. Groetsch (1977), Theorem 3.1.2) $T^\dagger$ is continuous if and only if $R(T)$ is closed. So the problem of solving the operator equation (1.0.1) in the sense of generalized inverse is also ill-posed if $R(T)$ is not closed.

In the thesis we say the problem (1.0.1) with $T \in \mathcal{B}(X,Y)$ is ill-posed, we mean that $T^\dagger$ is not continuous, i.e., the stability condition in the definition of well-posed problem is not satisfied.

1.2 REGULARIZATION

We can not make an unstable problem into stable problem, so one has to use the regularization techniques. Roughly, regularization means approximating an ill-posed problem by a family of neighbouring well-posed problems. We want to approximate the LRN-solution of minimal norm $\hat{x} = T^\dagger y$ of equation (1.0.1) for a specific right hand side y in the situation that the exact data y is not known precisely, but that only an approximation y^δ with $\|y - y^\delta\| \leq \delta$ is available.

In the ill-posed case, $T^\dagger y^\delta$ is not a good approximation of $T^\dagger y$ due to the unboundedness of $T^\dagger$ even if it exists. The idea is looking for some approximation, say x_α^δ depend continuously on the data y^δ with the property that as noise level δ decreases to zero and α is chosen appropriately, then x_α^δ tends to $\hat{x}$.

Definition 1.2.1. *(Engl et al. (1996)) Let* $T \in \mathcal{B}(X,Y)$ *and* $\alpha_0 \in (0, +\infty]$. *For every* $\alpha \in (0, \alpha_0)$, *let* $R_\alpha : Y \to X$ *be a continuous operator. The family* $\{R_\alpha\}$ *is called* ***regularization*** *if for all* $y \in D(T^\dagger)$ *there exists a parameter choice rule* $\alpha = \alpha(\delta, y^\delta)$ *such that*

$$\lim_{\delta \to 0} \sup\{\|R_\alpha y^\delta - T^\dagger y\| : y^\delta \in Y, \|y^\delta - y\| \leq \delta\} = 0 \tag{1.2.1}$$

holds. Here, $\alpha : \mathbb{R}^+ \times Y \to (0, \alpha_0)$ *is such that*

$$\lim_{\delta \to 0} \sup\{\alpha(\delta, y^\delta) : y^\delta \in Y, \|y^\delta - y\| \leq \delta\} = 0. \tag{1.2.2}$$

For a specific $y \in D(T^\dagger)$, a pair (R_α, α) is called a regularization method if (1.2.1) *and* (1.2.2) *hold.*

The quality of a regularization method is determined by the asymptotic of $\|\hat{x} - R_\alpha y^\delta\|$ as δ tends to 0.

Definition 1.2.2. *(Engl et al. (1996)) Let α be a parameter choice rule according to Definition* 1.2.1. *If α depends only on δ, i.e., $\alpha = \alpha(\delta)$, then α is called an* ***a-priori parameter choice rule****. Otherwise, α is called an* ***a-posteriori parameter choice rule****.*

The well known regularization method used for approximating $\hat{x}$ is the so called Tikhonov regularization (Engl (1987a,b); Engl et al. (1996); Engl and Neubauer (1985b, 1987, 1985a); George and Nair (1998, 1994b); Groetsch (1984, 1983); Schock (1984b,a); Tikhonov and Arsenin (1977)) in which a minimizer of the functional

$$J_\alpha(x) := \|Tx - y^\delta\|^2 + \alpha\|x\|^2, \; \alpha > 0 \tag{1.2.3}$$

is taken as an approximation for $\hat{x}$, here $\alpha > 0$ is a regularization parameter. It is known that, if T is bounded linear operator then the minimizer x_α^δ of $J_\alpha(x)$ is given by

$$x_\alpha^\delta = (T^* T + \alpha I)^{-1} T^* y^\delta \tag{1.2.4}$$

and if $\hat{x} \in R((T^* T)^\nu)$, $0 < \nu \leq 1$, then

$$\|\hat{x} - x_\alpha^\delta\| \leq c_1 \, \alpha^\nu + c_2 \left(\frac{\delta}{\sqrt{\alpha}} \right).$$

Throughout this thesis $c, c_1, c_2, \ldots$ denote generic positive constants which may take different values at different places.

The choice of regularization parameter α is an important problem in regularization theory of ill-posed Problems. It is known that (Engl et al. (1996); Groetsch (1984); Nair (2009)) if

$$\alpha(\delta) = c \, \delta^{\frac{2}{2\nu+1}}, \tag{1.2.5}$$

then

$$\|\hat{x} - x_\alpha^\delta\| = O\left(\delta^{\frac{2\nu}{2\nu+1}}\right) \tag{1.2.6}$$

and this rate is optimal for such a choice of $\alpha(\delta)$. But since ν is unknown, the choice (1.2.5) is not possible. Therefore one has to consider a-posteriori parameter choice strategy, i.e., $\alpha = \alpha(\delta, y^\delta)$ is determined during the course of computation of x_α^δ. Well known methods in this regard are the discrepancy principles (i.e., α is chosen as the solution of the equation given in the discrepancy principles).

1. Morozov (Morozov (1984, 1968); Groetsch (1983))

$$\|Tx_\alpha^\delta - y^\delta\| = \delta. \tag{1.2.7}$$

2. Arcangeli (Arcangeli (1966); George and Nair (1998); Nair (1992); Groetsch and Schock (1984)) $\|Tx_\alpha^\delta - y^\delta\| = \frac{\delta}{\sqrt{\alpha}}$.

3. Schock (Schock (1984b,a)) $\|Tx_\alpha^\delta - y^\delta\| = \frac{\delta^p}{\alpha^q} \quad p > 0\,, q > 0.$

4. Engl (Engl (1987a,b)) $\|T^*Tx_\alpha^\delta - T^*y^\delta\| = \frac{\delta^p}{\alpha^q} \quad p > 0\,, q > 0.$

If T is a bounded linear positive self-adjoint operator, then one can consider the solution w_α^δ of the equation

$$(T + \alpha\, I)\, x = y^\delta \tag{1.2.8}$$

as an approximation to $\hat{x}$. The above regularization method is called Lavrentiev or simplified regularization method. Note that w_α^δ is the minimizer of the functional

$$J_\alpha(x) = \langle Tx, x\rangle - 2\langle y, x\rangle + \alpha\,\langle x, x\rangle. \tag{1.2.9}$$

If $\hat{x} \in R(T^\nu)$, $0 < \nu \leq 1$, then it is known that (George and Nair (1993))

$$\|w_\alpha^\delta - \hat{x}\| = O(\delta^{\frac{\nu}{\nu+1}}). \tag{1.2.10}$$

To improve the order in (1.2.6) and (1.2.10), many authors studied Tikhonov regularization method (Lu et al. (2010); Natterer (1984); Tautenhahn (1996, 1993); Neubauer (1988)) and Lavrentiev regularization method (George and Nair (1997); George et al. (2013)) in the setting of Hilbert scales.

1.3 HILBERT SCALES

Definition 1.3.1. *(cf. Nair (2015), Definition* 1.1*) A family of Hilbert spaces* X_s, $s \in \mathbb{R}$, *is said to be* ***Hilbert scale*** *if* $t > s$ *implies* $X_t \subseteq X_s$ *and the inclusion is a continuous embedding, i.e., there exists* $c_{s,t} > 0$ *such that*

$$\|x\|_s \leq c_{s,t} \|x\|_t \quad \forall x \in X_t. \tag{1.3.1}$$

The construction of Hilbert scales are given by first defining X_s for $s \geq 0$, and then defining X_s for $s < 0$ using the concept of a Gelfand triple. Let V be a dense subspace of Hilbert space X with norm $\|\cdot\|$. Suppose V also a Hilbert space with respect to a norm $\|\cdot\|_V$ such that the inclusion of V into X is continuous, i.e., there exists $c > 0$ such that

$$\|x\| \leq c \|x\|_V \quad \forall x \in V.$$

For $x \in X$, let

$$\|x\|_* = \sup\{|\langle v, x\rangle| : v \in V, \|v\|_V \leq 1\}.$$

Then $\|\cdot\|_*$ is a norm on X and it is weaker than the original norm $\|\cdot\|$. Let $\tilde{V}$ be the completion of X with respect to the norm $\|\cdot\|_*$.

Definition 1.3.2. *(cf. Nair (2015), Definition* 1.2*) The triple* $(V, X, \tilde{V})$ *is called a* ***Gelfand triple****.*

Theorem 1.3.3. *(cf. Nair (2015), Theorem* 1.4*) The space* $\tilde{V}$ *is linearly isometric with* V', $V \subseteq X \subseteq \tilde{V}$ *and the inclusions are continuous embeddings.*

Example 1.3.4. *(cf. Nair (2015), Example* 2.2 *) Let* $\{v_n : n \in \mathbb{N}\}$ *be an orthonormal basis of a separable Hilert space* X*. Let* (ω_n) *be a sequence of positive real numbers with* $\omega_n \to 0$*. For* $t \geq 0$*, let*

$$X_t = \left\{ x \in X : \sum_{n=1}^{\infty} \frac{|\langle x, v_n\rangle|^2}{\omega_n^{2t}} < \infty \right\}.$$

Then X_t is a Hilbert space with inner product

$$\langle x, y\rangle_t = \sum_{n=1}^{\infty} \frac{\langle x, v_n\rangle \langle v_n, y\rangle}{\omega_n^{2t}}.$$

The corresponding norm $\|x\|_t$ is given by

$$\|x\|_t^2 = \langle x, x\rangle_t = \sum_{n=1}^{\infty} \frac{|\langle x, v_n\rangle|^2}{\omega_n^{2t}}.$$

Observe that

$$\|x\| \leq \|x\|_t \quad \forall x \in X_t,\ t > 0.$$

Thus, (X_t, X, X_{-t}), with $X_0 = X$ and $X_{-t} = \tilde{X}_t$, is a Gelfand triple for each $t > 0$, and $\{X_t : t \in \mathbb{R}\}$ is a Hilbert Scale.

Example 1.3.5. *(cf. Nair (2015), Example 2.4) For $t \geq 0$, the sobolev space*

$$H^t(\mathbb{R}^k) = \left\{ f \in L^2(\mathbb{R}^k) : \int_{\mathbb{R}^k} (1 + |x|^2)^t \, |\hat{f}(x)|^2 \, dx < \infty \right\}$$

is a Hilbert space with inner product

$$\langle f, g\rangle_t = \int_{\mathbb{R}^k} (1 + |x|^2)^t \, \hat{f}(x) \, \overline{\hat{g}(x)} \, dx$$

and the corresponding norm

$$\|f\|_t = \left[\int_{\mathbb{R}^k} (1 + |x|^2)^t \, |\hat{f}(x)|^2 \, dx \right]^{1/2}.$$

For $t < 0$, $H^t(\mathbb{R}^k)$ is defined via Gelfand triple. It can be shown that for $t < s$ the inclusion $H_s \subseteq H_t$ is continuous , and hence $\{H^t(\mathbb{R}^k) : t \in \mathbb{R}\}$ is a Hilbert scale.

Natterer (1984) introduce the notion of Hilbert scales in Tikhonov regularization to get better order in the error estimate. Natterer's idea was to look for a modification of the Tikhonov regularization which yield an approximation of the LRN-solution which minimizes the function

$$x \to \|x\|_s,$$

where $\|\cdot\|_s$ for $s > 0$ is the norm on the Hibert space X_s corresponding to a Hilbert scale $\{X_s : s \in \mathbb{R}\}$ for which the Interpolation inequality

$$\|u\|_s \leq \|u\|_r^{1-\lambda} \, \|u\|_t^{\lambda}$$

holds for $r \leq s \leq t$. This purpose was served by considering the minimizer $x^{\delta}_{\alpha,s}$ of the functional $J_\alpha(x)$ with

$$J_\alpha(x) = \|Tx - y^\delta\|^2 + \alpha\|x\|_s^2, \quad x \in X, \quad \alpha > 0.$$

Natterer showed that if T satisfies

$$\|Tx\| \geq c\|x\|_{-a}, \quad \forall x \in X$$

for some $a > 0$ and $c > 0$, and if $\hat{x} \in X_t$ where $0 \leq t \leq 2s+a$ and the regularization parameter α is chosen such that $\alpha \sim \delta^{\frac{2(a+s)}{a+t}}$, then

$$\|x^{\delta}_{\alpha,s} - \hat{x}\| = O(\delta^{\frac{t}{t+a}}).$$

Thus, higher smoothness requirement on $\hat{x}$ and with higher level of regularization gives higher order of convergence.

In Chapter 2, we studied Lavrentiev regularization method for non-linear ill-posed equation in the setting of Hilbert scale.

It is known that the term $\alpha\,\|x\|^2$ in (1.2.3) and $\alpha\langle x, x\rangle$ in (1.2.9) over smooth the solution $\hat{x}$. So many authors (Klann and Ramlau (2008); Hochstenbach and Reichel (2011); Hochstenbach et al. (2015); Reddy (2018)) studied fractional or weighted Tikhonov regularization method to reduce the oversmoothing.

1.4 WEIGHTED REGULARIZATION METHOD

Let T be a compact operator, $(\sigma_n; u_n, v_n)$ be the singular system of T and G_α be real valued functions satisfying the following conditions

$$\begin{aligned}
&\sup_n |G_\alpha(\sigma_n)\,\sigma_n^{-1}| = c(\alpha) < \infty,\\
&\lim_{\alpha\to 0} G_\alpha(\sigma_n) = 1 \quad \text{pointwise in } \sigma_n,\\
&|G_\alpha(\sigma_n)| \leq c \quad \forall \alpha, \sigma_n.
\end{aligned}$$

Then the family $\{R_\alpha\}$ of operators given by

$$R_\alpha\,(y) = \sum_{\sigma_n>0} G_\alpha(\sigma_n)\,\langle y, v_n\rangle\,u_n \tag{1.4.1}$$

becomes regularization. The functions G_α are called Filter function.

Definition 1.4.1 (cf.Klann and Ramlau (2008))**.** *Let $\beta \in [0,1]$ and $G_\alpha : \mathbb{R}_+ \to \mathbb{R}$ denote a filter function. Then $G_\alpha^\beta(x) = (G_\alpha(x))^\beta$ is called the fractional filter function with parameter β.*

For a given filter function G_α and $\beta \in [0,1]$ the operator $R_{\alpha,\beta} : Y \to X$ is given by

$$R_{\alpha,\beta}(y) = \sum_{\sigma_n > 0} G_\alpha^\beta(\sigma_n) \langle y, v_n \rangle u_n$$

is called the fractional filter operator with parameter β.

The Tikhonov method can be written in the form (1.4.1) with a filter function $G_\alpha(\sigma) = \frac{\sigma}{\sigma^2+\alpha}$. Klann and Ramlau (2008) consider the family of filter functions

$$G_{KR}(\sigma) = \frac{\sigma^{2\beta-1}}{(\sigma^2+\alpha)^\beta}$$

with parameter $\beta > \frac{1}{2}$.

In (Hochstenbach et al. (2015)) the LRN-solution of minimal norm $\hat{x}$ is approximated by $x_{\alpha,\beta}$, which are minimizer of the functionals

$$J_{\alpha,\beta}(x) := \|A\,x - y^\delta\|_W^2 + \alpha\,\|x\|^2, \tag{1.4.2}$$

where $\|y\|_W = \|(A\,A^T)^{\frac{\beta-1}{4}}\,y\|$ and $A \in \mathbb{R}^{m\times n}$. The normal equation associated with (1.4.2) is given by

$$((A^T A)^{\frac{\beta+1}{2}} + \alpha\, I)\, x = (A^T A)^{\frac{\beta-1}{2}} A^T y^\delta.$$

The filter function for fractional Tikhonov regularization (1.4.2) with $\beta > 0$ is given by

$$G_{\alpha,\beta}(\sigma) = \frac{\sigma^\beta}{\sigma^{\beta+1}+\alpha}.$$

Note that the standard Tikhonov regularization is recovered for $\beta = 1$.

In this thesis, except Chapter 2, we deal with weighted regularization methods for linear ill-posed equations.

1.5 OUTLINE OF THE THESIS

The rest of the thesis is organized as follows.

In chapter 2, we considered a derivative-free iterative method for approximately solving non-linear ill-posed equations involving a monotone operator in the setting of Hilbert scales.

In Chapter 3, we considered the fractional Tikhonov regularization (FTR) method and used Schock-type discrepancy principle for choosing regularization parameter α. We showed that FTR method gives better error estimate than that of Tikhonov regularization method.

In Chapter 4, we considered weighted simplified regularization method for approximately solving the equation $Ax = y$, where $A : X \to X$ is a positive self-adjoint operator and considered three discrepancy principles to choose the regularization parameter α. We obtained an optimal order error estimate under a general Hölder type source condition.

In Chapter 5, we considered the finite dimensional realization for weighted simplified regularization consider in the Chapter 4, and for choosing regularization parameter α we considered the adaptive parameter choice method considered by Pereverzev and Schock (2005). We obtained an optimal order error estimate under a general Hölder type source condition.

Chapter 6 gives conclusion of the thesis and future work.

CHAPTER 2

DERIVATIVE FREE REGULARIZATION METHOD FOR NON-LINEAR ILL-POSED EQUATIONS IN HILBERT SCALES

2.1 INTRODUCTION

In this chapter, we consider the problem of approximating a solution $\hat{x}$ of the non-linear equation

$$F(x) = y, \tag{2.1.1}$$

where $F : D(F) \subset X \longrightarrow X$ is a non-linear monotone operator. Recall (cf. Alber and Ryazantseva (2006)), that F is said to be monotone if

$$\langle F(x) - F(y), x - y \rangle \geq 0$$

for all $x, y \in D(F)$.

A typical example of (2.1.1) is the parameter identification problem in an elliptic PDE (Hofmann et al. (2016)); i.e., to find the source term q in the elliptic boundary value problem

$$\begin{aligned} -\Delta u + \xi(u) &= q \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.1.2}$$

from measurement of u in Ω. Here $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz continuously differentiable monotonically increasing function and $\Omega \subseteq \mathbb{R}^3$ is a smooth domain. The corresponding forward operator in this case is $F : H^2(\Omega) \longrightarrow H^2(\Omega)$ defined by

$$F(q) = u.$$

Note that the equation (2.1.1) is in general ill-posed in the sense that the solution $\hat{x}$ of (2.1.1) is not depending continuously on the data y. We assume that the available data, $y^\delta \in X$ is such that

$$\|y - y^\delta\| \leq \delta$$

and equation (2.1.1) is ill-posed. Therefore one has to use regularization methods for approximating $\hat{x}$. Since F is monotone, one may use Lavrentiev regularization method (Tautenhahn (2002); George and Nair (2008); Hofmann et al. (2016)), in which the solution x_α^δ of the equation

$$F(x) + \alpha(x - x_0) = y^\delta \tag{2.1.3}$$

is taken as an approximation for $\hat{x}$ where x_0 is some initial guess. Note that a closed form solution for (2.1.3) is not easy to find for non-linear F. Therefore, many authors (Tautenhahn (2002); Alber and Ryazantseva (2006); George and Nair (2008)) considered iterative methods to find an approximation for x_α^δ. George and Nair (2017) considered a derivative-free iterative method defined for $n = 0, 1, 2, \ldots$ by

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - \beta \left[F(x_{n,\alpha}^\delta) + \alpha(x_{n,\alpha}^\delta - x_0) - y^\delta \right], \tag{2.1.4}$$

where β is a scaling parameter and α is a regularization parameter for approximating x_α^δ. It is known that (Tautenhahn (1998, 2002); Hofmann et al. (2016)), the optimal order error estimate for Lavrentiev regularization is

$$\|x_\alpha^\delta - \hat{x}\| = O\left(\delta^{\frac{\nu}{\nu+1}}\right) \tag{2.1.5}$$

under the source condition

$$x_0 - \hat{x} \in R\left(F'(x_0)^\nu\right), \;\; 0 < \nu \leq 1,$$

or

$$x_0 - \hat{x} \in R\left(F'(\hat{x})^{\nu}\right), \quad 0 < \nu \leq 1.$$

In order to improve the convergence rate in (2.1.5), many authors considered iterative regularization method for (2.1.1) in the setting of Hilbert scales (Tautenhahn (1996); George and Nair (1997); Tautenhahn (1998); Neubauer (2000); Egger and Neubauer (2005); George et al. (2013)). Here we consider Lavrentiev regularization method for (2.1.1) in the setting of Hilbert scales. We also consider an inverse free, derivative-free iterative method for approximating $\hat{x}$ in the setting of a Hilbert scales.

The rest of the Chapter is organized as follows: Preliminaries are given in Section 2.2, the method and its convergence analysis are given in Section 2.3. Error bounds are given in Section 2.4, parameter choice strategies are given in Section 2.5. Implementation of the adaptive parameter choice is given in Section 2.6 and the numerical experiments are given in Section 2.7. Finally, the Chapter ends with a conclusion in Section 2.8.

2.2 PRELIMINARIES

In this study, we consider a Hilbert scale $\{X_s\}_{s\in\mathbb{R}}$ generated by a strictly positive definite, unbounded, densely defined, self-adjoint operator $L : D(L) \subseteq X \to X$. That is L satisfies:

$$\langle Lx, x\rangle > 0, \ \forall x \in D(L)$$

$D(L)$ is dense in X and

$$\|Lx\| \geq \|x\|, \, x \in D(L).$$

Recall (cf. George and Nair (1997)) that the space X_t is the completion of

$$D := \bigcap_{k=0}^{\infty} D(L^k)$$

with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle u, v\rangle_t = \langle L^t u, v\rangle, \ u, v \in D.$$

Moreover, $\{X_s\}_{s\in\mathbb{R}}$ satisfies the Definition 1.3.1 (cf. Engl et al. (1996); George and Nair (1997)). Next we show that the equation

$$F(x) + \alpha L^s(x - x_0) = y^\delta \tag{2.2.1}$$

has unique solution $x^\delta_{\alpha,s}$. We need the following definition for our proof.

Definition 2.2.1. *(cf. Alber and Ryazantseva (2006), Definition* 1.1.42*) An operator $A : D(A) \subseteq X \to X$ is said to be coercive if there exists a function $c(t)$ defined on $[0, \infty)$ such that $c(t) \to \infty$ as $t \to \infty$, and the inequality*

$$\langle A(x), x\rangle \geq c(\|x\|)\,\|x\|$$

holds for all $x \in D(A)$.

Next, we prove that the operator $T := F + \alpha L^s$ is coercive. This can be seen as follows:

$$\begin{aligned}
\langle T(x), x\rangle &= \langle F(x) + \alpha L^s(x), x\rangle \\
&= \langle F(x) - F(0) + \alpha L^s(x), x - 0\rangle + \langle F(0), x\rangle \\
&= \langle F(x) - F(0), x - 0\rangle + \langle \alpha L^s(x), x\rangle + \langle F(0), x\rangle \\
&\geq \alpha\|x\|_s^2 - \|F(0)\|\|x\| \quad \text{(by the monotonicity of } F) \\
&\geq \alpha\,\frac{1}{c_{0,s}}\|x\|^2 - \|F(0)\|\|x\| \quad \text{(by (1.3.1))}
\end{aligned}$$

and hence

$$\lim_{\|x\|\to\infty} \frac{\langle T(x), x\rangle}{\|x\|} \geq \lim_{\|x\|\to\infty} \alpha\,\frac{1}{c_{0,s}}\|x\| - \|F(0)\| = \infty.$$

That is $T = F + \alpha L^s$ is coercive. Further,

$$\langle T(x) - T(y), x - y\rangle = \langle F(x) - F(y), x - y\rangle + \alpha\langle L^s(x - y), x - y\rangle \geq \alpha\frac{1}{c_{0,s}}\|x - y\|^2,$$

i.e., T is strongly monotone. So by Minty-Browder Theorem (Alber and Ryazantseva (2006), page 54) for given $\alpha > 0$, (2.2.1) has unique solution $x^\delta_{\alpha,s}$ for any $y^\delta \in X$.

Let $r_0 = \|x_0 - \hat{x}\|_s$. The following Lemmas is used to prove our main results.

Lemma 2.2.2. *Let $x^{\delta}_{\alpha,s}$ be the solution of (2.2.1) and $x_{\alpha,s}$ is the solution of*

$$F(x) + \alpha L^s(x - x_0) = y. \tag{2.2.2}$$

Then

$$\|x^{\delta}_{\alpha,s} - x_{\alpha,s}\|_s \le c_{0,s}\,\frac{\delta}{\alpha}$$

and

$$\|x_{\alpha,s} - \hat{x}\|_s \le \|x_0 - \hat{x}\|_s.$$

In particular,

$$\|x^{\delta}_{\alpha,s} - x_0\|_s \le c_{0,s}\,\frac{\delta}{\alpha} + 2r_0. \tag{2.2.3}$$

Proof. Observe that by (2.2.1) and (2.2.2), we have

$$F(x^{\delta}_{\alpha,s}) - F(x_{\alpha,s}) + \alpha L^s(x^{\delta}_{\alpha,s} - x_{\alpha,s}) = y^{\delta} - y.$$

Hence

$$\langle F(x^{\delta}_{\alpha,s}) - F(x_{\alpha,s}), x^{\delta}_{\alpha,s} - x_{\alpha,s}\rangle + \alpha\langle L^s(x^{\delta}_{\alpha,s} - x_{\alpha,s}), x^{\delta}_{\alpha,s} - x_{\alpha,s}\rangle = \langle y^{\delta} - y, x^{\delta}_{\alpha,s} - x_{\alpha,s}\rangle.$$

By using (1.3.1) and the monotonicity of F, we have

$$\begin{aligned} \alpha\,\|x^{\delta}_{\alpha,s} - x_{\alpha,s}\|_s^2 &\le \delta\,\|x^{\delta}_{\alpha,s} - x_{\alpha,s}\| \\ &\le \delta\, c_{0,s}\,\|x^{\delta}_{\alpha,s} - x_{\alpha,s}\|_s. \end{aligned}$$

Thus,

$$\|x^{\delta}_{\alpha,s} - x_{\alpha,s}\|_s \le c_{0,s}\,\frac{\delta}{\alpha}.$$

Again, since $y = F(\hat{x})$, we have

$$F(x_{\alpha,s}) + \alpha\, L^s\,(x_{\alpha,s} - x_0) = F(\hat{x}),$$

so that

$$F(x_{\alpha,s}) - F(\hat{x}) + \alpha\, L^s\,(x_{\alpha,s} - x_0) = 0,$$

i.e.,

$$F(x_{\alpha,s}) - F(\hat{x}) + \alpha\, L^s\,(x_{\alpha,s} - \hat{x}) = \alpha L^s(x_0 - \hat{x}).$$

Hence

$$\langle F(x_{\alpha,s}) - F(\hat{x}), x_{\alpha,s} - \hat{x}\rangle + \alpha\langle L^s(x_{\alpha,s} - \hat{x}), x_{\alpha,s} - \hat{x}\rangle = \alpha\langle L^s(x_0 - \hat{x}), x_{\alpha,s} - \hat{x}\rangle.$$

Again, using the monotonicity of F, we have

$$\begin{aligned} \alpha\|x_{\alpha,s} - \hat{x}\|_s^2 &\leq \alpha\|L^{\frac{s}{2}}(x_{\alpha,s} - \hat{x})\|\|L^{\frac{s}{2}}(x_0 - \hat{x})\| \\ &\leq \alpha\|x_{\alpha,s} - \hat{x}\|_s\|x_0 - \hat{x}\|_s. \end{aligned}$$

Thus,

$$\|x_{\alpha,s} - \hat{x}\|_s \leq \|x_0 - \hat{x}\|_s.$$

Now (2.2.3) follows from the triangle inequality:

$$\|x_{\alpha,s}^\delta - x_0\|_s \leq \|x_{\alpha,s}^\delta - x_{\alpha,s}\|_s + \|x_{\alpha,s} - \hat{x}\|_s + \|\hat{x} - x_0\|_s.$$

This completes the proof.

□

Remark 2.2.3. *Note that by (1.3.1) and (2.2.3), we have*

$$\|x_{\alpha,s}^\delta - x_0\| \leq c_{0,s}\,\|x_{\alpha,s}^\delta - x_0\|_s \leq c_{0,s}\left(c_{0,s}\frac{\delta}{\alpha} + 2r_0\right)$$

i.e., $x_{\alpha,s}^\delta \in B(x_0, R)$, *where*

$$R = c_{0,s}\left(c_{0,s}\frac{\delta}{\alpha} + 2r_0\right).$$

2.3 THE METHOD AND THE CONVERGENCE ANALYSIS

Let $\rho = c_{0,s}(c_{0,s} + 1)(c_{0,s} + 2r_0)$. We assume that the following conditions hold:

(i) $\bar{B}(x_0, \rho) \subseteq D(F)$,

(ii) F has self-adjoint Fréchet derivative $F'(x)$ for every $x \in \bar{B}(x_0, \rho)$,

(iii) there exists $\beta_0 > 0$ such that

$$\|L^{-\frac{s}{2}} F'(x) L^{-\frac{s}{2}}\| \leq \beta_0 \quad \text{for all } \ x \in \bar{B}(x_0, \rho).$$

(iv) there exist positive constants d_1, d_2, b such that

$$d_1 \|x\|_{-b} \leq \|F'(y)x\| \leq d_2 \|x\|_{-b} \quad \text{for all } y \in \bar{B}(x_0, \rho) \text{ and } x \in X.$$

Let $f(t) := \min\{d_1^t, d_2^t\}$, $g(t) := \max\{d_1^t, d_2^t\}$, $t \in \mathbb{R}$ and $|t| \leq 1$. Further, let $M_{s,y} := L^{-\frac{s}{2}} F'(y) L^{-\frac{s}{2}}$ for $y \in \bar{B}(x_0, \rho)$.

We shall make use of the following proposition, the proof of which is analogous to the proof of Proposition 3.1 in (George and Nair (1997)).

Proposition 2.3.1. *(cf. George and Nair (1997), Proposition* 3.1*) For* $s > 0$ *and* $|\nu| \leq 1$,

$$f\left(\frac{\nu}{2}\right) \|x\|_{-\frac{\nu(s+b)}{2}} \leq \|M_{s,y}^{\nu/2} x\| \leq g\left(\frac{\nu}{2}\right) \|x\|_{-\frac{\nu(s+b)}{2}}, \ x \in X \text{ and } y \in \bar{B}(x_0, \rho).$$

The method: Let $\delta \in (0, d]$ and $\alpha \in [\delta, a)$. We define the sequence $\{x_{n,\alpha,s}^\delta\}$ iteratively for $n = 0, 1, 2, 3, \ldots$ by

$$x_{n+1,\alpha,s}^\delta = x_{n,\alpha,s}^\delta - \beta \left[L^{-s}(F(x_{n,\alpha,s}^\delta) - y^\delta) + \alpha(x_{n,\alpha,s}^\delta - x_0)\right], \tag{2.3.1}$$

where $x_{0,\alpha,s}^\delta = x_0$ and $\beta := \frac{1}{\beta_0 + a}$. We observe that if $\{x_{n,\alpha,s}^\delta\}$ is converges, then the limit $x_{\alpha,s}^\delta$ (say), is the solution of (2.2.1). Next, we prove the main results of this Section.

Theorem 2.3.2. *For each* $\delta \in (0, d]$ *and* $\alpha \in [\delta, a)$, *the sequence* $\{x_{n,\alpha,s}^\delta\}$ *is in* $\bar{B}(x_0, \rho)$ *and it converges to* $x_{\alpha,s}^\delta$. *Further,*

$$\|x_{n,\alpha,s}^\delta - x_{\alpha,s}^\delta\| \leq k\, q_{\alpha,s}^n,$$

where $q_{\alpha,s} := 1 - \beta\alpha$ *and* $k \geq c_{0,s}(c_{0,s} + 2r_0)$ *with* $\beta := \frac{1}{\beta_0 + a}$.

Proof. Clearly, we have $x_{0,\alpha,s}^\delta = x_0 \in \bar{B}(x_0, \rho)$. Also, since $\rho \geq c_{0,s} R$ by (2.2.3), we have $x_{\alpha,s}^\delta \in \bar{B}(x_0, \rho)$. By the Fundamental Theorem of Integration, we have

$$F(x) - F(u) = \left[\int_0^1 F'(u + \theta(x - u))\, d\theta\right](x - u)$$

whenever x and u are in a ball contained in $D(F)$. We show iteratively that $x^\delta_{n,\alpha,s} \in \bar{B}(x_0,\rho)$, the operator

$$A_{n,\theta} := \int_0^1 F'(x^\delta_{\alpha,s} + \theta(x^\delta_{n,\alpha,s} - x^\delta_{\alpha,s}))d\theta$$

is a well defined, positive self-adjoint operator and

$$\|x^\delta_{n+1,\alpha,s} - x^\delta_{\alpha,s}\|_s \leq (1-\beta\alpha)\|x^\delta_{n,\alpha,s} - x^\delta_{\alpha,s}\|_s$$

for $n = 0,1,2,\ldots$, which will complete the proof, since

$$\|x_0 - x^\delta_{\alpha,s}\| \leq c_{0,s}\|x_0 - x^\delta_{\alpha,s}\|_s \leq c_{0,s}R \leq \rho.$$

Formally, by (1.3.1), we have

$$x^\delta_{n+1,\alpha,s} - x^\delta_{\alpha,s} = x^\delta_{n,\alpha,s} - x^\delta_{\alpha,s} - \beta\left[L^{-s}(F(x^\delta_{n,\alpha,s}) - F(x^\delta_{\alpha,s})) + \alpha(x^\delta_{n,\alpha,s} - x^\delta_{\alpha,s})\right].$$

Since

$$F(x^\delta_{n,\alpha,s}) - F(x^\delta_{\alpha,s}) = A_{n,\theta}(x^\delta_{n,\alpha,s} - x^\delta_{\alpha,s}),$$

we have

$$x^\delta_{n+1,\alpha,s} - x^\delta_{\alpha,s} = [I - \beta(L^{-s}A_{n,\theta} + \alpha I)](x^\delta_{n,\alpha,s} - x^\delta_{\alpha,s}). \qquad (2.3.2)$$

Now, let $n = 0$. We have already seen that $\|x_0 - x^\delta_{\alpha,s}\| < \rho$ so that $x^\delta_{\alpha,s} \in \bar{B}(x_0,\rho)$ and $A_{0,\theta}$ is a well defined positive self-adjoint operator with $\|L^{-\frac{s}{2}}A_{0,\theta}L^{-\frac{s}{2}}\| \leq \beta_0$.

Next assume that for some $n \geq 0$, $x^\delta_{n,\alpha,s} \in \bar{B}(x_0,\rho)$ and $A_{n,\theta}$ is a well defined positive self-adjoint operator with $\|L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}\| \leq \beta_0$. Then from (2.3.2),

$$L^{\frac{s}{2}}(x^\delta_{n+1,\alpha,s} - x^\delta_{\alpha,s}) = \left[I - \beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}} + \alpha I)\right] \; L^{\frac{s}{2}}(x^\delta_{n,\alpha,s} - x^\delta_{\alpha,s}),$$

so

$$\|x^\delta_{n+1,\alpha} - x^\delta_{\alpha,s}\|_s \leq \|I - \beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}} + \alpha I)\| \, \|(x^\delta_{n,\alpha} - x^\delta_{\alpha,s})\|_s.$$

Since $L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}$ and $I - \beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}} + \alpha I)$ are positive self-adjoint operator, we have

$$\begin{aligned}\|I - \beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}} + \alpha I)\| &= \sup_{\|x\|=1} |\langle [I - \beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}} + \alpha I)]x, x\rangle| \\ &= \sup_{\|x\|=1} |(1-\beta\alpha) - \beta\langle L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}x, x\rangle|\end{aligned}$$

and since $\|L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}\| \leq \beta_0$ for all $n \in \mathbb{N}$ and $\beta = \frac{1}{\beta_0+a}$, we have

$$0 \leq \beta\langle L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}x, x\rangle \leq \beta\|L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}\| \leq \beta\beta_0 < 1 - \beta\alpha$$

for all $\alpha \in (0, a)$. Therefore,

$$\|I - \beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}} + \alpha I)\| \leq 1 - \beta\alpha.$$

Thus,

$$\|x^{\delta}_{n+1,\alpha,s} - x^{\delta}_{\alpha,s}\|_s \leq (1 - \beta\alpha)\|x^{\delta}_{n,\alpha} - x^{\delta}_{\alpha,s}\|_s.$$

Hence,

$$\|x^{\delta}_{n+1,\alpha,s} - x^{\delta}_{\alpha,s}\| \leq c_{0,s}\,\|x^{\delta}_{n+1,\alpha,s} - x^{\delta}_{\alpha,s}\|_s \leq c_{0,s}\,\|x_0 - x^{\delta}_{\alpha,s}\|_s$$

and

$$\begin{aligned}
\|x^{\delta}_{n+1,\alpha,s} - x_0\| &\leq c_{0,s}\|x^{\delta}_{n+1,\alpha,s} - x_0\|_s \\
&\leq c_{0,s}\left[\|x^{\delta}_{n+1,\alpha,s} - x^{\delta}_{\alpha,s}\|_s + \|x^{\delta}_{\alpha,s} - x_0\|_s\right] \\
&\leq c_{0,s}(c_{0,s}+1)\|x_0 - x^{\delta}_{\alpha,s}\|_s \leq c_{0,s}(c_{0,s}+1)(2r_0 + c_{0,s}) \leq \rho.
\end{aligned}$$

Thus, $x^{\delta}_{n+1,\alpha,s} \in \bar{B}(x_0, \rho)$. Also, for $0 \leq \theta \leq 1$,

$$\begin{aligned}
\|[x^{\delta}_{\alpha,s} + \theta(x^{\delta}_{n+1,\alpha,s} - x^{\delta}_{\alpha,s})] - x_0\| &= \|(x^{\delta}_{\alpha,s} - x_0) + \theta(x^{\delta}_{n+1,\alpha,s} - x^{\delta}_{\alpha,s})\| \\
&\leq \|x^{\delta}_{\alpha,s} - x_0\| + \theta\|x^{\delta}_{n+1,\alpha,s} - x^{\delta}_{\alpha,s}\| \\
&\leq c_{0,s}[\|x^{\delta}_{\alpha,s} - x_0\|_s + \theta\|x^{\delta}_{n+1,\alpha,s} - x^{\delta}_{\alpha,s}\|_s] \\
&\leq c_{0,s}(c_{0,s}+1)\|x^{\delta}_{\alpha,s} - x_0\|_s \\
&\leq c_{0,s}(c_{0,s}+1)(2r_0 + c_{0,s}) \\
&\leq \rho.
\end{aligned}$$

Hence, $A_{n+1,\theta}$ is a well defined positive self-adjoint operator with $\|L^{-\frac{s}{2}}A_{n+1,\theta}L^{-\frac{s}{2}}\| \leq \beta_0$. This completes the proof. □

2.4 ERROR BOUNDS UNDER SOURCE CONDITIONS

In order to obtain estimate for $\|x_{\alpha,s}^\delta - \hat{x}\|$, we have to impose some non-linearity conditions on F and assume that $x_0 - \hat{x}$ belongs to some source set. We use the following two assumptions to obtain an error estimate for $\|x_{\alpha,s}^\delta - \hat{x}\|$.

Assumption 2.4.1. *There exists a constant $k_0 \geq 0$ such that for every $x \in \bar{B}(x_0, \rho)$ and $v \in X$, there exists an element $\Phi(x, x_0, v) \in X$ such that*

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x, x_0, v)$$

and

$$\|\Phi(x, x_0, v)\| \leq k_0 \|v\| \|x - x_0\|.$$

for all $x, v \in \bar{B}(x_0, \rho)$.

Assumption 2.4.2. *There exists some $E > 0, t > 0$ such that $x_0 - \hat{x} \in X_t$ and*

$$\|x_0 - \hat{x}\|_t \leq E.$$

Theorem 2.4.3. *Let $x_{\alpha,s}^\delta$ be the solution of (2.2.1), let $x_{\alpha,s}$ be solutions of (2.2.2) respectively, and let Assumption 2.4.1 and Assumption 2.4.2 with $t \leq s + b$ hold. Further, suppose*

$$\rho\, k_0 < \frac{f\left(\frac{s}{s+b}\right)}{g\left(\frac{s}{s+b}\right)}.$$

Then, we have the following estimates:

$$(a)\ \text{We have} \quad \|x_{\alpha,s}^\delta - x_{\alpha,s}\| \leq \frac{c_{-s,0}}{f(\frac{s}{s+b}) - g(\frac{s}{s+b})\rho k_0} \frac{\delta}{\alpha^{\frac{b}{s+b}}}.$$

$$(b)\ \text{We have} \quad \|x_{\alpha,s} - \hat{x}\| \leq \frac{g(\frac{s-t}{s+b})}{f(\frac{s}{s+b}) - g(\frac{s}{s+b})k_0\rho} E\alpha^{\frac{t}{s+b}}.$$

$$(c)\ \text{In particular, for } \alpha = \delta^{\frac{s+b}{b+t}},\ \text{we have} \quad \|x_{\alpha,s}^\delta - \hat{x}\| = O\left(\delta^{\frac{t}{t+b}}\right).$$

Proof. Let $A_s = \int_0^1 F'(x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}))d\theta$. Then, since

$$F(x_{\alpha,s}^\delta) - F(x_{\alpha,s}) + \alpha L^s(x_{\alpha,s}^\delta - x_{\alpha,s}) = y^\delta - y,$$

we have

$$(A_s + \alpha L^s)(x_{\alpha,s}^\delta - x_{\alpha,s}) = y^\delta - y.$$

In particular,

$$(F'(x_0) + \alpha L^s)(x^\delta_{\alpha,s} - x_{\alpha,s}) = y^\delta - y + (F'(x_0) - A_s)(x^\delta_{\alpha,s} - x_{\alpha,s}).$$

Therefore, we have

$$\begin{aligned} x^\delta_{\alpha,s} - x_{\alpha,s} &= (F'(x_0) + \alpha L^s)^{-1}\left[y^\delta - y + (F'(x_0) - A_s)(x^\delta_{\alpha,s} - x_{\alpha,s})\right] \\ &= (F'(x_0) + \alpha L^s)^{-1}\left[y^\delta - y - F'(x_0)\int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x^\delta_{\alpha,s} - x_{\alpha,s}), x^\delta_{\alpha,s} - x_{\alpha,s})d\theta\right] \end{aligned}$$

and hence

$$\begin{aligned} \|x^\delta_{\alpha,s} - x_{\alpha,s}\| &\leq \|(F'(x_0) + \alpha L^s)^{-1}(y^\delta - y)\| \\ &\quad + \left\|(F'(x_0) + \alpha L^s)^{-1}F'(x_0)\int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x^\delta_{\alpha,s} - x_{\alpha,s}), x^\delta_{\alpha,s} - x_{\alpha,s})d\theta\right\| \\ &= \Gamma_1 + \Gamma_2, \end{aligned}$$

where

$\Gamma_1 = \|(F'(x_0) + \alpha L^s)^{-1}(y^\delta - y)\|$, and

$$\Gamma_2 = \left\|(F'(x_0) + \alpha L^s)^{-1}F'(x_0)\int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x^\delta_{\alpha,s} - x_{\alpha,s}), x^\delta_{\alpha,s} - x_{\alpha,s})d\theta\right\|.$$

Note that by Proposition 2.3.1, we have

$$\begin{aligned} \Gamma_1 &= \|(F'(x_0) + \alpha L^s)^{-1}(y^\delta - y)\| \\ &= \|L^{-\frac{s}{2}}(L^{-\frac{s}{2}}F'(x_0)L^{-\frac{s}{2}} + \alpha I)^{-1}L^{-\frac{s}{2}}(y^\delta - y)\| \\ &\leq \frac{1}{f\left(\frac{s}{s+b}\right)}\|B_s^{\frac{s}{s+b}}(B_s + \alpha I)^{-1}L^{-\frac{s}{2}}(y^\delta - y)\| \\ &\leq \frac{c_{-s,0}}{f\left(\frac{s}{s+b}\right)}\frac{\delta}{\alpha^{\frac{b}{s+b}}}, \end{aligned}$$

where

$$B_s = L^{-\frac{s}{2}}F'(x_0)L^{-\frac{s}{2}}.$$

Again by Proposition 2.3.1, we have

$$\begin{aligned}
\Gamma_2 &= \left\| L^{-\frac{s}{2}}(B_s+\alpha I)^{-1}L^{-\frac{s}{2}}F'(x_0)\int_0^1 \Phi(x_0, x_{\alpha,s}+\theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})d\theta \right\| \\
&\leq \frac{1}{f\left(\frac{s}{s+b}\right)} \left\| B_s^{\frac{s}{s+b}}(B_s+\alpha I)^{-1}B_s L^{\frac{s}{2}}\int_0^1 \Phi(x_0, x_{\alpha,s}+\theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})d\theta \right\| \\
&\leq \frac{1}{f\left(\frac{s}{s+b}\right)} \left\| (B_s+\alpha I)^{-1}B_s B_s^{\frac{s}{s+b}} L^{\frac{s}{2}}\int_0^1 \Phi(x_0, x_{\alpha,s}+\theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})d\theta \right\| \\
&\leq \frac{g\left(\frac{s}{s+b}\right)}{f\left(\frac{s}{s+b}\right)} \left\| \int_0^1 \Phi(x_0, x_{\alpha,s}+\theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})d\theta \right\| \\
&\leq \frac{g\left(\frac{s}{s+b}\right)}{f\left(\frac{s}{s+b}\right)} k_0 \int_0^1 \|x_0 - x_{\alpha,s} - \theta(x_{\alpha,s}^\delta - x_{\alpha,s})\| \|x_{\alpha,s}^\delta - x_{\alpha,s}\| d\theta \\
&\leq \frac{g\left(\frac{s}{s+b}\right)}{f\left(\frac{s}{s+b}\right)} \rho\, k_0 \,\|x_{\alpha,s}^\delta - x_{\alpha,s}\|.
\end{aligned}$$

The last step follows from the fact that $x_{\alpha,s}, x_{\alpha,s}^\delta \in B(x_0,\rho)$ and hence

$$x_{\alpha,s}+\theta(x_{\alpha,s}^\delta - x_{\alpha,s}) \in B(x_0,\rho).$$

This proves (a). To prove (b), we notice that since $y = F(\hat{x})$, we have by (2.2.2)

$$F(x_{\alpha,s}) - F(\hat{x}) + \alpha L^s(x_{\alpha,s} - x_0) = 0. \tag{2.4.1}$$

Let

$$A = \int_0^1 F'(\hat{x}+\theta(x_{\alpha,s}-\hat{x}))d\theta.$$

Then by (2.4.1) we have

$$(A+\alpha L^s)(x_{\alpha,s}-\hat{x}) = \alpha L^s(x_0 - \hat{x})$$

or

$$(F'(x_0)+\alpha L^s)(x_{\alpha,s}-\hat{x}) = (F'(x_0)-A)(x_{\alpha,s}-\hat{x}) + \alpha L^s(x_0-\hat{x}).$$

Therefore, we have

$$x_{\alpha,s}-\hat{x} = (F'(x_0)+\alpha L^s)^{-1}[(F'(x_0)-A)(x_{\alpha,s}-\hat{x}) + \alpha L^s(x_0-\hat{x})].$$

Hence, using Assumptions 2.4.1 and 2.4.2, we have

$$\begin{aligned}
x_{\alpha,s}-\hat{x} &= L^{-\frac{s}{2}}(B_s+\alpha I)^{-1}L^{-\frac{s}{2}}\left[-F'(x_0)\int_0^1 \Phi(x_0,\hat{x}+\theta(x_{\alpha,s}-\hat{x}),x_{\alpha,s}-\hat{x})d\theta\right]\\
&\quad +\alpha L^{-\frac{s}{2}}(B_s+\alpha I)^{-1}L^{-\frac{s}{2}}L^{s}(x_0-\hat{x})\\
&= L^{-\frac{s}{2}}(B_s+\alpha I)^{-1}L^{-\frac{s}{2}}\left[-F'(x_0)\int_0^1 \Phi(x_0,\hat{x}+\theta(x_{\alpha,s}-\hat{x}),x_{\alpha,s}-\hat{x})d\theta\right]\\
&\quad +\alpha L^{-\frac{s}{2}}(B_s+\alpha I)^{-1}L^{\frac{s}{s}}(x_0-\hat{x}).\\
\|x_{\alpha,s}-\hat{x}\| &\leq \left\|L^{-\frac{s}{2}}(B_s+\alpha I)^{-1}L^{-\frac{s}{2}}[F'(x_0)\int_0^1 \Phi(x_0,\hat{x}+\theta(x_{\alpha,s}-\hat{x}),x_{\alpha,s}-\hat{x})d\theta\right\|\\
&\quad +\alpha\|L^{-\frac{s}{2}}(B_s+\alpha I)^{-1}L^{\frac{s}{2}}(x_0-\hat{x})\|\\
&\leq \frac{1}{f\left(\frac{s}{s+b}\right)}\left\|B_s^{\frac{s}{s+b}}(B_s+\alpha I)^{-1}B_sL^{\frac{s}{2}}\int_0^1 \Phi(x_0,\hat{x}+\theta(x_{\alpha,s}-\hat{x}),x_{\alpha,s}-\hat{x})d\theta\right\|\\
&\quad +\frac{1}{f\left(\frac{s}{s+b}\right)}\alpha\|B_s^{\frac{s}{s+b}}(B_s+\alpha I)^{-1}L^{\frac{s}{2}}(x_0-\hat{x})\|\\
&\leq \frac{1}{f\left(\frac{s}{s+b}\right)}\left\|(B_s+\alpha I)^{-1}B_sB_s^{\frac{s}{s+b}}L^{\frac{s}{2}}\int_0^1 \Phi(x_0,\hat{x}+\theta(x_{\alpha,s}-\hat{x}),x_{\alpha,s}-\hat{x})d\theta\right\|\\
&\quad +\frac{1}{f\left(\frac{s}{s+b}\right)}\alpha\|B_s^{\frac{s}{s+b}}(B_s+\alpha I)^{-1}L^{\frac{s}{2}}(x_0-\hat{x})\|\\
&\leq \frac{g\left(\frac{s}{s+b}\right)}{f\left(\frac{s}{s+b}\right)}k_0\|x_0-\hat{x}-\theta(x_{\alpha,s}-\hat{x})\|\|x_{\alpha,s}-\hat{x}\|\\
&\quad +\frac{1}{f\left(\frac{s}{s+b}\right)}\alpha\|(B_s+\alpha I)^{-1}B_s^{\frac{t}{s+b}}\|\|B_s^{\frac{s-t}{s+b}}L^{\frac{s}{2}}(x_0-\hat{x})\|\\
&\leq \frac{g\left(\frac{s}{s+b}\right)}{f\left(\frac{s}{s+b}\right)}k_0\,\rho\,\|x_{\alpha,s}-\hat{x}\|+\frac{g\left(\frac{s-t}{s+b}\right)}{f\left(\frac{s}{s+b}\right)}\alpha^{\frac{t}{s+b}}\|x_0-\hat{x}\|_t\\
&\leq \frac{g\left(\frac{s}{s+b}\right)}{f\left(\frac{s}{s+b}\right)}k_0\,\rho\,\|x_{\alpha,s}-\hat{x}\|+\frac{g\left(\frac{s-t}{s+b}\right)}{f\left(\frac{s}{s+b}\right)}E\,\alpha^{\frac{t}{s+b}}.
\end{aligned}$$

This completes the proof of (b). Now (c) follows from (a) and (b).

2.5 A PRIORI CHOICE OF THE PARAMETER

Note that by (a) and (b) of Theorem 2.4.3, we have

$$\|x^{\delta}_{\alpha,s}-\hat{x}\|\leq C\left(\frac{\delta}{\alpha^{\frac{b}{s+b}}}+\alpha^{\frac{t}{s+b}}\right), \tag{2.5.1}$$

where

$$C = \max\left\{ \frac{c_{-s,0}}{f\left(\frac{s}{s+b}\right) - g\left(\frac{s}{s+b}\right)\,\rho\, k_0}, \frac{g\left(\frac{s-t}{s+b}\right)\,E}{f\left(\frac{s}{s+b}\right) - g\left(\frac{s}{s+b}\right)\,k_0\,\rho} \right\}. \tag{2.5.2}$$

Further, observe that the error $\frac{\delta}{\alpha^{\frac{b}{s+b}}} + \alpha^{\frac{t}{s+b}}$ in (2.5.1) is of optimal order if $\alpha_\delta := \alpha(t,\delta)$ satisfies

$$\frac{\delta}{\alpha^{\frac{b}{s+b}}} = \alpha^{\frac{t}{s+b}}.$$

That is $\alpha_\delta = \delta^{\frac{s+b}{t+b}}$. Hence by (2.5.1) we have the following Theorem.

Theorem 2.5.1. *Let the assumptions in Theorem 2.3.2 and Theorem 2.4.3 hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \delta^{\frac{s+b}{t+b}}$. Let n_δ be such that*

$$n_\delta := \min\left\{ n : q^n_{\alpha,s} \le \frac{\delta}{\alpha^{\frac{b}{s+b}}} \right\}.$$

Then

$$\|x^\delta_{n_\delta,\alpha,s} - \hat{x}\| = O\left(\delta^{\frac{t}{t+b}}\right).$$

2.5.1 Adaptive Scheme and Stopping Rule

Pereverzev and Schock (2005) introduced the adaptive selection of the parameter strategy, we modified the adaptive method suitably for the situation for choosing the regularization parameter α. For convenience, take $x^\delta_{i,\alpha,s} := x^\delta_{n_i,\alpha_i,s}$. Let $i \in \{0,1,2,\ldots,N\}$ and $\alpha_i = \mu^i\alpha_0$, where $\mu > 1$ and $\alpha_0 > \delta$. Let

$$l := \max\left\{ i : \alpha_i^{\frac{t}{s+b}} \le \frac{\delta}{\alpha_i^{\frac{b}{s+b}}} \right\} < N \quad \text{and} \tag{2.5.3}$$

$$k := \max\left\{ i : \|x^\delta_{i,\alpha,s} - x^\delta_{j,\alpha,s}\| \le 4\bar{C}\frac{\delta}{\alpha_j^{\frac{b}{s+b}}}, j = 0,1,2,\ldots,i-1 \right\} \tag{2.5.4}$$

where $\bar{C} = C + k$, C as in (2.5.2) and k as in Theorem 2.3.2. Now we have the following theorem.

Theorem 2.5.2. *Assume that there exists $i \in \{0,1,\cdots,N\}$ such that*

$$\alpha_i^{\frac{t}{s+b}} \le \frac{\delta}{\alpha^{\frac{b}{s+b}}}.$$

Let the assumptions of Theorem 2.3.2 and Theorem 2.4.3 be fulfilled, and let l and k be as in (2.5.3) and (2.5.4) respectively. Let

$$n_i = \min\left\{ n : q^n_{\alpha_i,s} \le \frac{\delta}{\alpha_i^{\frac{b}{s+b}}} \right\}.$$

Then $l \le k$ and

$$\|x^\delta_{n_k,\alpha,s} - \hat{x}\| \le 6\,\bar{C}\,\mu^{\frac{b}{s+b}}\,\delta^{\frac{t}{t+b}}.$$

Proof. To prove $l \le k$, it is enough to show that, for each $i \in \{1, 2, \ldots N\}$,

$$\alpha_i^{\frac{t}{s+b}} \le \frac{\delta}{\alpha_i^{\frac{b}{s+b}}} \implies \|x^\delta_{n_i,\alpha,s} - x^\delta_{n_j,\alpha,s}\| \le 4\,\bar{C}\,\frac{\delta}{\alpha^{\frac{b}{s+b}}},\ \forall j = 0, 1, 2, \ldots i-1.$$

For $j < i$, we have

$$\begin{aligned}
\| x^\delta_{n_i,\alpha,s} - x^\delta_{n_j,\alpha,s} \| &\le \| x^\delta_{i,\alpha,s} - \hat{x} \| + \| \hat{x} - x^\delta_{j,\alpha,s} \| \\
&\le \bar{C}\left(\alpha_i^{\frac{t}{s+b}} + \frac{\delta}{\alpha_i^{\frac{b}{s+b}}} \right) + \bar{C}\left(\alpha_j^{\frac{t}{s+b}} + \frac{\delta}{\alpha_j^{\frac{b}{s+b}}} \right) \\
&\le 2\,\bar{C}\,\alpha_i^{\frac{t}{t+b}} + 2\,\bar{C}\,\frac{\delta}{\alpha_j^{\frac{b}{s+b}}} \\
&\le 4\,\bar{C}\,\frac{\delta}{\alpha_j^{\frac{b}{s+b}}}.
\end{aligned}$$

Thus, the relation $l \le k$ is proved. Observe that

$$\| \hat{x} - x^\delta_{n_k,\alpha,s} \| \le \| \hat{x} - x^\delta_{n_l,\alpha,s} \| + \| x^\delta_{n_k,\alpha,s} - x^\delta_{n_l,\alpha,s} \|,$$

where

$$\| \hat{x} - x^\delta_{n_l,\alpha,s} \| \le \bar{C}\left(\alpha_l^{\frac{t}{s+b}} + \frac{\delta}{\alpha_l^{\frac{b}{s+b}}} \right) \le 2\,\bar{C}\,\frac{\delta}{\alpha_l^{\frac{b}{s+b}}}.$$

Now since $l \le k$, we have

$$\| x^\delta_{n_k,\alpha,s} - x^\delta_{n_l,\alpha,s} \| \le 4\,\bar{C}\,\frac{\delta}{\alpha_l^{\frac{b}{s+b}}}.$$

Hence

$$\| \hat{x} - x^\delta_{n_k,\alpha,s} \| \le 6\,\bar{C}\,\frac{\delta}{\alpha_l^{\frac{b}{s+b}}}$$

Now, since $\alpha_\delta^{\frac{b}{s+b}} = \delta^{\frac{b}{t+b}} \le \alpha_{l+1}^{\frac{b}{s+b}} \le \mu^{\frac{b}{s+b}} \alpha_l{}^{\frac{b}{s+b}}$, it follows that

$$\frac{\delta}{\alpha_l^{\frac{b}{s+b}}} \le \mu^{\frac{b}{s+b}} \frac{\delta}{\alpha_\delta^{\frac{b}{s+b}}} = \mu^{\frac{b}{s+b}} \delta^{\frac{t}{t+b}}.$$

This completes the proof.

2.6 IMPLEMENTATION OF ADAPTIVE CHOICE RULE

The balancing algorithm associated with the choice of the parameter specified in Theorem 2.5.2 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \ldots, N$.

2.6.1 Algorithm

1. Set $i = 0$.
2. Choose $n_i := \min \left\{ n : q^n_{\alpha_i,s} \le \frac{\delta}{\alpha_i^{\frac{b}{s+b}}} \right\}$.
3. Solve $x_{i,\alpha,s} := x^\delta_{n_i,\alpha_i,s}$ by using the iteration (2.3.1).
4. If $\|x_{i,\alpha,s} - x_{j,\alpha,s}\| > 4\bar{C}\frac{\delta}{\alpha_j^{\frac{b}{s+b}}}, j < i$, then take $k = i - 1$ and return $x_{k,\alpha,s}$.
5. Else set $i = i + 1$ and go to 2.

2.7 NUMERICAL EXPERIMENTS

In this section, we present a numerical experiment for the elliptic boundary-value problem 2.1.2 and compare the results of method (2.3.1) with that of method (2.1.4). Let us define the linear operator

$$L : H^2 \cap H^1_0[0,1] \subset L^2[0,1] \mapsto L^2[0,1]$$

by $Lx = -x''$. Then L is densely defined, self-adjoint and positive definite (Jin (2000)) and the Hilbert scale $\{X\}_s$ generated by L is given by

$$X_s = \left\{ x \in H^s[0,1] : x^{(2l)}(0) = x^{(2l)}(1) = 0, l = 0, 1, \dots, \left[\frac{s}{2} - \frac{1}{4} \right] \right\}$$

for any $s \in \mathbb{R}$, where $H^s[0,1]$ is the usual Sobolev space and

$$\|x\|_s = \int_0^1 |x^{(s)}(t)| dt$$

for all $s = 0, 1, 2, \dots$. We have taken $s = b = 2$ in our computation. Table 2.1 and 2.2 gives the number of iterations, alpha and the relative error.

Table 2.1: Table showing the number of iterations, alpha and the error for $\mu = 1.15$ $\delta = 1/153, \beta = 0.25$.

Function	Method (2.3.1)				Method in (2.1.4)			
	k	n_k	$\alpha(k)$	$\frac{\|\hat{x} - x^\delta_{n,\alpha_k,s}\|}{\|x^\delta_{n,\alpha_k,s}\|}$	k	n_k	$\alpha(k)$	$\frac{\|\hat{x} - x^\delta_{n,\alpha_k}\|}{\|x^\delta_{n,\alpha_k}\|}$
$\hat{x} = \min\{x, 1-x\}$, $x \in [0,1]$	5	31	0.0829	0.0074	7	24	0.0312	0.3021
$\hat{x} = x^2$ if $0.2 < x < 0.7$, else $\hat{x} = x$	5	32	0.0954	0.0093	6	27	0.0474	0.4655

Table 2.2: Table showing the number of iterations, alpha and the error for $\mu = 1.25$ $\delta = 1/590, \beta = 0.25$.

Function	Method (2.3.1)				Method in (2.1.4)			
	k	n_k	$\alpha(k)$	$\frac{\|\hat{x} - x^\delta_{n,\alpha_k,s}\|}{\|x^\delta_{n,\alpha_k,s}\|}$	k	n_k	$\alpha(k)$	$\frac{\|\hat{x} - x^\delta_{n,\alpha_k}\|}{\|x^\delta_{n,\alpha_k}\|}$
$\hat{x} = \min\{x, 1-x\}$, $x \in [0,1]$	23	15	0.0080	0.0085	20	16	0.0100	0.0926
$\hat{x} = x^2$ if $0.2 < x < 0.7$, else $\hat{x} = x$	12	20	0.0245	0.0086	15	18	0.0157	0.1548

Remark 2.7.1. *From the tables and figures, one can see that the method (2.3.1) gives a better approximation than the method (2.1.4).*

2.8 CONCLUSION

In this chapter, we considered a derivative-free iterative method for approximately solving ill-posed equations involving a monotone operator in the setting of Hilbert scales. We obtained an optimal order error estimate under a general Hölder-type source condition. Also we considered the adaptive parameter choice strategy considered by Pereverzev and Schock (2005), for choosing the regularization parameter.

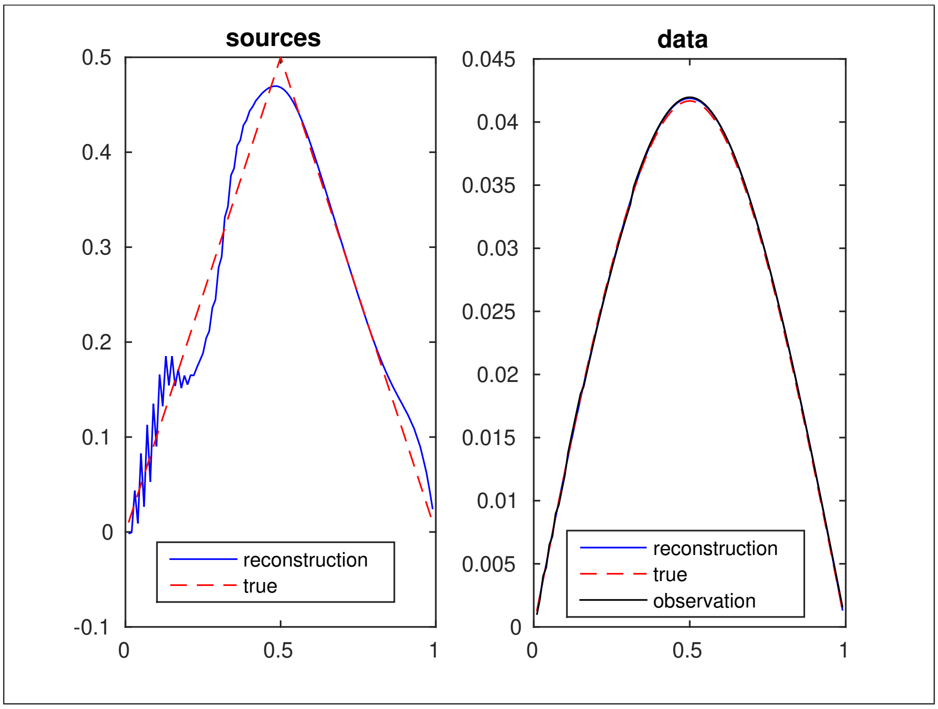

Figure 2.1: Exact and approximate data and solution of method (2.3.1) for $\hat{x} = \min\{x, 1-x\}$, where $\mu = 1.15$ $\delta = 1/153, \beta = 0.25$.

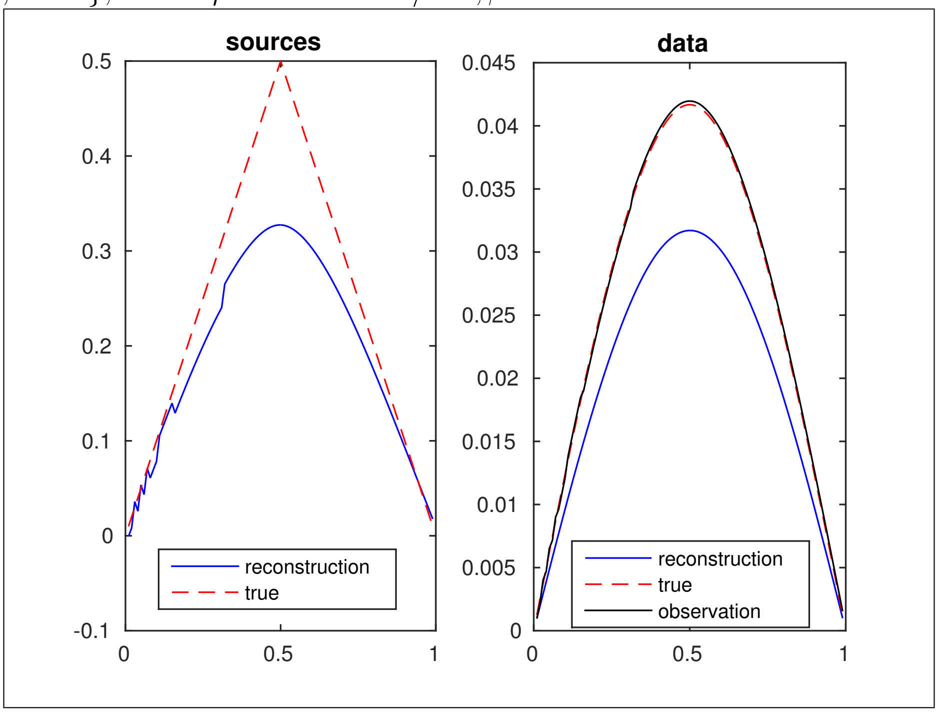

Figure 2.2: Exact and approximate data and solution of method (2.1.4) for $\hat{x} = \min\{x, 1-x\}$, where $\mu = 1.15$ $\delta = 1/153, \beta = 0.25$.

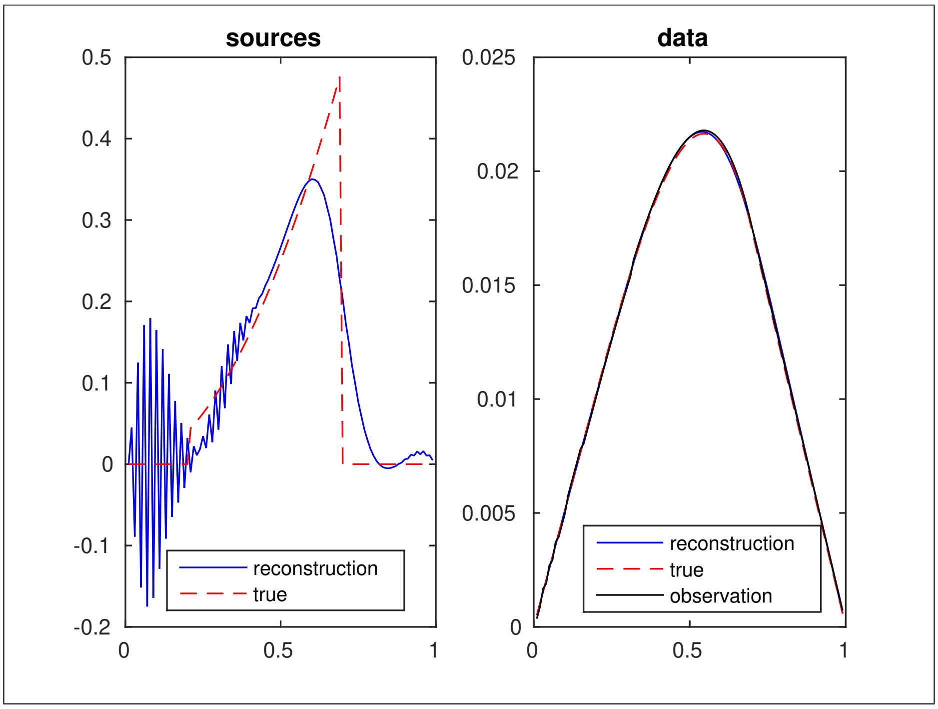

Figure 2.3: Exact and approximate data and solution of method (2.3.1) for $\hat{x} = x^2$ if $0.2 < x < 0.7$, else $\hat{x} = x$, where $\mu = 1.15$ $\delta = 1/153, \beta = 0.25$.

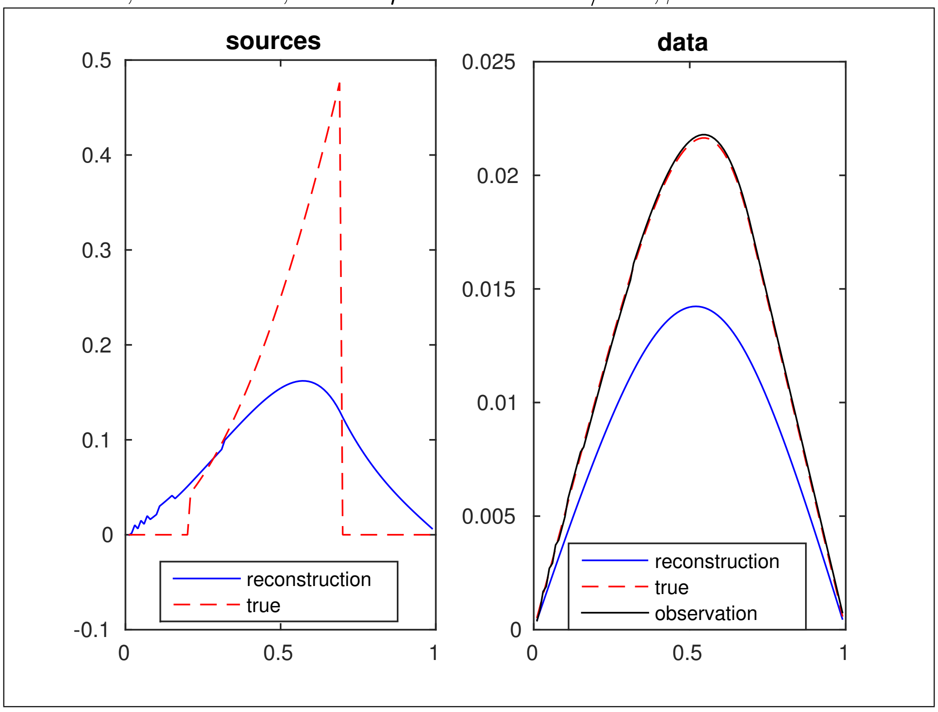

Figure 2.4: Exact and approximate data and solution of method (2.1.4) for $\hat{x} = x^2$ if $0.2 < x < 0.7$, else $\hat{x} = x$, where $\mu = 1.15$ $\delta = 1/153, \beta = 0.25$.

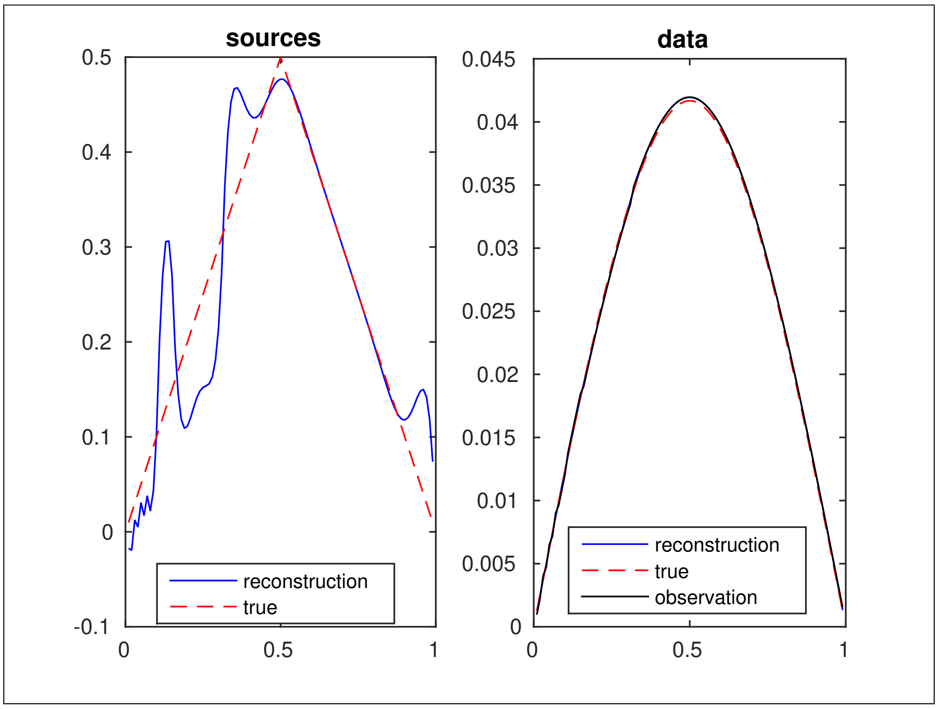

Figure 2.5: Exact and approximate data and solution of method (2.3.1) for $\hat{x} = \min\{x, 1-x\}$, where $\mu = 1.25$ $\delta = 1/590, \beta = 0.25$.

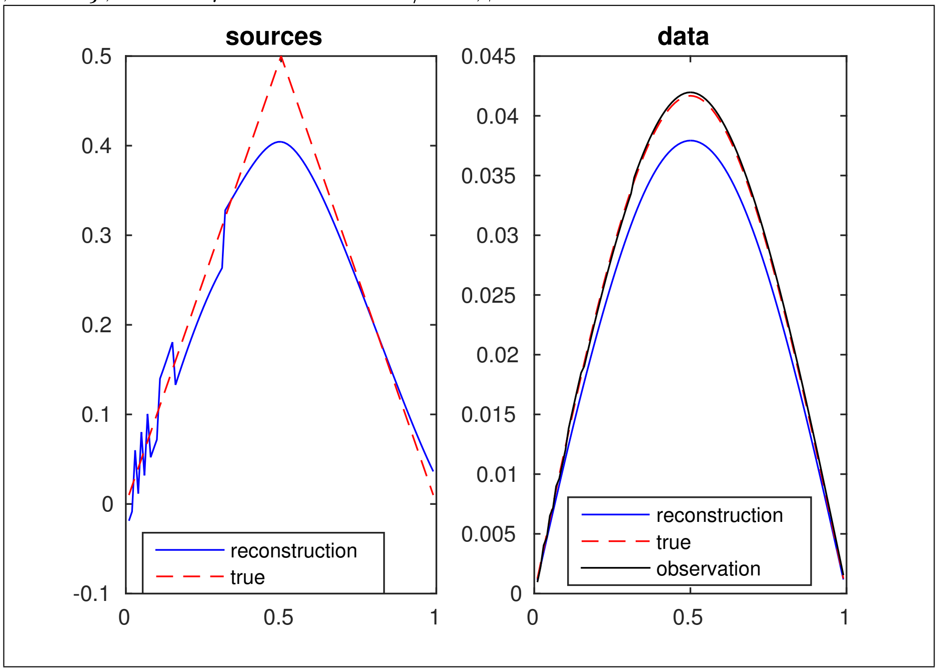

Figure 2.6: Exact and approximate data and solution of method (2.1.4) for $\hat{x} = \min\{x, 1-x\}$, where $\mu = 1.25$ $\delta = 1/590, \beta = 0.25$.

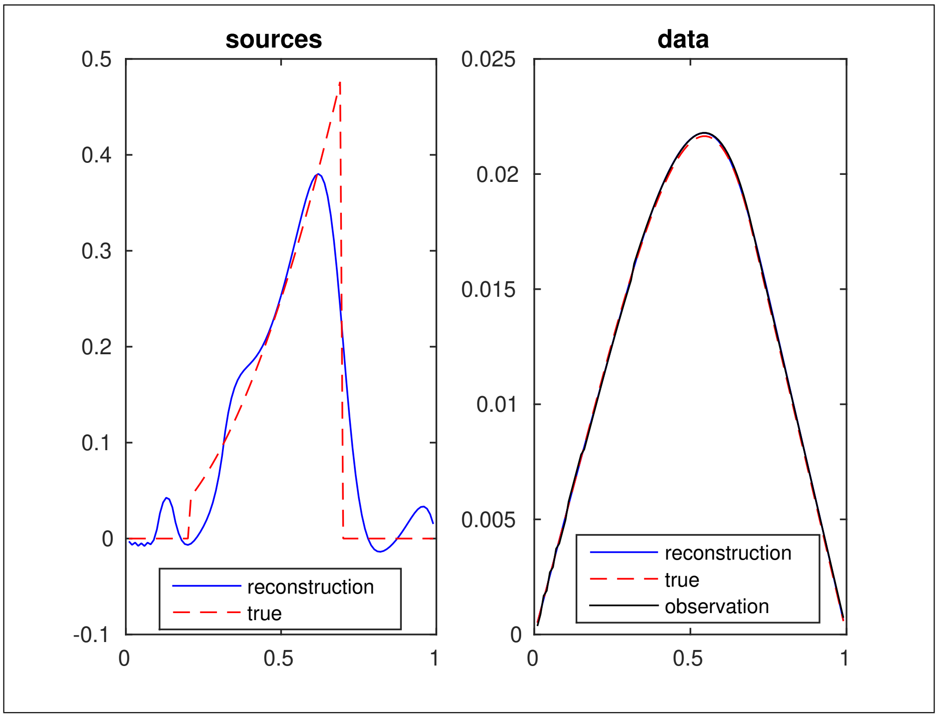

Figure 2.7: Exact and approximate data and solution of method (2.3.1) for $\hat{x} = x^2$ if $0.2 < x < 0.7$, else $\hat{x} = x$, where $\mu = 1.25$ $\delta = 1/590, \beta = 0.25$.

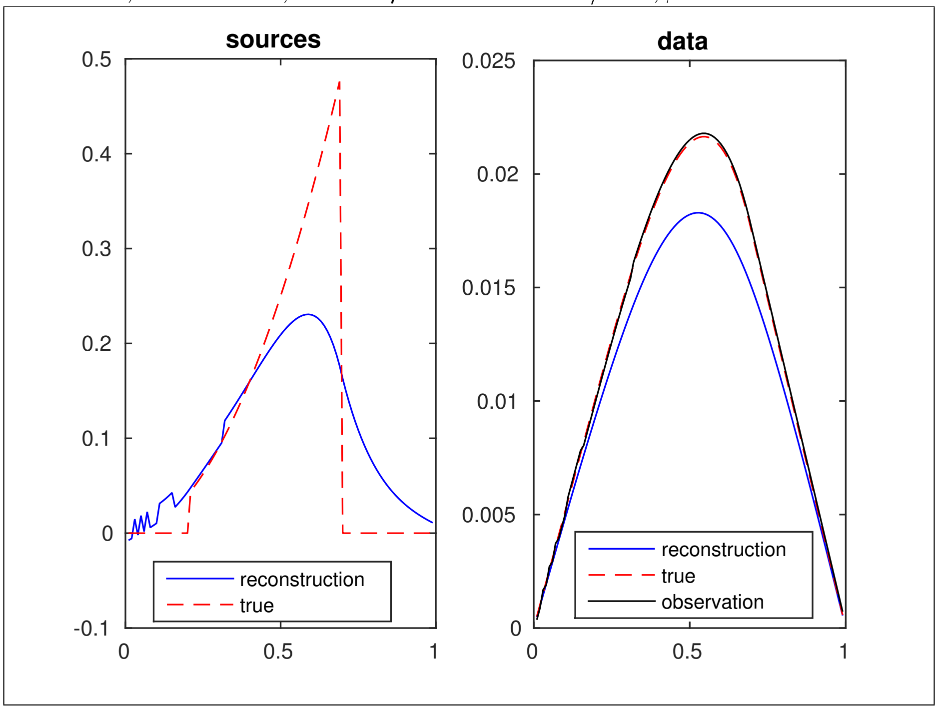

Figure 2.8: Exact and approximate data and solution of method (2.1.4) for $\hat{x} = x^2$ if $0.2 < x < 0.7$, else $\hat{x} = x$, where $\mu = 1.25$ $\delta = 1/590, \beta = 0.25$.

CHAPTER 3

DISCREPANCY PRINCIPLES FOR FRACTIONAL TIKHONOV REGULARIZATION

3.1 INTRODUCTION

Let $T : X \to Y$ be a bounded linear operator with non closed range $R(T)$ and $y \in R(T) + R(T)^{\perp}$. In this Chapter, we consider the problem of approximating the minimal norm least square solution $\hat{x} = T^{\dagger}y$ of ill-posed operator equation

$$Tx = y \tag{3.1.1}$$

with the help of well-posed equations (Engl et al. (1996); Groetsch (1977, 1984); Guacaneme (1988); Nair (2009)). As, already mentioned in Chapter 2, we assume that the available data y^{δ} satisfies

$$\|y - y^{\delta}\| \leq \delta. \tag{3.1.2}$$

Since (3.1.1) is ill-posed, one has to use some regularization method to approximate $\hat{x}$.

It is known that the solution x_{α}^{δ} of (1.2.4) over smooths the solution $\hat{x}$, and to overcome this problem many authors considered fractional or weighted Tikhonov regularization method (Bianchi et al. (2015); Bianchi and Donatelli (2017); Gerth et al. (2015); Hochstenbach and Reichel (2011); Huckle and Sedlacek (2012); Hochstenbach et al. (2015); Klann and Ramlau (2008); Reddy (2018)) for approximating $\hat{x}$. In this method, the minimizer of the functional

$$J_{\alpha,w}(x) := \|Tx - y^\delta\|_W^2 + \alpha\|x\|^2 \tag{3.1.3}$$

is taken as an approximation for $\hat{x}$. Here

$$\|y\|_W = \|W^{\frac{\beta}{2}} y\|_Y, \quad \beta \in \left[-\frac{1}{2}, 0\right] \tag{3.1.4}$$

with $W = (TT^*)$ is the weighted semi-norm. Reddy (2018) consider the Engl type discrepancy principle for choosing the regularization parameter α. Precisely, Reddy consider the following discrepancy principles

$$G(\alpha, y^\delta) := \|\alpha\big((T^*T)^{\frac{\beta+1}{2}} + \alpha I\big)^{-1} (T^*T)^{\frac{\beta-1}{2}} T^* y^\delta\|^2 = \tau_1 \frac{\delta^p}{\alpha^q}, \ \tau_1 > 0$$

and

$$G_1(\alpha, y^\delta) := \|T^*Tx^\delta_{\alpha,\beta} - T^*y^\delta\|^2 = \frac{\delta^p}{\alpha^q} \quad p > 0\,, q > 0, \ \alpha > 0$$

for choosing the regularization parameter α for weighted Tikhonov regularization with the weighted semi-norm

$$\|y\|_W = \|W^{\frac{\beta-1}{4}} y\|_Y,$$

for some parameter $0 \le \beta \le 1$. Throughout this Chapter $x^\delta_{\alpha,\beta}$ is the minimizer of (3.1.3).

In this Chapter, we consider the fractional or weighted Tikhonov regularization method (3.1.3) with the weighted semi-norm (3.1.4) and studied the parameter choice strategy for choosing the parameter α in (3.1.3), namely, the Schock type (Schock (1984b)) discrepancy principle

$$G(\alpha, y^\delta) := \|Tx^\delta_{\alpha,\beta} - y^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0\,, q > 0. \tag{3.1.5}$$

The rest of the Chapter is organized as follows. In Section 3.2, we consider the error analysis of the proposed method and in Section 3.3 we consider the Schock-type discrepancy principle. Numerical examples are provided in Section 3.4 and the Chapter ends with concluding remarks in Section 3.5.

3.2 ERROR ANALYSIS

Let $x^\delta_{\alpha,\beta}$ be the minimizer of (3.1.3) with semi norm (3.1.4). Then

$$x^\delta_{\alpha,\beta} = \Big((T^*T)^{1+\beta} + \alpha I\Big)^{-1}(T^*T)^\beta T^* y^\delta. \tag{3.2.1}$$

Let

$$x_{\alpha,\beta} = \Big((T^*T)^{1+\beta} + \alpha I\Big)^{-1}(T^*T)^\beta T^* y \tag{3.2.2}$$

and

$$\hat{x} \in R\Big((T^*T)^\nu\Big), \quad 0 < \nu \leq 1+\beta. \tag{3.2.3}$$

We need the following result for our error analysis.

Proposition 3.2.1. *(cf. Louis (1989), Proposition 3.4.3) Let $x^\delta_{\alpha,\beta}$ and $x_{\alpha,\beta}$ be as in* (3.2.1) *and* (3.2.2), *respectively. Let $\hat{x}$ satisfies* (3.2.3). *Then*

$$(i) \quad \|x^\delta_{\alpha,\beta} - x_{\alpha,\beta}\| \leq c_1 \frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}},$$

$$(ii) \quad \|\hat{x} - x_{\alpha,\beta}\| \leq c_2\, \alpha^{\frac{\nu}{1+\beta}}.$$

In particular, we have

$$(iii) \quad \|\hat{x} - x^\delta_{\alpha,\beta}\| \leq c_1 \frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}} + c_2\, \alpha^{\frac{\nu}{1+\beta}}.$$

Proof. Note that by (3.2.1) and (3.2.2) we have

$$\begin{aligned}
\|x^\delta_{\alpha,\beta} - x_{\alpha,\beta}\| &= \left\|\Big((T^*T)^{1+\beta} + \alpha I\Big)^{-1}(T^*T)^\beta T^*(y - y^\delta)\right\| \\
&\leq \sup_{\lambda\in\sigma(T^*T)} \left|\frac{\lambda^{(\beta+\frac{1}{2})}}{\lambda^{1+\beta}+\alpha}\right| \|y - y^\delta\| \\
&\leq c_1 \frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}}.
\end{aligned}$$

This proves (i). Again by (3.2.2), we have

$$\begin{aligned}
||\hat{x} - x_{\alpha,\beta}|| &= \left\| \hat{x} - \left((T^* T)^{1+\beta} + \alpha I \right)^{-1} (T^* T)^{\beta} T^* y \right\| \\
&= \left\| \left((T^* T)^{1+\beta} + \alpha I \right)^{-1} \alpha \, \hat{x} \right\| \\
&= \left\| \alpha \left((T^* T)^{1+\beta} + \alpha I \right)^{-1} (T^* T)^{\nu} z \right\|, \quad \nu \leq 1 + \beta \\
&\leq \sup_{\lambda \in \sigma(T^* T)} \left| \frac{\alpha \lambda^{\nu}}{\lambda^{1+\beta} + \alpha} \right| \|z\| \\
&\leq c_2 \, \alpha^{\frac{\nu}{1+\beta}}.
\end{aligned}$$

This proves (ii), now (iii) follows from (i) and (ii).

□

Remark 3.2.2. *(cf. Bianchi et al. (2015), Proposition 10) Observe that,* $\frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}}$ *is decreasing for* $\beta \in [-\frac{1}{2}, 0]$*, whereas* $\alpha^{\frac{\nu}{1+\beta}}$ *is increasing for* $\beta \in [-\frac{1}{2}, 0]$*. Therefore, one has to choose* $\beta \in [-\frac{1}{2}, 0]$*, such that* $\frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}} = \alpha^{\frac{\nu}{1+\beta}}$ *in order to obtain an optimal order error estimate for* $\|\hat{x} - x^{\delta}_{\alpha,\beta}\|$*. For a fixed,* $\delta > 0$*,* $\nu > 0$ *and* $\alpha \in [\delta^{\frac{2}{2\nu+1}}, \delta^{\frac{1}{2\nu+1}}]$*, the best possible choice for* β *is*

$$\beta = \left(\frac{2\nu + 1}{2} \right) \frac{\log \alpha}{\log \delta} - 1.$$

In this case $\beta \in [-\frac{1}{2}, 0]$ *and* $\alpha = \delta^{\frac{2(\beta+1)}{2\nu+1}}$*. In Section* 3.3*, we study Schock-type discrepancy principle for choosing* α *in (3.2.1), for a fixed* β*.*

3.3 SCHOCK-TYPE DISCREPANCY PRINCIPLE

In this Section, we consider Schock-type discrepancy principle for choosing the regularization parameter α. We obtained error estimate for $\|\hat{x} - x^{\delta}_{\alpha,\beta}\|$, when the parameter α is chosen according to the Schock-type discrepancy principle and the error estimate is of order optimal if the parameters are chosen properly. Interested readers may refer Section 3 in (Hochstenbach and Reichel (2011)) for a detailed discussion about the dependance of the parameter α and β.

Next, we prove some lemmas to prove our main result in this section.

Lemma 3.3.1. *Let* $p > 0, q > 0$ *and* y^δ *satisfies* (3.1.2). *Then there exists a unique* α *such that* (3.1.5) *holds.*

Proof. Consider the function $H(\alpha, y^\delta) = \alpha^q\, G(\alpha, y^\delta)$. By spectral radius theorem, we have

$$H(\alpha, y^\delta) = \alpha^{2q} \int_0^{\|T\|^2} \left(\frac{\alpha\,\lambda}{\lambda^{2\beta+1} + \alpha}\right)^2 d\|E_\lambda y^\delta\|^2$$

where E_λ is the spectral family of $T\,T^*$. Hence, $H(\alpha, y^\delta)$ is strictly increasing, continuous, $H(\alpha, y^\delta)$ tends to ∞ as α tends to ∞ and tends to 0 as α tends to 0. The result now follows from the intermediate value theorem.

□

Lemma 3.3.2. *Let* $\alpha = \alpha(\delta)$ *be the unique solution of (3.1.5). If* $y \neq 0$, *then* $\alpha(\delta) \to 0$ *as* $\delta \to 0$. *Furthermore, for* $0 < \delta \leq \frac{\|y\|}{2}$, $\alpha(\delta) = O(\delta^{\frac{p}{q+1}})$.

Proof. Suppose $\alpha(\delta)$ does not converges to 0 as δ tends to 0. Then there exists a sequence (δ_n) with $\delta_n \to 0$ and $\alpha(\delta_n) \to r > 0$. From (3.1.5) we have

$$\delta^p = \alpha^q \|Tx^\delta_{\alpha,\beta} - y^\delta\|.$$

In particular, we have

$$\delta_n^p = \alpha(\delta_n)^q \left\| T\left((T^*\,T)^{1+\beta} + \alpha(\delta_n) I\right)^{-1} (T^*\,T)^\beta T^* y^{\delta_n} - y^{\delta_n}\right\|.$$

As δ_n tends to 0, we get

$$0 = r^q \left\| T\left((T^*\,T)^{1+\beta} + rI\right)^{-1} (T^*\,T)^\beta T^* y - y\right\|$$

and hence

$$\left((T\,T^*)^{1+\beta} + rI\right)^{-1} [(T\,T^*)^{1+\beta} y - (T\,T^*)^{1+\beta} y - r\,y] = 0$$

i.e., $r\,y = 0$ implies $y = 0$. Which is a contradiction to our assumption. Thus $\alpha(\delta) \to 0$ as $\delta \to 0$. Now by (3.1.5) and (3.2.1), we have

$$\begin{aligned}
\|y^\delta\| - \frac{\delta^p}{\alpha^q} &= \|y^\delta\| - \|Tx^\delta_{\alpha,\beta} - y^\delta\| \\
&\leq \|Tx^\delta_{\alpha,\beta}\| \\
&= \frac{1}{\alpha}\|T(\alpha x^\delta_{\alpha,\beta})\| \\
&\leq \frac{\|T\|}{\alpha}\|\alpha x^\delta_{\alpha,\beta}\| \\
&= \frac{\|T\|}{\alpha}\|(T^*T)^\beta T^*(Tx^\delta_{\alpha,\beta} - y^\delta)\| \\
&\leq \frac{\|T\|^{2(1+\beta)}}{\alpha}\|Tx^\delta_{\alpha,\beta} - y^\delta\| \\
&\leq \frac{\|T\|^{2(1+\beta)}}{\alpha^{q+1}}\delta^p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\|y\|}{2} \leq \|y\| - \delta &\leq \|y\| - \|y - y^\delta\| \leq \|y^\delta\| \\
&\leq \frac{\|T\|^{2(1+\beta)} + \alpha}{\alpha^{q+1}}\delta^p
\end{aligned}$$

and hence

$$\alpha^{q+1} \leq \frac{2(\|T\|^{2(1+\beta)} + \alpha)}{\|y\|}\delta^p.$$

From this the result follows.

□

Hereafter, we assume that $0 < \delta \leq \frac{\|y\|}{2}$.

Lemma 3.3.3. *Suppose $\hat{x} \in R((T^*\,T)^\nu)$, $0 < \nu \leq 1 + \beta$. Then*

$$\left\|\alpha\left((T\,T^*)^{1+\beta} + \alpha I)\right)^{-1} T\hat{x}\right\| = O(\alpha^w),$$

where $w = \min\left\{1,\ \frac{2\nu+1}{2(1+\beta)}\right\}$.

Proof. Let $T = U\,(T^*T)^{1/2}$ be the polar decomposition of T, where U is a unitary operator, and let $z \in X$ be such that $\hat{x} = (T^*T)^\nu z$. Then we have

$$\begin{aligned}
\left\|\alpha\Big((T\,T^*)^{1+\beta} + \alpha I\Big)^{-1} T\hat{x}\right\| &= \left\|\alpha T\Big((T^*\,T)^{1+\beta} + \alpha I)\Big)^{-1} \hat{x}\right\| \\
&= \left\|\alpha U\,(T^*T)^{1/2}\Big((T^*\,T)^{1+\beta} + \alpha I\Big)^{-1} (T^*T)^\nu\, z\right\| \\
&\leq \sup_{\lambda\in\sigma(T^*T)} \left\{\frac{\alpha\,\lambda^{(\nu+\frac{1}{2})}}{\lambda^{1+\beta}+\alpha}\right\} \|z\| \\
&\leq \begin{cases} c_1\ \alpha^{\frac{2\nu+1}{2(1+\beta)}} & if\ \nu < \beta + \frac{1}{2} \\ c_2\ \alpha & if\ \nu \geq \beta + \frac{1}{2} \end{cases} \\
&= O(\alpha^w).
\end{aligned}$$

□

Lemma 3.3.4. *Let $\hat{x} \in R((T^*\,T)^\nu)$, $0 < \nu \leq 1 + \beta$. Suppose*

$$w = \min\left\{1, \frac{2\nu+1}{2(1+\beta)}\right\},\ \ \frac{p}{q+1} \leq \min\left\{\frac{1}{w}, \frac{2}{\left[1 + \frac{2\nu}{1+\beta} + \frac{1-w}{q}\right]}\right\}$$

and $\alpha = \alpha(\delta)$ is chosen according to (3.1.5). Then

$$\frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}} = O(\delta^\mu),\ \ \mu = 1 - \frac{p}{q+1}\left[\frac{1}{2(1+\beta)} + \frac{1-w}{2q(1+\beta)}\right].$$

Proof. Using (3.1.5) and (3.2.1), we have

$$\begin{aligned}
\|Tx^\delta_{\alpha,\beta} - y^\delta\| &= \|T((T^*T)^{1+\beta} + \alpha I)^{-1}(T^*T)^\beta T^* y^\delta - y^\delta\| \\
&= \|((TT^*)^{1+\beta} + \alpha I)^{-1}(TT^*)^\beta TT^* y^\delta - y^\delta\| \\
&= \|\alpha((TT^*)^{1+\beta} + \alpha I)^{-1} y^\delta\| \\
&\leq \|\alpha((TT^*)^{1+\beta} + \alpha I)^{-1}(y^\delta - y)\| + \|\alpha((TT^*)^{1+\beta} + \alpha I)^{-1} y\|,
\end{aligned}$$

where $\|\alpha((TT^*)^{1+\beta} + \alpha I)^{-1}(y^\delta - y)\| \leq \delta$ and by Lemma 3.3.3, we have

$$\|\alpha((TT^*)^{1+\beta} + \alpha I)^{-1} y\| = O(\alpha^w).$$

That is, we have by Lemma 3.3.2

$$\frac{\delta^p}{\alpha^q} = \|Tx^\delta_{\alpha,\beta} - y^\delta\| \leq \delta + c\ \delta^{\frac{pw}{q+1}}.$$

Hence

$$\begin{aligned}
\frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}} &= \delta^{1-\frac{p}{2q(1+\beta)}} \left[\frac{\delta^p}{\alpha^q}\right]^{\frac{1}{2q(1+\beta)}} \\
&\leq \delta^{\frac{2q(1+\beta)-p}{2q(1+\beta)}} \left[\delta + c\delta^{\frac{pw}{q+1}}\right]^{\frac{1}{2q(1+\beta)}} \\
&\leq \left[\delta^{2q(1+\beta)-p+1} + c\delta^{2q(1+\beta)-p+\frac{pw}{q+1}}\right]^{\frac{1}{2q(1+\beta)}} \\
&\leq \delta^{\frac{2q(1+\beta)+1-p}{2q(1+\beta)}} + c\delta^{1-\frac{p}{q+1}\left[\frac{1}{2(1+\beta)}+\frac{1-w}{2q(1+\beta)}\right]}.
\end{aligned}$$

This completes the proof.

□

Now, we state the main theorem of this Section.

Theorem 3.3.5. *Let* $\hat{x} \in R((T^* T)^\nu)$, $0 < \nu \leq 1 + \beta$ *and* $0 < \delta \leq \frac{\|y\|}{2}$. *Suppose*

$$w = \min\left\{1, \frac{2\nu+1}{2(1+\beta)}\right\}, \ \frac{p}{q+1} \leq \min\left\{\frac{1}{w}, \frac{2}{\left[1 + \frac{2\nu}{1+\beta} + \frac{1-w}{q}\right]}\right\}$$

and $\alpha = \alpha(\delta)$ *is chosen according to (3.1.5). Then*

$$\|\hat{x} - x^\delta_{\alpha,\beta}\| = O(\delta^\rho), \ \rho = \min\left\{\mu, \frac{p\nu}{(q+1)(1+\beta)}\right\}.$$

In particular, if

$$\frac{p}{q+1} = \frac{2(1+\beta)}{2\nu + 1 + \frac{1-w}{q}},$$

then

$$\|\hat{x} - x^\delta_{\alpha,\beta}\| = O\left(\delta^{\frac{2\nu}{2\nu+1+\frac{1-w}{q}}}\right).$$

Proof. The proof of the first part follows from Proposition 3.2.1, Lemma 3.3.2 and Lemma 3.3.3 and the second part follows by noting that

$\mu = \frac{p\nu}{(q+1)(1+\beta)}$ if and only if $\frac{p}{q+1} = \frac{2(1+\beta)}{2\nu+1+\frac{1-w}{q}}$.

□

Remark 3.3.6. *1. Note that we obtained the optimal rate* $O(\delta^{\frac{2\nu}{2\nu+1}})$ *for* $\beta + \frac{1}{2} \leq \nu \leq 1 + \beta$ *by choosing* $\frac{p}{q+1} = \frac{2(1+\beta)}{2\nu+1}$.

3.4 NUMERICAL EXAMPLES

In this section, we pick up two examples for the numerical discussion to validate our theoretical results. The discrete version of the operator T is taken from the Regularization Toolbox by Hansen (2007). We take the singular value decomposition (SVD)

$$T = U\Sigma V^T, \tag{3.4.1}$$

where $U = [u_1, u_2, \ldots, u_n] \in \mathbb{R}^{n\times n}$ and $V = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and

$$\Sigma = diag[\lambda_1, \lambda_2, \ldots, \lambda_n] \in \mathbb{R}^{n\times n},$$

whose singular values are ordered according to

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0.$$

Substituting the SVD (3.4.1) into (3.2.1) and (3.1.5) yield

$$x^\delta_{\alpha,\beta} = V(\Sigma^{2(\beta+1)} + \alpha I)^{-1}\Sigma^{2\beta+1}U^T y^\delta \tag{3.4.2}$$

and

$$G(\alpha, y^\delta) := \|\alpha U(\Sigma^{2(\beta+1)} + \alpha I)^{-1}U^T y^\delta\|^2 = \frac{\delta^{2p}}{\alpha^{2q}}. \tag{3.4.3}$$

We adopted the Newton's method to solve above nonlinear equation for α with different values β, δ and q with $q = p - 1$. Precisely, we use the Newton's iteration $\alpha_m = \alpha_{m-1} - \frac{f(\alpha_{m-1})}{f'(\alpha_{m-1})}$, $m = 1, 2, 3, \ldots$ with an initial guess α_0, where $f(\alpha) = \alpha^{2(q+1)}\|U(\Sigma^{2(\beta+1)} + \alpha I)^{-1}U^T y^\delta\|^2 - \delta^{2p}$ for solving (3.4.3).

Relative errors $E_{\alpha,\beta} := \left(\frac{\|x^\delta_{\alpha,\beta} - \hat{x}\|}{\|\hat{x}\|}\right)$, and α are presented in the tables for different values of β, p, n (size of the mesh) and noise level δ. In each figure, plot (a) contains the computed solution (C.S) and exact solution (exact sol.) for method (3.1.5) plot (b) contains the exact data and noise data; and plots (c) contains the computed solution (C.S) and exact solution (exact sol.) for method (1.2.7).

Example 3.4.1. We choose *Foxgood* example from the Regularization Toolbox by Hansen (2007) with n points. It is defined as follows:

$$[Tx](s) := \int_0^1 \sqrt{s^2+t^2}x(t)dt = y(s),\ 0 \le s \le 1 \tag{3.4.4}$$

with noise free data $y(s) = \frac{1}{3}((1+s^2)^{3/2} - s^3)$ and solution $\hat{x}(t) = t$. The exact data contaminated by introducing random noise level $\delta = 0.05$ and 0.01. In Table 3.1 and Table 3.2 respectively, we present the relative errors as well as α values using discrepancy principle (3.1.5) and (1.2.7) (i.e. Morozov's) with $x^{\delta}_{\alpha,\beta}$ replacing x^{δ}_{α}, for different values of β, δ, p and n; and the best reconstruction happened at $\beta = -0.15$. Plots of *Foxgood* example for different values of δ, n and β are given in Fig:3.1 - Fig: 3.18, with captions.

Table 3.1: Relative errors of Foxgood example using Schock-type discrepancy principle.

β		$n=100$		$n=500$		$n=1000$	
		$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$
	α	$5.5642e-03$	$1.0888e-03$	$2.2502e-03$	$4.4600e-04$	$1.6203e-03$	$3.3702e-04$
0	$E_{\alpha,\beta}$	$5.5000e-02$	$3.8270e-02$	$3.0538e-02$	$1.4404e-02$	$3.7297e-02$	$2.5219e-02$
	p	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$
	α	$8.3551e-03$	$1.5829e-03$	$1.7749e-03$	$1.3546e-03$	$1.6727e-03$	$4.7055e-04$
-0.15	$E_{\alpha,\beta}$	$2.5289e-02$	$1.4220e-02$	$1.3625e-02$	$1.0526e-02$	$1.3909e-02$	$7.8764e-03$
	p	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	2	$2(\beta+1)$
	α	$6.5768e-03$	$4.4953e-02$	$5.2176e-03$	$1.4110e-03$	$2.7313e-03$	$1.0424e-03$
-0.25	$E_{\alpha,\beta}$	$3.4220e-02$	$1.2546e-02$	$2.7996e-02$	$1.1207e-02$	$2.1871e-02$	$1.0552e-02$
	p	$2(\beta+1)$	$2(\beta+1)$	2	$2(\beta+1)$	2	$2(\beta+1)$
	α	$2.4995e-02$	$5.4093e-03$	$1.1585e-02$	$2.0394e-03$	$9.0796e-03$	$1.8336e-03$
-0.35	$E_{\alpha,\beta}$	$5.0517e-02$	$2.7007e-02$	$3.0893e-02$	$1.3559e-02$	$3.1528e-02$	$1.8870e-02$
	p	2	2	2	$2(\beta+1)$	3	3

Remark 3.4.1. *From Table 3.1 and Table 3.2 (also see the Fig :*3.1 *to Fig:*3.18*), one can see that Schock-type discrepancy principle gives better results that of Morozov's discrepancy principle.*

Example 3.4.2. We consider *Shaw* example from the Regularization Toolbox by Hansen (2007) with n points. It is defined as follows:

$$[Tx](s) := \int_{-\pi}^{\pi} k(s,t)x(t)dt = y(s),\ -\pi \le s \le \pi, \tag{3.4.5}$$

where $k(s,t) = (\cos(s)+\cos(t))^2(\frac{\sin(u)}{u})^2$ and $u = \pi(\sin(s)+\sin(t))$. The solution $\hat{x}$ is given by $\hat{x}(t) = a_1 \exp(-c_1(t-t_1)^2) + a_2 \exp(-c_2(t-t_2)^2)$. We have introduced

Table 3.2: Relative errors of Foxgood example using Morozov's discrepancy principle.

β		$n = 100$		$n = 500$		$n = 1000$	
		$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$
0	α	$2.115917e-02$	$2.768679e-02$	$2.124497e-02$	$2.799350e-02$	$2.138771e-02$	$2.758640e-02$
	$E_{\alpha,\beta}$	$2.094633e-01$	$2.220851e-01$	$2.411659e-01$	$2.259165e-01$	$2.095151e-01$	$2.248150e-01$
	p	1	1	1	1	1	1
-0.15	α	$2.124176e-02$	$2.761705e-02$	$2.130926e-02$	$2.794182e-02$	$2.130313e-02$	$2.746932e-02$
	$E_{\alpha,\beta}$	$1.132394e-01$	$1.957755e-01$	$1.531136e-01$	$1.812302e-01$	$1.533963e-01$	$1.840433e-01$
	p	1	1	1	1	1	1
-0.25	α	$2.109719e-02$	$2.653435e-02$	$2.214646e-02$	$2.687400e-02$	$2.123106e-02$	$2.710374e-02$
	$E_{\alpha,\beta}$	$8.446746e-02$	$1.391500e-01$	$1.180537e-01$	$1.514214e-01$	$1.448442e-01$	$1.458631e-01$
	p	1	1	1	1	1	1
-0.35	α	$2.096463e-02$	$2.517483e-02$	$2.114938e-02$	$2.640352e-02$	$2.119726e-02$	$2.609331e-02$
	$E_{\alpha,\beta}$	$2.529874e-01$	$9.518773e-02$	$1.600357e-01$	$1.096781e-01$	$1.003286e-01$	$1.098752e-01$
	p	1	1	1	1	1	1

the random noise level $\delta = 0.05$ and 0.01 in the exact data. In Table 3.3 and Table 3.4 respectively, we present the relative errors as well as α values using discrepancy principle (3.1.5) and (1.2.7) for different values of β, p, n and δ; and the best reconstruction took place at $\beta = -0.15$ and -0.25. Plots of *Shaw* example for different values of δ, n and β are given in Fig:3.19 - Fig: 3.36, with captions.

Table 3.3: Relative errors of *Shaw* example using Schock-type discrepancy principle.

β		$n = 100$		$n = 500$		$n = 1000$	
		$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$
0	α	$5.1766e-03$	$9.8614e-04$	$2.2498e-03$	$4.8070e-04$	$1.6163e-03$	$3.3178e-04$
	$E_{\alpha,\beta}$	$8.9961e-02$	$7.4335e-02$	$8.0229e-02$	$6.1092e-02$	$7.5070e-02$	$5.6923e-02$
	p	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$
-0.15	α	$2.0193e-03$	$4.3750e-04$	$6.5349e-04$	$1.3105e-04$	$3.9097e-04$	$7.5181e-05$
	$E_{\alpha,\beta}$	$5.9217e-02$	$4.0732e-02$	$4.4175e-02$	$3.2936e-02$	$4.2000e-02$	$2.7380e-02$
	p	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$	$2(\beta+1)$
-0.25	α	$1.4227e-02$	$2.4848e-02$	$4.8714e-03$	$4.9989e-04$	$1.6835e-03$	$3.3915e-04$
	$E_{\alpha,\beta}$	$7.2247e-02$	$3.8546e-02$	$5.2663e-02$	$3.7078e-02$	$3.5475e-02$	$2.9615e-02$
	p	$2(\beta+1)$	$2(\beta+1)$	2	2	2	2
-0.35	α	$1.7544e-02$	$1.0803e-03$	$4.1877e-02$	$1.2057e-03$	$3.0734e-03$	$3.2087e-04$
	$E_{\alpha,\beta}$	$7.7159e-02$	$4.7244e-02$	$5.3164e-02$	$3.5334e-02$	$4.8028e-03$	$3.0082e-02$
	p	2	2	2	2	2	2

Remark 3.4.2. *From Table 3.3 and Table 3.4 (also see the Fig :*3.19 *to Fig:*3.36*), one can see that Schock-type discrepancy principle gives better results that of Morozov's discrepancy principle.*

Table 3.4: Relative errors of *Shaw* example using Morozov's discrepancy principle.

β		$n = 100$		$n = 500$		$n = 1000$	
		$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$
	α	$2.114466e-02$	$2.725112e-02$	$2.128403e-02$	$2.818781e-02$	$2.134104e-02$	$2.856375e-02$
0	$E_{\alpha,\beta}$	$1.627051e-01$	$1.651075e-01$	$1.688977e-01$	$1.661034e-01$	$1.654460e-01$	$1.668177e-01$
	p	1	1	1	1	1	1
	α	$2.111252e-02$	$2.947128e-02$	$2.139182e-02$	$2.844304e-02$	$2.132938e-02$	$2.850540e-02$
-0.15	$E_{\alpha,\beta}$	$1.902385e-01$	$1.597346e-01$	$1.408892e-01$	$1.550010e-01$	$1.650700e-01$	$1.548001e-01$
	p	1	1	1	1	1	1
	α	$2.121291e-02$	$2.969247e-02$	$2.138218e-02$	$2.918355e-02$	$2.134521e-02$	$2.930549e-02$
-0.25	$E_{\alpha,\beta}$	$1.070037e-01$	$1.391788e-01$	$1.229891e-01$	$1.437794e-01$	$1.327417e-01$	$1.453733e-01$
	p	1	1	1	1	1	1
	α	$2.118527e-02$	$2.919855e-02$	$2.139795e-02$	$3.022258e-02$	$2.139716e-02$	$2.954102e-02$
-0.35	$E_{\alpha,\beta}$	$1.489167e-01$	$1.110051e-01$	$1.376022e-01$	$1.273378e-01$	$1.372250e-01$	$1.230665e-01$
	p	1	1	1	1	1	1

3.5 CONCLUDING REMARKS

In this Chapter, we considered, the Schock-type discrepancy principle for choosing the regularization parameter α in fractional Tikhonov regularization method for ill-posed problem. As mentioned in the Remark 3.2.2, it is not easy to choose $\beta \in [-\frac{1}{2}, 0]$ to obtain a better error estimate, but we observe (see Tables 3.1, 3.2, 3.3 and 3.4) that the relative errors $E_{\alpha,\beta} = \left(\frac{\|x^\delta_{\alpha,\beta} - \hat{x}\|}{\|\hat{x}\|}\right) < E_{\alpha,0} := \left(\frac{\|x^\delta_{\alpha} - \hat{x}\|}{\|\hat{x}\|}\right)$ holds, when α is chosen according to the Schock-type discrepancy principle and Morozov's discrepancy principle for $\beta \in [-\frac{1}{2}, 0)$. This shows that FTR method gives better error estimate than that of ordinary Tikhonov regularization method. The best possible choice of β is an open problem (Morigi et al. (2017)).

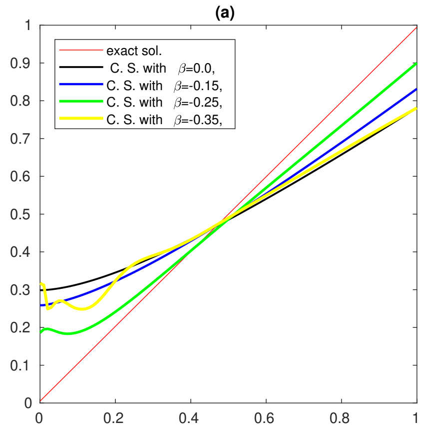

Figure 3.1: Solution with $\delta = 0.05$ and $n = 100$ for method (3.1.5).

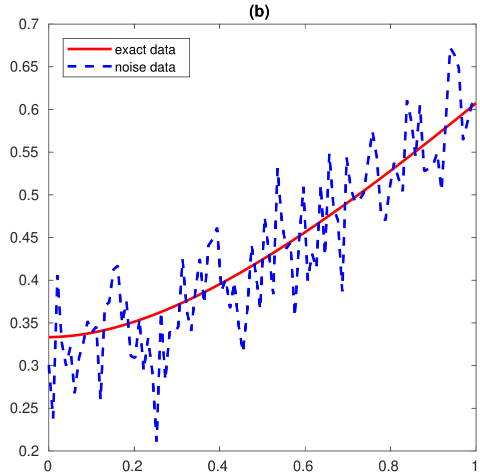

Figure 3.2: Data of *Foxgood* example with $\delta = 0.05$ and $n = 100$.

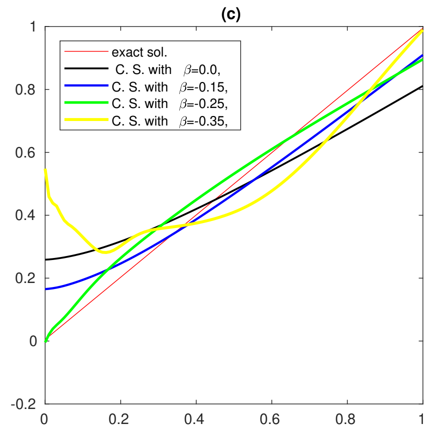

Figure 3.3: Solution with $\delta = 0.05$ and $n = 100$ for method (1.2.7).

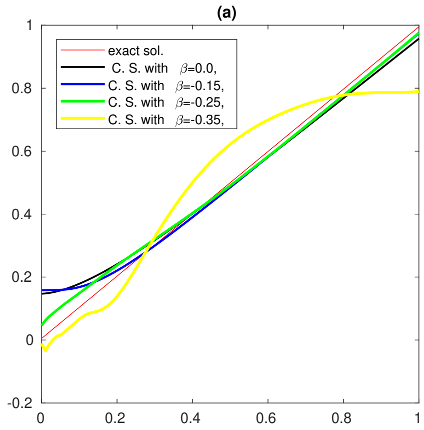

Figure 3.4: Solution with $\delta = 0.01$ and $n = 100$ for method (3.1.5).

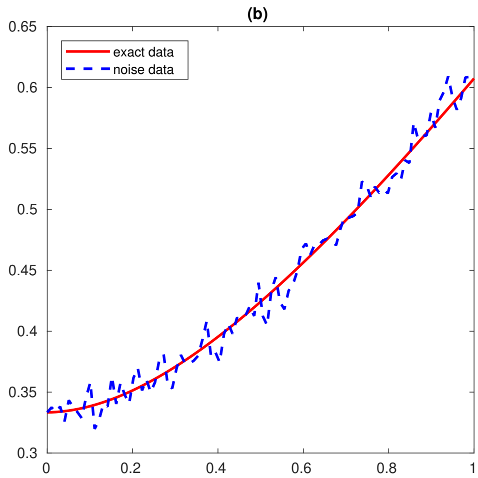

Figure 3.5: Data of *Foxgood* example with $\delta = 0.01$ and $n = 100$.

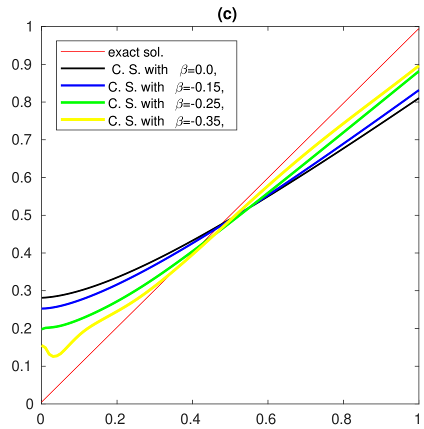

Figure 3.6: Solution with $\delta = 0.01$ and $n = 100$ for method (1.2.7).

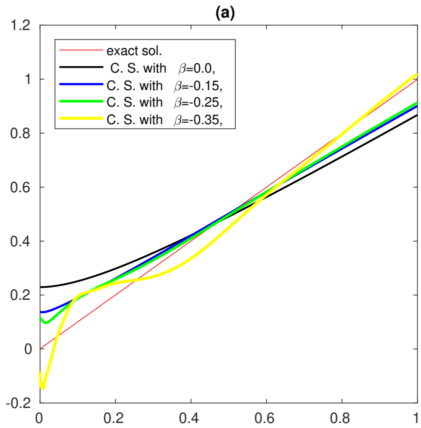

Figure 3.7: Solution with $\delta = 0.05$ and $n = 500$ for method (3.1.5).

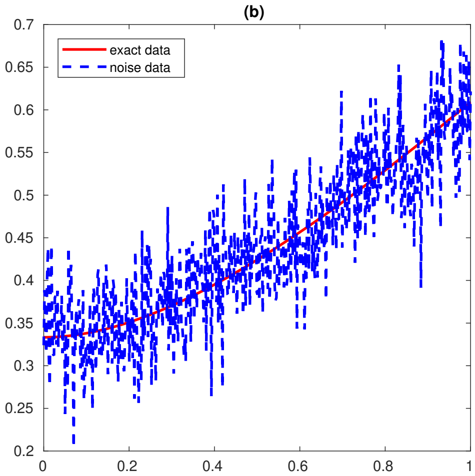

Figure 3.8: Data of *Foxgood* example with $\delta = 0.05$ and $n = 500$.

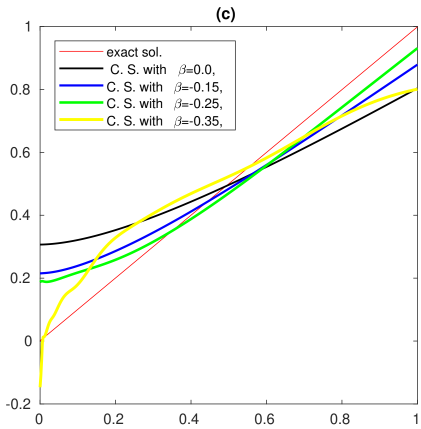

Figure 3.9: Solution with $\delta = 0.05$ and $n = 500$ for method (1.2.7).

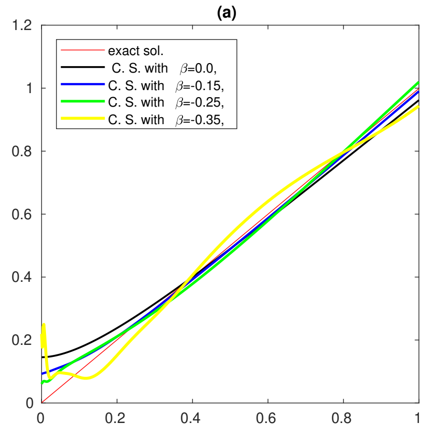

Figure 3.10: Solution with $\delta = 0.01$ and $n = 500$ for method (3.1.5).

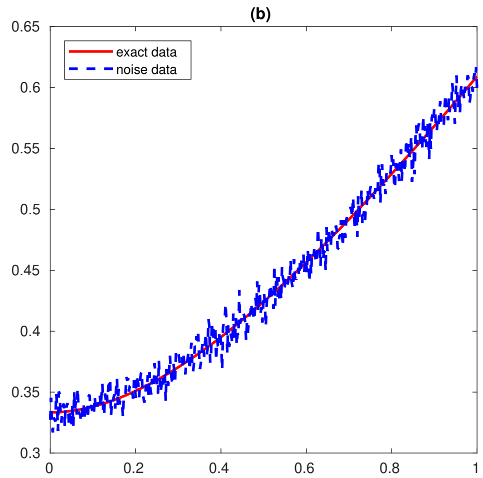

Figure 3.11: Data of *Foxgood* example with $\delta = 0.05$ and $n = 500$.

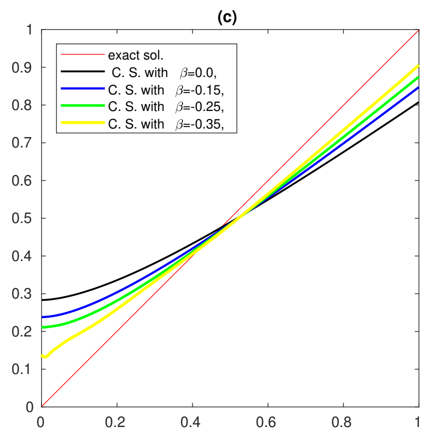

Figure 3.12: Solution with $\delta = 0.01$ and $n = 500$ for method (1.2.7).

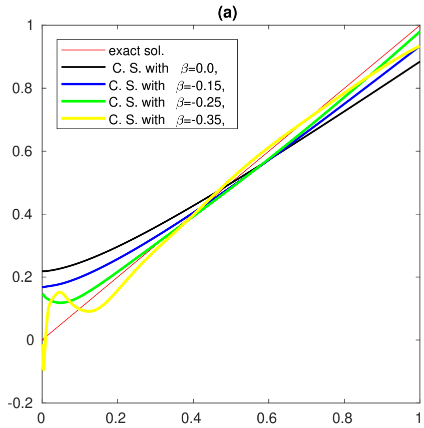

Figure 3.13: Solution with $\delta = 0.05$ and $n = 1000$ for method (3.1.5).

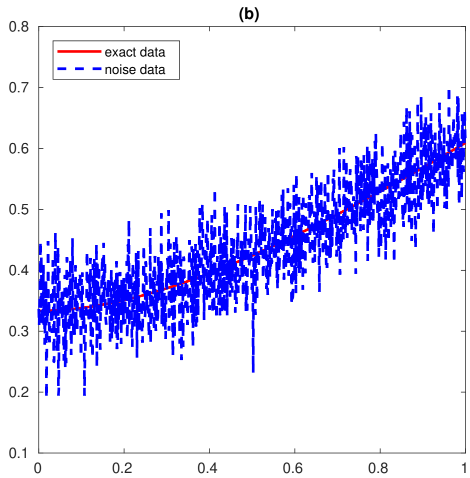

Figure 3.14: Data of *Foxgood* example with $\delta = 0.05$ and $n = 1000$.

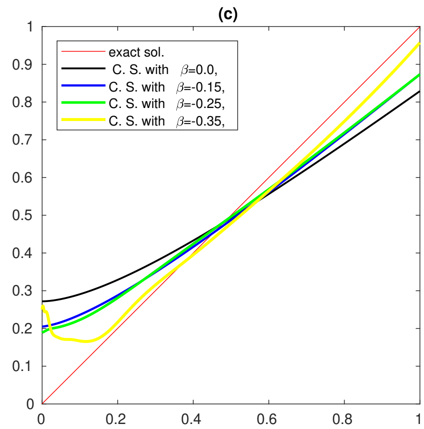

Figure 3.15: Solution with $\delta = 0.05$ and $n = 1000$ for method (1.2.7).

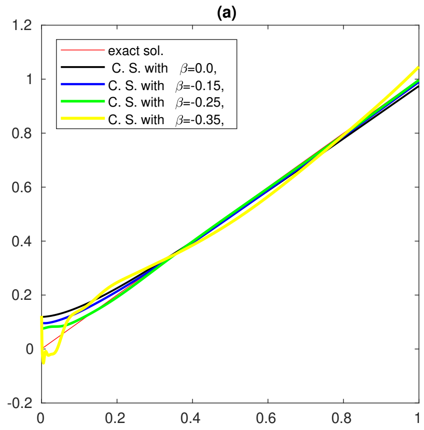

Figure 3.16: Solution with $\delta = 0.01$ and $n = 1000$ for method (3.1.5).

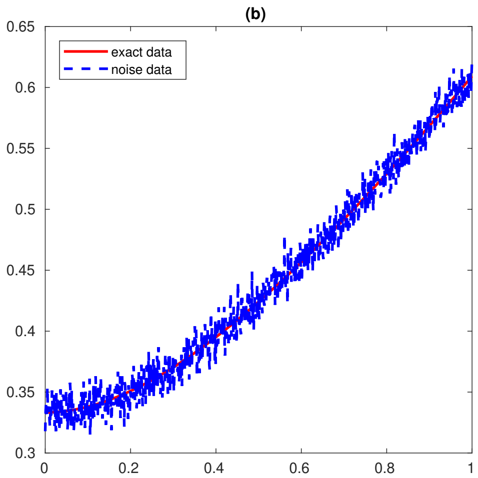

Figure 3.17: Data of *Foxgood* example with $\delta = 0.01$ and $n = 1000$.

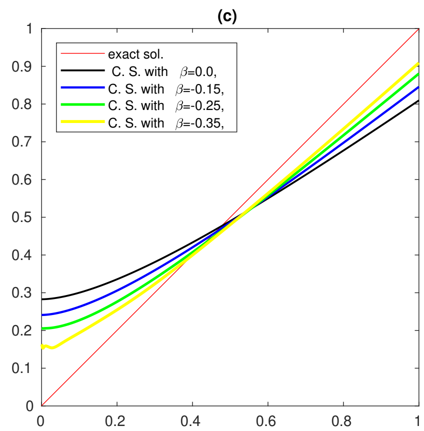

Figure 3.18: Solution with $\delta = 0.01$ and $n = 1000$ for method (1.2.7).

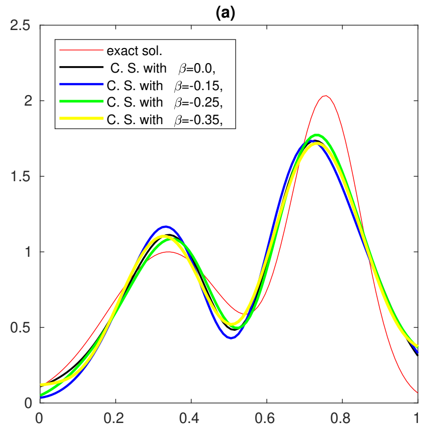

Figure 3.19: Solution with $\delta = 0.05$ and $n = 100$ for method (3.1.5).

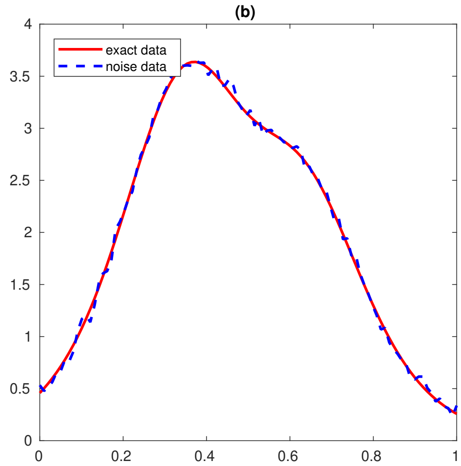

Figure 3.20: Data of *Shaw* example with $\delta = 0.05$ and $n = 100$.

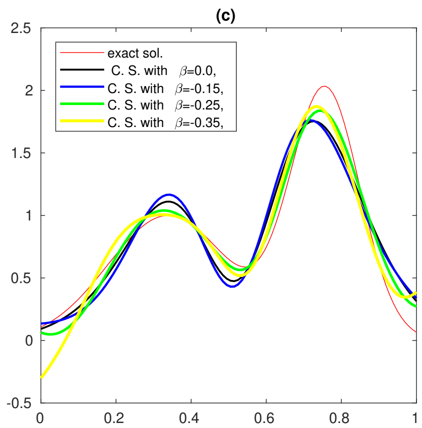

Figure 3.21: Solution with $\delta = 0.05$ and $n = 100$ for method (1.2.7).

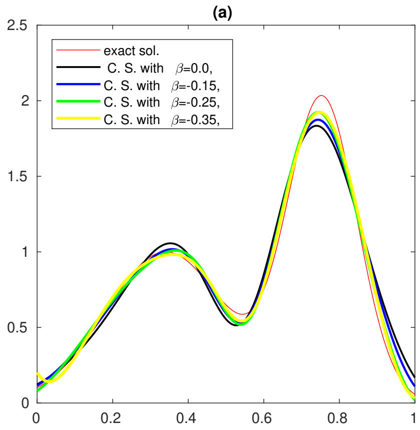

Figure 3.22: Solution with $\delta = 0.01$ and $n = 100$ for method (3.1.5).

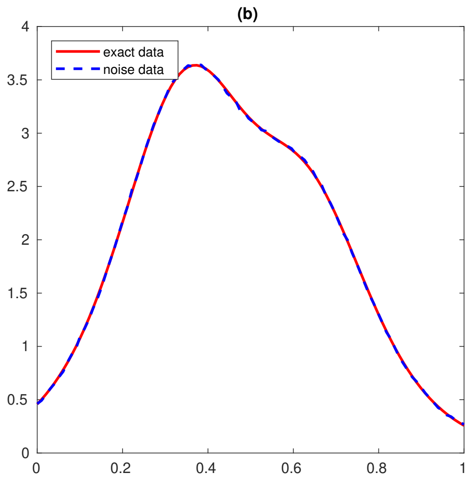

Figure 3.23: Data of *Shaw* example with $\delta = 0.01$ and $n = 100$.

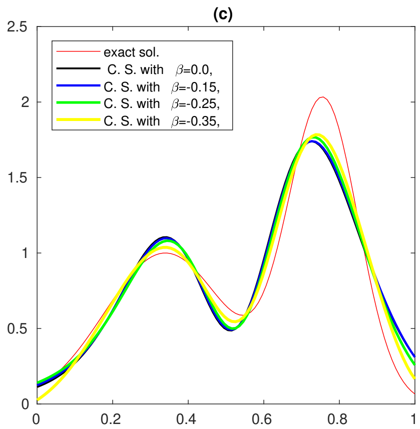

Figure 3.24: Solution with $\delta = 0.01$ and $n = 100$ for method (1.2.7).

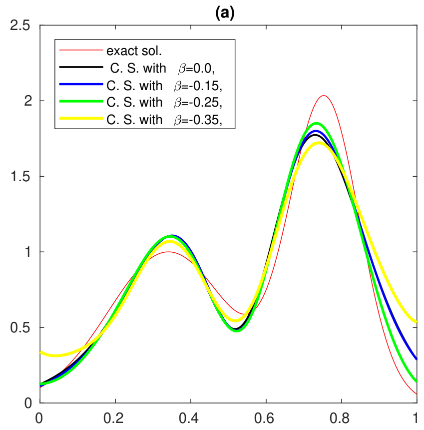

Figure 3.25: Solution with $\delta = 0.05$ and $n = 500$ for method (3.1.5).

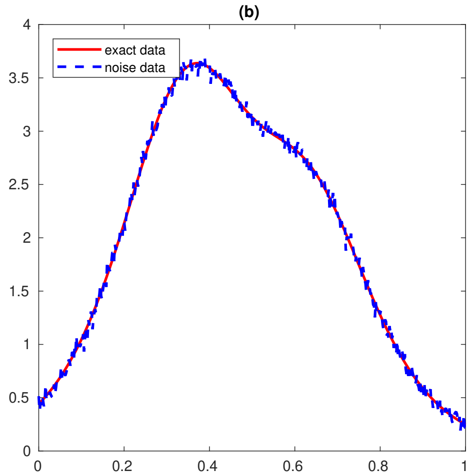

Figure 3.26: Data of *Shaw* example with $\delta = 0.05$ and $n = 500$.

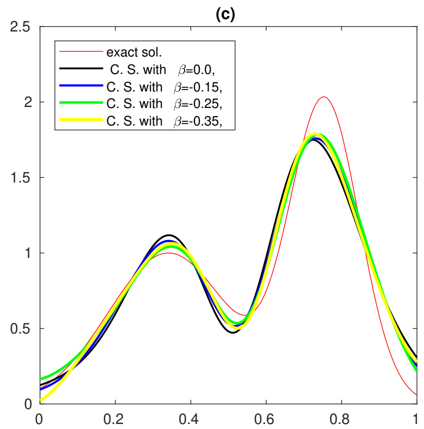

Figure 3.27: Solution with $\delta = 0.05$ and $n = 500$ for method (1.2.7).

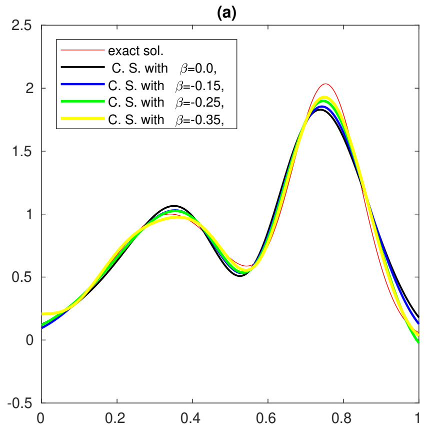

Figure 3.28: Solution with $\delta = 0.01$ and $n = 500$ for method (3.1.5).

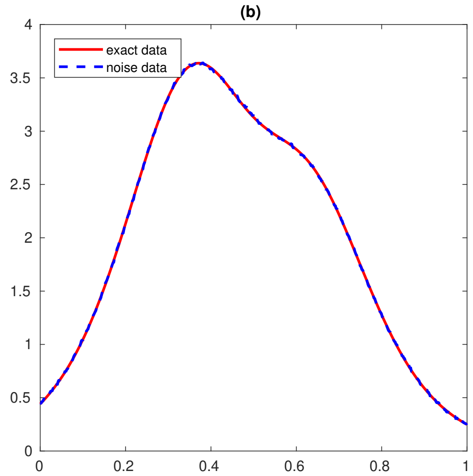

Figure 3.29: Data of *Shaw* example with $\delta = 0.01$ and $n = 500$.

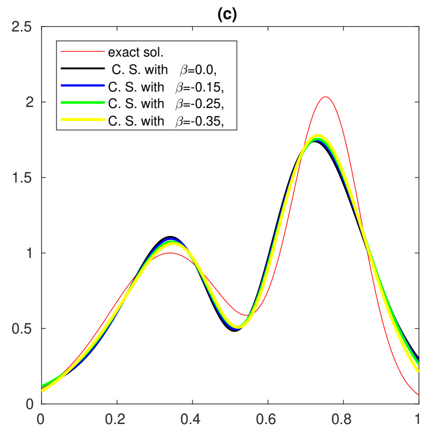

Figure 3.30: Solution with $\delta = 0.01$ and $n = 500$ for method (1.2.7).

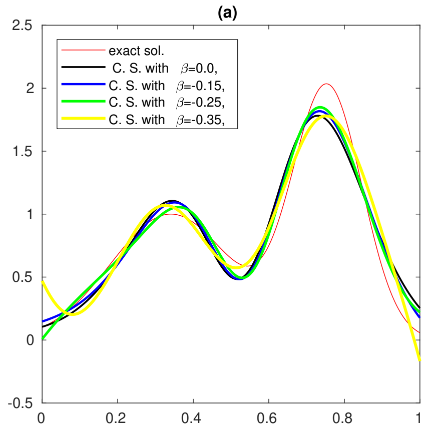

Figure 3.31: Solution with $\delta = 0.05$ and $n = 1000$ for method (3.1.5).

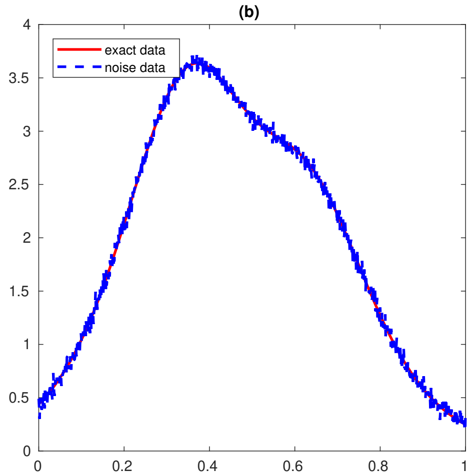

Figure 3.32: Data of *Shaw* example with $\delta = 0.05$ and $n = 1000$.

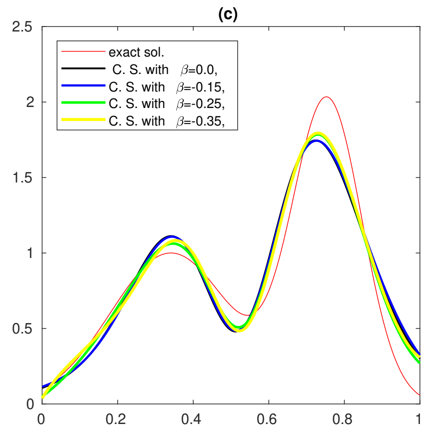

Figure 3.33: Solution with $\delta = 0.05$ and $n = 1000$ for method (1.2.7).

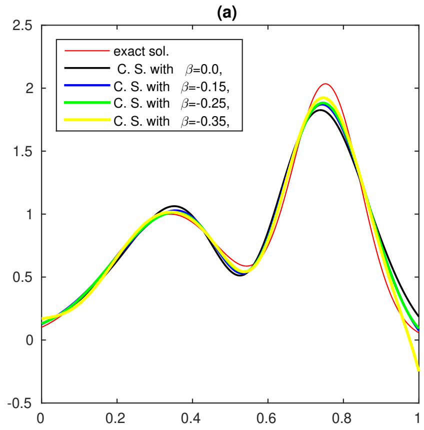

Figure 3.34: Solution with $\delta = 0.01$ and $n = 1000$ for method (3.1.5).

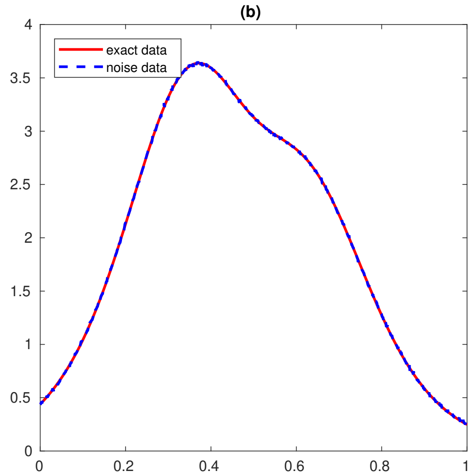

Figure 3.35: Data of *Shaw* example with $\delta = 0.01$ and $n = 1000$.

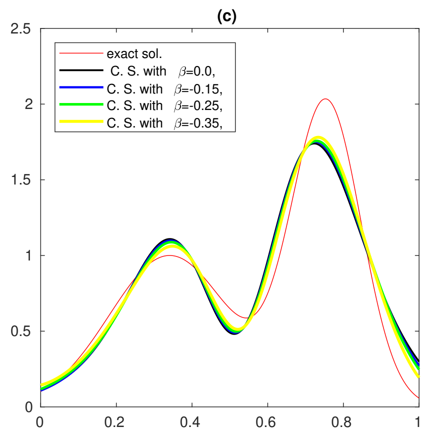

Figure 3.36: Solution with $\delta = 0.01$ and $n = 1000$ for method (1.2.7).

CHAPTER 4

PARAMETER CHOICE STRATEGIES FOR WEIGHTED SIMPLIFIED REGULARIZATION METHOD

4.1 INTRODUCTION

In this chapter, we consider the operator equation

$$Ax = y, \tag{4.1.1}$$

where $A : X \longrightarrow X$ is a positive self- adjoint operator. Precisely, we studied the weighted or fractional simplified regularization method, in which the minimizer $w_{\alpha,\beta}^{\delta}$ of the functional

$$J_{\alpha}^{\beta}(x) = \langle Ax, x \rangle - 2\,\langle y, x \rangle + \alpha\,\langle A^{\beta}x, x \rangle, \quad \alpha > 0,$$

where $\beta \in [0, 1)$, is taken as an approximation for the solution $\hat{x}$ of (4.1.1). The minimizer of above functional $w_{\alpha,\beta}$, satisfies the operator equation

$$(A^{1-\beta} + \alpha\, I)\, x = A^{-\beta}\, y. \tag{4.1.2}$$

Let $w_{\alpha,\beta}^{\delta}$ be the solution of

$$(A^{1-\beta} + \alpha\, I)\, x = A^{-\beta}\, Q\, y^{\delta}. \tag{4.1.3}$$

Where Q is the orthogonal projection onto $\overline{R(A)}$

Remark 4.1.1. *We define A^{-1} as (cf. (Groetsch, 1977, Theorem 3.2.2)) follow. Let $\{U_\rho(x)\}$ is a net of continuous real-valued function on $[0, \|A\|]$ such that $\{xU_\rho(x)\}$ is uniformly bounded and $\lim_\rho U_\rho(x) = x^{-1}$ for $x \neq 0$ then*

$$x = \lim_\rho U_\rho(A)z$$

for all $z = Ax \in R(A)$. For example one may define $A^{-1} = \int_0^\infty e^{-Au}du$.

Note that, if $Qy^\delta \notin R(A)$, then for $Qy^\delta \in \overline{R(A)} - R(A)$, one can find $\tilde{y}^\delta \in R(A)$ such that $\|\tilde{y}^\delta - Qy^\delta\| \leq \epsilon$ for any $\epsilon > 0$. Therefore, we may take $\tilde{y}^\delta$ in place of y^δ with $\delta = \delta + \epsilon$ (because $\|\tilde{y}^\delta - y\| \leq \|\tilde{y}^\delta - Qy^\delta\| + \|Qy^\delta - y\| \leq \delta + \epsilon$), in (3.1.2). So without loss of generality we assume that $Qy^\delta \in R(A)$ and $A^{-\beta}Qy^\delta$ is well defined.

As we mention in the earlier Chapters, one of the main constrain in regularization methods is the choice of the regularization parameter α. Discrepancy principles are considered for choosing the regularization parameter.

For simplified regularization method, in (George and Nair (1993)), the following discrepancy principle was considered

$$D(\alpha, x) := \alpha^{2p+2} \left\langle (A + \alpha I)^{-2p-2} Qx, Qx \right\rangle = c\delta^2, \;\; c > 1 \qquad (4.1.4)$$

and in (George and Nair (1994a)), the following discrepancy principle was considered

$$\|Aw_\alpha^\delta - y^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0. \qquad (4.1.5)$$

In this study, we consider the analogues of the discrepancy principles (4.1.4) (see Section 4.3) and (4.1.5) (see Section 4.4) for weighted or fractional simplified regularization method. We also consider the adaptive parameter choice method considered by Pereverzev and Schock (2005) for choosing the regularization parameter α in (4.1.2).

The rest of Chapter is organized as follows. In Section 4.2 we provide error estimates for for $\|w_{\alpha,\beta}^\delta - w_{\alpha,\beta}\|$ and $\|w_{\alpha,\beta} - \hat{x}\|$. In Section 4.3 and Section 4.4 we considered the modified form of discrepancy principles (4.1.4) and (4.1.5), respectively and in Section 4.5 we consider the adaptive parameter choice strategy for weighted or fractional simplified regularization method. Numerical example is given in Section 4.6 and conclusion in Section 4.7.

4.2 ERROR ESTIMATES

In this Section, we obtain the error estimates for $\|w^\delta_{\alpha,\beta} - w_{\alpha,\beta}\|$ and $\|w_{\alpha,\beta} - \hat{x}\|$ under the assumption (3.1.2) and

$$\hat{x} \in X := \{x : x = A^\nu z,\ \|z\| \le \rho\}, \quad 0 < \nu \le 1 - \beta. \tag{4.2.1}$$

Proposition 4.2.1. *Suppose y^δ satisfies (3.1.2) and $\hat{x}$ satisfies (4.2.1). Then*

(i) $\|w_{\alpha,\beta} - \hat{x}\| = O\left(\alpha^{\frac{\nu}{1-\beta}}\right)$,
and

(ii) $\|w_{\alpha,\beta} - w^\delta_{\alpha,\beta}\| = O\left(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}}\right)$.
In particular,

(iii) $\|w^\delta_{\alpha,\beta} - \hat{x}\| \le c_1 \frac{\delta}{\alpha^{\frac{1}{1-\beta}}} + c_2\, \alpha^{\frac{\nu}{1-\beta}}$.

Proof. By (4.1.2) and (4.2.1), we have

$$\begin{aligned} \|\hat{x} - w_{\alpha,\beta}\| &= \|\alpha\,(A^{1-\beta} + \alpha I)^{-1}\,\hat{x}\| \\ &= \|\alpha\,(A^{1-\beta} + \alpha I)^{-1}\,A^\nu\, z\| \\ &\le \sup_{\lambda>0} \left|\frac{\alpha\,\lambda^\nu}{(\lambda^{1-\beta} + \alpha)}\right| \|z\| \\ &= O\left(\alpha^{\frac{\nu}{1-\beta}}\right). \end{aligned}$$

Similarly, by (4.1.3) and (4.1.2), we have

$$\begin{aligned} \|w_{\alpha,\beta} - w^\delta_{\alpha,\beta}\| &= \|(A^{1-\beta} + \alpha)^{-1}\,A^{-\beta}\,Q(y - y^\delta)\| \\ &\le \delta \sup_{\lambda>0} \left|\frac{\lambda^{-\beta}}{(\lambda^{1-\beta} + \alpha)}\right| \\ &= O\left(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}}\right). \end{aligned}$$

Hence we proved (i) and (ii). Now (iii) follows from (i) and (ii). This completes the proof.

□

4.3 DISCREPANCY PRINCIPLE -I

In this section, we consider the discrepancy principle studied in (George and Nair (1993)) suitably modified for choosing the regularization parameter α in (4.1.4). For $\alpha > 0,\ \beta < p \leq 1$ and $x \in X$, let

$$D_p(\alpha, x) := \alpha^{2p+2} \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{-\beta} Qx, A^{-\beta} Qx \right\rangle .$$

The following lemma is used for proving our main results in this Section.

Lemma 4.3.1. *For each non-zero $x \in X$, $\beta < p \leq 1$, the map $\alpha \to D_p(\alpha, x)$ is continuous, strictly increasing,*

$$\lim_{\alpha \to 0} D_p(\alpha, x) = 0 \quad and \quad \lim_{\alpha \to \infty} D_p(\alpha, x) = \|A^{-\beta} Qx\|^2.$$

In particular, if $y^\delta \notin N(A)$ and y^δ satisfies

$$\|y - y^\delta\| < \delta < \frac{\|A^{-\beta} Q y^\delta\|}{\sqrt{c}} \tag{4.3.1}$$

for some $c > 1$, then the equation

$$D_p(\alpha, y^\delta) = c\delta^2 \tag{4.3.2}$$

has unique solution $\alpha = \alpha(\delta)$ such that $\alpha(\delta) \to 0$ as $\delta \to 0$.

Proof. Let $\{E_\lambda\}$ be the spectral family of the operator A. Then we have

$$D_p(\alpha, x) = \int \frac{\alpha^{2p+2}\, \lambda^{-2\beta}}{(\lambda^{1-\beta} + \alpha)^{2p+2}}\, d\left\langle E_\lambda\, Q\, x, Q\, x \right\rangle .$$

Note that the map $\alpha \to f_p(\alpha, \lambda) = (\alpha^{2p+2}\, \lambda^{-2\beta})/(\lambda^{1-\beta} + \alpha)^{2p+2}$ is strictly increasing for each $\lambda > 0$, and satisfies $f_p(\alpha, \lambda) \to 0$ as $\alpha \to 0$ and $f_p(\alpha, \lambda) \to \lambda^{-2\beta}$ as $\alpha \to \infty$. Hence the result follows from the Dominated convergence theorem and by the intermediate value theorem the equation (4.3.2) has unique solution $\alpha = \alpha(\delta)$. Proof of $\alpha(\delta) \to 0$ as $\delta \to 0$ follows as in (Schock (1984b), Lemma 1).

□

Lemma 4.3.2. *Suppose that $y \neq 0$, y^δ satisfies (4.3.1), $c_3 = (\sqrt{c} - 1)^2$, $c_4 = (\sqrt{c} + 1)^2$ and $\alpha = \alpha(\delta)$ is chosen according to (4.3.2). Then*

$$c_3\, \delta^2 \leq\ D_p(\alpha(\delta), y)\ \leq c_4\, \delta^2.$$

Proof. For $\alpha > 0$, $\beta < p \leq 1$, let $B_\alpha = \alpha^{p+1} (A^{1-\beta} + \alpha I)^{-p-1}$. Then, for each non-zero $x \in X$, we have $\|B_\alpha A^{-\beta} Qx\|^2 = D_p(\alpha, x)$. Therefore,

$$\begin{aligned} D_p(\alpha, y)^{\frac{1}{2}} &= \|B_\alpha A^{-\beta} y\| \\ &\geq \|B_\alpha A^{-\beta} Q y^\delta\| - \|B_\alpha A^{-\beta} Q(y - y^\delta)\| \\ &\geq \sqrt{c}\,\delta - \delta, \end{aligned}$$

and

$$\begin{aligned} D_p(\alpha, y)^{\frac{1}{2}} &= \|B_\alpha A^{-\beta} y\| \\ &\leq \|B_\alpha A^{-\beta} Q y^\delta\| + \|B_\alpha A^{-\beta} Q(y - y^\delta)\| \\ &\leq \sqrt{c}\,\delta + \delta. \end{aligned}$$

This completes the proof.

□

Theorem 4.3.3. *Let $y \neq 0$, y^δ satisfies (4.3.1), $\hat{x}$ satisfies (4.2.1) and $\alpha = \alpha(\delta)$ is chosen according to (4.3.2). Then, $w^\delta_{\alpha(\delta),\beta} \to \hat{x}$ as $\delta \to 0$.*

Proof. By (4.1.2) we have,

$$\|\hat{x} - w_{\alpha,\beta}\| = \|\alpha\,(A^{1-\beta} + \alpha I)^{-1}\hat{x}\| = \|R_{\alpha,\beta}\hat{x}\|, \tag{4.3.3}$$

where $R_{\alpha,\beta} = \alpha\,(A^{1-\beta} + \alpha I)^{-1}$. Then, in order to prove the theorem, by (4.3.3) and (ii) of Proposition 4.2.1, it is enough to prove that

$$(1) R_{\alpha(\delta),\beta}\,\hat{x} \to 0 \text{ as } \delta \to 0,$$

and

$$(2) \frac{\delta}{\alpha^{\frac{1}{1-\beta}}} \to 0 \text{ as } \delta \to 0.$$

Note that $\|R_{\alpha,\beta}\| \leq 1$ for all $\alpha > 0$ and for every $u \in R(A)$,

$$\begin{aligned} \|R_{\alpha,\beta}\, u\| &= \|R_{\alpha,\beta}\, Av\| \\ &\leq \sup_{\lambda > 0} \left| \frac{\alpha\,\lambda}{\lambda^{1-\beta} + \alpha} \right| \|v\| \\ &\leq c_4\ \alpha^{\frac{1}{1-\beta}}\, \|v\| \end{aligned}$$

for some $v \in X$. Therefore, $R_{\alpha,\beta}\, u \to 0$ as $\alpha \to 0$ for every u in a dense subspace of the Hilbert space $N(A)^{\perp}$ and as a consequence of the uniform boundedness principle we obtain (1). To prove (2) let

$$C_\alpha = \alpha^p \,\left(A^{1-\beta} + \alpha\, I\right)^{-p-1}\, A^{1-\beta}, \quad \alpha > 0.$$

Then for all $u \in R(A^p)$,

$$\begin{aligned}
\|C_\alpha\, u\| &= \|C_\alpha\, A^p\, v\| \\
&= \alpha^p \|(A^{1-\beta} + \alpha\, I)^{-p-1}\, A^{1-\beta}\, A^p\, v\| \\
&\le \alpha^p \sup_{\lambda > 0} \left| \frac{\lambda^{1-\beta+p}}{(\lambda^{1-\beta} + \alpha)^{p+1}} \right| \|v\| \\
&\le c_4\; \alpha^{\frac{p}{1-\beta}}\, \|v\|
\end{aligned}$$

for some $v \in X$. Since $\|C_\alpha\| \le 1$ for all $\alpha > 0$ and $R(A^p)$ is dense in $N(A)^{\perp}$, by the uniform boundedness principle, we obtain $C_{\alpha(\delta)}\, x \to 0$ as $\delta \to 0$. Now by Lemma 4.3.2,

$$\begin{aligned}
c_3\, \delta^2 &\le D_p(\alpha, y) \\
&= \alpha^{2p+2} \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{-\beta} y, A^{-\beta} y \right\rangle \\
&= \alpha^{2p+2} \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{-\beta}\, A\hat{x}, A^{-\beta}\, A\hat{x} \right\rangle \\
&= \alpha^{2p+2} \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{2(1-\beta)} \hat{x}, \hat{x} \right\rangle \\
&= \alpha^2\, \|C_\alpha\, \hat{x}\|^2 \\
&\le c_4^2\; \alpha^{\frac{2+2(p-\beta)}{1-\beta}}\, \|v\|^2.
\end{aligned}$$

Since $p > \beta$, we have,

$$\frac{\delta^2}{\alpha^{\frac{2}{1-\beta}}} \le \frac{c_4^2 \|v\|^2}{c_3} \alpha^{\frac{2(p-\beta)}{1-\beta}} \to 0 \quad as \quad \delta \to 0,$$

this proves (2).

□

Lemma 4.3.4. *Let $y \neq 0$, y^δ satisfies (4.3.1), $\hat{x}$ satisfies (4.2.1) and $\alpha = \alpha(\delta)$ be chosen according to (4.3.2). Then, we have the following:*

$$(i) \quad \alpha = O\left(\delta^{\frac{1}{p+1}}\right),$$

$$(ii) \quad \frac{\delta}{\alpha^{\frac{1}{1-\beta}}} = O\left(\delta^{\frac{\nu-\beta}{1-\beta+\nu}}\right), \ \beta \in [0, \nu).$$

Proof. By Lemma 4.3.2, for all sufficiently small $\alpha > 0$, we have

$$\begin{aligned} c_4\,\delta^2 &\geq D_p(\alpha, y) \\ &= \alpha^{2p+2}\ \|(A^{1-\beta} + \alpha I)^{-p-1} A^{-\beta} y\|^2 \\ &\geq \alpha^{2p+2}\ \frac{\|A^{-\beta} y\|^2}{\|A^{1-\beta} + \alpha I\|^{2(p+1)}} \\ &\geq c_5\ \alpha^{2p+2}, \end{aligned}$$

for some constant c_5. Thus $\alpha = O\left(\delta^{\frac{1}{p+1}}\right)$, this proves (i).

By (4.2.1), there exists $z \in X$ such that $\hat{x} = A^\nu z$, so that $y = A\hat{x} = A^{1+\nu} z$. Therefore by Lemma 4.3.2, we have

$$\begin{aligned} c_3\,\delta^2 &\leq D_p(\alpha, y) \\ &= \alpha^{2p+2}\ \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{-\beta} y, A^{-\beta} y \right\rangle \\ &= \alpha^{2p+2}\ \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{-\beta}\, A^{1+\nu} z, A^{-\beta}\, A^{1+\nu} z \right\rangle \\ &= \alpha^{2p+2}\, \|(A^{1-\beta} + \alpha I)^{-p-1} A^{1+\nu-\beta} z\|^2 \\ &\leq \alpha^{2p+2}\ \sup_{\lambda > 0} \left| \frac{\lambda^{2(1+\nu-\beta)}}{(\lambda^{1-\beta} + \alpha)^{2(p+1)}} \right| \\ &= O\left(\alpha^{\frac{2(1+\nu-\beta)}{1-\beta}}\right). \end{aligned}$$

Hence for $\beta \in [0, \nu)$, we have $\delta = O\left(\alpha^{\frac{1+\nu-\beta}{1-\beta}}\right)$, this proves (ii).

□

Combining the results in Proposition 4.2.1 and Lemma 4.3.4, we have the following Theorem.

Theorem 4.3.5. *Let y^δ satisfy (4.3.1), $\alpha = \alpha(\delta)$ chosen according to (4.3.2) and $\hat{x}$ satisfies (4.2.1). Then, for $\beta \in [0, \nu)$*

$$\|\hat{x} - w^\delta_{\alpha,\beta}\| = O\left(\delta^{\frac{\nu-\beta}{(1+p)\,(1-\beta)}}\right).$$

□

4.4 DISCREPANCY PRINCIPLE -II

In this section, we consider the discrepancy principle studied in (George and Nair (1994a)), suitably modified for choosing the regularization parameter α in (4.1.5). Precisely, for given $r > 0, q > 0$, we choose α such that

$$\|A^{-\beta}(Aw^{\delta}_{\alpha,\beta} - Qy^{\delta})\| = \frac{\delta^r}{\alpha^q}. \tag{4.4.1}$$

Let

$$\phi(\alpha) = \alpha^{2q}\ \|A^{-\beta}(Aw^{\delta}_{\alpha,\beta} - Qy^{\delta})\|^2, \quad \alpha > 0.$$

Lemma 4.4.1. *The function $\phi(\alpha)$ is continuous and strictly increasing for $\alpha > 0$, and satisfies $\lim_{\alpha\to 0}\phi(\alpha) = 0$ and $\lim_{\alpha\to\infty}\phi(\alpha) = \infty$. In particular, there exists unique $\alpha = \alpha(\delta)$ satisfying (4.4.1). Further $\alpha(\delta) \to 0$ as $\delta \to 0$.*

Proof. Observe that

$$\begin{aligned}
\phi(\alpha) &= \alpha^{2q}\,\|A^{-\beta}(Aw^{\delta}_{\alpha,\beta} - Qy^{\delta})\|^2, \quad \alpha > 0\\
&= \alpha^{2q}\,\|A^{-\beta}(A\,(A^{1-\beta} + \alpha\, I)^{-1}\,A^{-\beta}Qy^{\delta} - Qy^{\delta})\|^2\\
&= \alpha^{2q}\,\|\alpha\, A^{-\beta}\,(A^{1-\beta} + \alpha\, I)^{-1}Qy^{\delta}\|^2\\
&= \alpha^{2q}\int_0^{\|A\|}\left(\frac{\alpha\,\lambda^{-\beta}}{\lambda^{1-\beta} + \alpha}\right)^2 d\langle E_\lambda Qy^{\delta}, Qy^{\delta}\rangle,
\end{aligned}$$

where E_λ is spectral family of A.

Note that the map $\alpha \to f(\alpha,\lambda) = \alpha^2\,\lambda^{-2\beta}/(\lambda^{1-\beta} + \alpha)^2$ is strictly increasing. Thus $\phi(\alpha)$ is continuous, $\phi(\alpha) \to 0$ as $\alpha \to 0$, $\phi(\alpha) \to \infty$ as $\alpha \to \infty$ and $\phi(\alpha)$ is strictly increasing for $\alpha > 0$. By the intermediate value theorem the equation (4.4.1) has unique solution $\alpha = \alpha(\delta)$. Now, using the arguments similar to the ones in (Schock (1984b), Lemma 1), one can prove $\alpha(\delta) \to 0$ as $\delta \to 0$.

□

Theorem 4.4.2. *If $\alpha = \alpha(\delta)$ is chosen according to (4.4.1), then $\alpha = O\left(\delta^{\frac{r}{q+1}}\right)$. If, in addition, $r \le (q+1)(1-\beta)$, then $\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} = O\left(\delta^m\right),\ m = 1 - \frac{r}{(q+1)(1-\beta)}$, and $w^{\delta}_{\alpha,\beta} \to \hat{x}$ as $\delta \to 0$.*

Proof. Note that

$$\begin{aligned}
\|A^{-\beta}Qy^\delta\| - \frac{\delta^r}{\alpha^q} &= \|A^{-\beta}Qy^\delta\| - \|A^{1-\beta}\, w^\delta_{\alpha,\beta} - A^\beta Qy^\delta\| \\
&\le \|A^{1-\beta}\, w^\delta_{\alpha,\beta}\| \\
&= \frac{\|A^{1-\beta}(A^{1-\beta}\, w^\delta_{\alpha,\beta} - A^{-\beta}Qy^\delta)\|}{\alpha} \\
&\le \|A^{1-\beta}\|\, \frac{\delta^r}{\alpha^{q+1}},
\end{aligned}$$

so,

$$\begin{aligned}
\|A^{-\beta}Qy^\delta\| &\le \frac{\delta^r}{\alpha^q}\left(1 + \frac{\|A^{1-\beta}\|}{\alpha}\right) \\
&\le \frac{\delta^r}{\alpha^{q+1}}\left(\alpha + \|A^{1-\beta}\|\right) \\
\alpha^{q+1} &\le \delta^r\, \frac{\left(\alpha + \|A^{1-\beta}\|\right)}{\|A^{-\beta}Qy^\delta\|}.
\end{aligned}$$

This implies $\alpha = O\left(\delta^{\frac{r}{q+1}}\right)$.

Further, note that

$$\frac{\delta^r}{\alpha^q} = \|A^{1-\beta}w^\delta_{\alpha,\beta} - A^\beta Qy^\delta\| = \|\alpha\, w^\delta_{\alpha,\beta}\| \le \alpha\,(\|w^\delta_{\alpha,\beta} - w_{\alpha,\beta}\| + \|w_{\alpha,\beta}\|). \tag{4.4.2}$$

But by Proposition 4.2.1,

$$\|w_{\alpha,\beta} - w^\delta_{\alpha,\beta}\| = O\left(\frac{\delta}{\alpha^{\frac{1}{\beta+1}}}\right)$$

and $\|w_{\alpha,\beta}\| = \|(A^{1-\beta} + \alpha\, I)^{-1}\, A^{1-\beta}\, \hat{x}\| \le \|\hat{x}\|$. Therefore, we have

$$\frac{\delta^r}{\alpha^q} \le \alpha\left(c_2\frac{\delta}{\alpha^{\frac{1}{\beta+1}}} + \|\hat{x}\|\right) = c_2\alpha^{\frac{\beta}{\beta+1}}\,\delta + \alpha\,\|\hat{x}\|.$$

Now using the estimate $\alpha = O\left(\delta^{\frac{r}{q+1}}\right)$, we get

$$\begin{aligned}
\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} &= \delta^{1-\frac{r}{q(1-\beta)}} \left(\frac{\delta^r}{\alpha^q}\right)^{\frac{1}{q(1-\beta)}} \\
&\leq \delta^{1-\frac{r}{q(1-\beta)}} \left(c_2 \alpha^{\frac{\beta}{1-\beta}} \delta + \alpha \|\hat{x}\|\right)^{\frac{1}{q(1-\beta)}} \\
&\leq \left(c_2 \delta^{1+(1-\beta)q-r} \alpha^{\frac{\beta}{1-\beta}} + c_6 \delta^{(1-\beta)q-r+\frac{r}{q+1}}\right)^{\frac{1}{q(1-\beta)}} \\
&\leq \left(c_7 \delta^{1+(1-\beta)q-r+\frac{r\beta}{(q+1)(1-\beta)}} + c_6 \delta^{(1-\beta)q-r+\frac{r}{q+1}}\right)^{\frac{1}{q(1-\beta)}} \\
&= O\left(\delta^{1-\frac{r}{(q+1)(1-\beta)}}\right) \\
&= O\left(\delta^m\right)
\end{aligned}$$

where $m = 1 - \frac{r}{(q+1)(1-\beta)}$. So $w^{\delta}_{\alpha,\beta} \to \hat{x}$ follows as in Theorem 4.3.3.

□

Theorem 4.4.3. *Let $\hat{x}$ satisfies (4.2.1), $q > 0, r \leq (q+1)(1-\beta)$ and $\alpha = \alpha(\delta)$ be chosen according to (4.4.1). Then*

$$(i) \qquad \|\hat{x} - w^{\delta}_{\alpha,\beta}\| = O(\delta^s),$$

where $s = min\left\{\frac{r\nu}{(q+1)(1-\beta)}, 1 - \frac{r}{(q+1)(1-\beta)}\right\}$. For a fixed ν the best rate is obtained when $r = \frac{(q+1)(1-\beta)}{\nu+1}$ which gives $\alpha = O\left(\delta^{\frac{1-\beta}{\nu+1}}\right)$ and

$$(ii) \qquad \|\hat{x} - w^{\delta}_{\alpha,\beta}\| = O\left(\delta^{\frac{\nu}{\nu+1}}\right).$$

Proof. From Proposition 4.2.1, we have

$$\|\hat{x} - w^{\delta}_{\alpha,\beta}\| \leq c_2 \alpha^{\frac{\nu}{1-\beta}} + c_1 \frac{\delta}{\alpha^{\frac{1}{\beta+1}}},$$

so that the result in (i) follows from Theorem 4.4.2. If $r = \frac{(q+1)(1-\beta)}{\nu+1}$ then $\frac{r\nu}{(q+1)(1-\beta)} = 1 - \frac{r}{(q+1)(\beta+1)}$ so that $O\left(\alpha^{\frac{\nu}{1-\beta}}\right) = O\left(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}}\right) = O\left(\delta^{\frac{\nu}{\nu+1}}\right)$, proving (ii).

□

Remark 4.4.4. *1. Note that we obtained the optimal rate $O\left(\delta^{\frac{v}{v+1}}\right)$, by choosing $\frac{r}{q+1} = \frac{(1-\beta)}{v+1}$.*

2. *The discrepancy principle-I and discrepancy principle-II considered in Section* 4.3 *and in Section* 4.4*, can achieve the so-called better rates only when* p, q *and* r *are chosen depending on* ν *in the source condition. Unfortunately this* ν *is difficult to know in practical applications. So, we consider the adaptive selection of parameter, which is independent of* ν*, considered by Pereverzev and Schock (2005) in the next section.*

4.5 ADAPTIVE SELECTION OF THE PARAMETER

Note that by (*iii*) of Proposition 4.2.1, we have

$$\|w_{\alpha,\beta}^{\delta} - \hat{x}\| \leq C\, \left(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} + \alpha^{\frac{\nu}{1-\beta}}\right) \tag{4.5.1}$$

where

$$C = \max\{c_1, c_2\}.$$

Further, observe that the error $\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} + \alpha^{\frac{\nu}{1-\beta}}$ in (4.5.1) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies, $\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} = \alpha^{\frac{\nu}{1-\beta}}$. That is $\alpha_\delta = \delta^{\frac{1-\beta}{\nu+1}}$. In order to obtain the optimal order in (4.5.1), Pereverzev and Schock (2005), introduced the adaptive selection of the parameter strategy, we modified adaptive method suitably for the situation for choosing the parameter α. Let $i \in \{0, 1, 2, \ldots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$l := \max\left\{ i : \alpha_i^{\frac{\nu}{1-\beta}} \leq \frac{\delta}{\alpha_i^{\frac{1}{1-\beta}}} \right\} < N \quad \text{and} \tag{4.5.2}$$

$$k := \max\left\{ i : \|w_{\alpha_i,\beta}^{\delta} - w_{\alpha_j,\beta}^{\delta}\| \leq 4C\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}, j = 0, 1, 2, \ldots, i-1 \right\} \tag{4.5.3}$$

where $C = \max\{c_1, c_2\}$ where c_1, c_2 is as in Proposition 4.2.1. Now we have the following Theorem.

Theorem 4.5.1. *Assume that there exists* $i \in \{0, 1, \ldots, N\}$ *such that* $\alpha_i^{\frac{\nu}{1-\beta}} \leq \frac{\delta}{\alpha_i^{\frac{1}{1-\beta}}}$. *Let assumptions of Proposition 4.2.1 be fulfilled, and let* l *and* k *be as in (4.5.2) and (4.5.3) respectively. Then* $l \leq k$; *and*

$$\|w_{\alpha_k,\beta}^{\delta} - \hat{x}\| \leq 6\, C\, \mu^{\frac{\nu+1}{1-\beta}} \delta^{\frac{\nu}{\nu+1}}.$$

Proof. To prove $l \leq k$, it is enough to show that, for each $i \in \{1, 2, \ldots, N\}$, $\alpha_i^{\frac{\nu}{1-\beta}} \leq \frac{\delta}{\alpha_i^{\frac{1}{1-\beta}}} \Longrightarrow \|w^\delta_{\alpha_i,\beta} - w^\delta_{\alpha_j,\beta}\| \leq 4\,C\,\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}$, $\forall j = 0, 1, 2, \ldots, i-1$. For $j < i$, we have

$$\begin{aligned}
\| w^\delta_{\alpha_i,\beta} - w^\delta_{\alpha_j,\beta} \| &\leq \| w^\delta_{\alpha_i,\beta} - \hat{x} \| + \| \hat{x} - w^\delta_{\alpha_j,\beta} \| \\
&\leq C\left(\alpha_i^{\frac{\nu}{1-\beta}} + \frac{\delta}{\alpha_i^{\frac{1}{1-\beta}}}\right) + C\left(\alpha_j^{\frac{\nu}{1-\beta}} + \frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}\right) \\
&\leq 2\,C\,\alpha_i^{\frac{\nu}{1-\beta}} + 2\,C\,\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}} \\
&\leq 4\,C\,\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}.
\end{aligned}$$

Thus the relation $l \leq k$ is proved. Further note that

$$\| \hat{x} - w^\delta_{\alpha_k,\beta} \| \leq \| \hat{x} - w^\delta_{\alpha_l,\beta} \| + \| w^\delta_{\alpha_l,\beta} - w^\delta_{\alpha_k,\beta} \|$$

where

$$\| \hat{x} - w^\delta_{\alpha_l,\beta} \| \leq C\left(\alpha_l^{\frac{\nu}{1-\beta}} + \frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}}\right) \leq 2\,C\,\frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}}.$$

Now since $l \leq k$, we have

$$\| w^\delta_{\alpha_k,\beta} - w^\delta_{\alpha_l,\beta} \| \leq 4\,C\,\frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}}.$$

Hence

$$\| \hat{x} - w^\delta_{\alpha_k,\beta} \| \leq 6\,C\,\frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}}$$

Again, since $\alpha_\delta^{\frac{\nu+1}{1-\beta}} = \delta \leq \alpha_{l+1}^{\frac{\nu+1}{1-\beta}} \leq \mu^{\frac{\nu+1}{1-\beta}} \alpha_l{}^{\frac{\nu+1}{1-\beta}}$, it follows that

$$\frac{\delta}{\alpha_\delta^{\frac{1}{1-\beta}}} \leq \frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}} \leq \mu^{\frac{\nu+1}{1-\beta}} \alpha_l{}^{\frac{\nu}{(1-\beta)}} \leq \mu^{\frac{\nu+1}{1-\beta}} \alpha_\delta{}^{\frac{\nu}{(1-\beta)}} \leq \mu^{\frac{\nu+1}{1-\beta}} \delta^{\frac{\nu}{\nu+1}}.$$

This completes the proof.

4.5.1 Implementation of adaptive choice rule

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 4.5.1 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \ldots, N$.

4.5.2 Algorithm

1. Set $i = 0$.
2. Solve $w^{\delta}_{\alpha_i,\beta}$ by using (4.1.3).
3. If $\|w^{\delta}_{\alpha_i,\beta} - w^{\delta}_{\alpha_j,\beta}\| > 4C\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}, j = 0, 1, 2, \ldots, i-1$, then take $k = i - 1$ and return $w_{\alpha_k,\beta}$.
4. Else set $i = i + 1$ and go to 2.

4.6 NUMERICAL EXAMPLES

In this section, we consider an academic example for the numerical discussion to validate our theoretical results. The discrete version of the operator A is taken from the Regularization Toolbox by Hansen (2007).

We adopted the Newton's method to solve above non-linear equations (4.3.2) and (4.4.1) for α with different values β, δ, p, r and q with $q = r - 1$. Relative errors $E_{\alpha,\beta} := \left(\frac{\|w^{\delta}_{\alpha,\beta} - \hat{x}\|}{\|\hat{x}\|}\right)$, and α are presented in the tables for different values of β, p, r, n (size of the mesh) and noise level δ.

Example 4.6.1 Let

$$[Tx](s) := \int_{-\pi}^{\pi} k(s,t)x(t)dt = g(s), \ -\pi \leq s \leq \pi,$$

where $k(s,t) = (cos(s) + cos(t))^2 \left(\frac{sin(u)}{u}\right)^2$ and $u = \pi(sin(s) + sin(t))$. We take $A := T^*T$ and $y = T^*g$ for our computation. The solution $\hat{x}$ is given by $\hat{x} =$

$a_1 exp(-c_1(t-t_1))^2) + a_2 exp(-c_2(t-t_2)^2))$. We have introduced the random noise level $\delta = 0.05$ and 0.01 in the exact data. Relative errors and α values are showcased in Tables 4.1–4.3 obtained using discrepancy principle-I, discrepancy principle-II, and the adaptive method respectively, for different values of β, p, r, n and δ.

Table 4.1: Relative errors for discrepancy principle-I.

β		$n = 100$		$n = 500$		$n = 1000$	
		$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$
	α	1.038235e − 01	5.633752e − 02	5.709557e − 02	2.983508e − 02	4.201687e − 02	2.288945e − 02
0	$E_{\alpha,\beta}$	1.846841e − 01	1.777739e − 01	1.802147e − 01	1.670909e − 01	1.683492e − 01	1.637891e − 01
	p	1/2	1/2	1/2	1/2	1/2	1/2
	α	9.034614e − 02	4.018162e − 02	4.900657e − 02	2.153913e − 02	3.784975e − 02	1.482163e − 02
0	$E_{\alpha,\beta}$	1.867070e − 01	1.707974e − 01	1.737618e − 01	1.622070e − 01	1.696070e − 01	1.573365e − 01
	p	2/3	2/3	2/3	2/3	2/3	2/3
	α	6.318497e − 02	9.587215e − 03	3.307382e − 02	4.056634e − 03	2.709670e − 02	2.331297e − 03
0	$E_{\alpha,\beta}$	1.849394e − 01	1.481517e − 01	1.675791e − 01	1.365604e − 01	1.670293e − 01	1.182621e − 01
	p	1	1	1	1	1	1
	α	1.121729e − 01	5.931568e − 02	5.762544e − 02	3.140849e − 02	4.407745e − 02	2.381447e − 02
0.15	$E_{\alpha,\beta}$	1.928669e − 01	1.718287e − 01	1.643411e − 01	1.573068e − 01	1.595378e − 01	1.523359e − 01
	p	1/2	1/2	1/2	1/2	1/2	1/2
	α	9.368723e − 02	4.350537e − 02	5.386323e − 02	2.154639e − 02	3.938599e − 02	1.351661e − 02
0.15	$E_{\alpha,\beta}$	1.807037e − 01	1.641363e − 01	1.695588e − 01	1.483044e − 01	1.603532e − 01	1.370803e − 01
	p	2/3	2/3	2/3	2/3	2/3	2/3
	α	6.392676e − 02	7.187664e − 03	3.210264e − 02	4.068169e − 03	1.954892e − 02	2.769513e − 03
0.15	$E_{\alpha,\beta}$	1.705366e − 01	1.104975e − 01	1.571158e − 01	1.028908e − 01	1.428269e − 01	8.778747e − 02
	p	1	1	1	1	1	1
	α	1.027793e − 01	5.891488e − 02	5.830293e − 02	3.198677e − 02	4.494245e − 02	2.389980e − 02
0.25	$E_{\alpha,\beta}$	1.590265e − 01	1.609385e − 01	1.590501e − 01	1.475253e − 01	1.531484e − 01	1.373195e − 01
	p	1/2	1/2	1/2	1/2	1/2	1/2
	α	8.613665e − 02	4.458716e − 02	5.541284e − 02	2.258585e − 02	3.180878e − 02	1.609262e − 02
0.25	$E_{\alpha,\beta}$	1.508893e − 01	1.538537e − 01	1.638523e − 01	1.377718e − 01	1.208649e − 01	1.284374e − 01
	p	2/3	2/3	2/3	2/3	2/3	2/3
	α	6.610678e − 02	1.128888e − 02	3.754577e − 02	5.500000e − 03	3.255848e − 02	3.733918e − 03
0.25	$E_{\alpha,\beta}$	1.719273e − 01	1.155649e − 01	1.508187e − 01	1.028954e − 01	1.563724e − 01	7.240657e − 02
	p	1	1	1	1	1	1
	α	1.044218e − 01	5.891491e − 02	5.656391e − 02	3.188663e − 02	4.605740e − 02	2.391486e − 02
0.35	$E_{\alpha,\beta}$	1.828589e − 01	1.537426e − 01	1.445362e − 01	1.266817e − 01	1.520494e − 01	1.175264e − 01
	p	1/2	1/2	1/2	1/2	1/2	1/2
	α	8.151534e − 02	4.521998e − 02	5.598205e − 02	2.376004e − 02	3.951507e − 02	1.718729e − 02
0.35	$E_{\alpha,\beta}$	1.338753e − 01	1.401163e − 01	1.544583e − 01	1.219481e − 01	1.385935e − 01	1.110965e − 01
	p	2/3	2/3	2/3	2/3	2/3	2/3
	α	5.576333e − 02	1.137152e − 02	2.465876e − 02	4.966749e − 03	1.593763e − 02	3.733918e − 03
0.35	$E_{\alpha,\beta}$	1.315137e − 01	9.130048e − 02	1.301862e − 01	5.466374e − 02	1.022811e − 01	7.240657e − 02
	p	1	1	1	1	1	1

4.7 CONCLUSION

In this chapter, we considered three parameter choice strategies for weighted simplified regularization method for ill-posed equations involving positive self-adjoint

Table 4.2: Relative errors for discrepancy principle-II.

β		$n=100$		$n=500$		$n=1000$	
		$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$
0	α	$2.409365e-02$	$4.960682e-03$	$1.823065e-02$	$4.054100e-03$	$1.626371e-02$	$3.482412e-03$
	$E_{\alpha,\beta}$	$1.617371e-01$	$1.319166e-01$	$1.621281e-01$	$1.355377e-01$	$1.590535e-01$	$1.284244e-01$
	r	3	3	3	3	3	3
0.15	α	$2.388507e-02$	$4.958752e-03$	$1.821108e-02$	$4.052624e-03$	$1.625577e-02$	$3.480592e-03$
	$E_{\alpha,\beta}$	$1.260072e-01$	$1.088610e-01$	$1.378881e-01$	$1.022840e-01$	$1.457864e-01$	$9.255840e-02$
	r	3	3	3	3	3	3
0.25	α	$2.382287e-02$	$4.958898e-03$	$1.823654e-02$	$4.050538e-03$	$1.622840e-02$	$3.479817e-03$
	$E_{\alpha,\beta}$	$1.427260e-01$	$9.135510e-02$	$1.487823e-01$	$7.232072e-02$	$1.083420e-01$	$8.011626e-02$
	r	3	3	3	3	3	3
0.35	α	$1.648290e-02$	$3.568452e-03$	$1.103797e-02$	$2.563328e-03$	$9.250479e-03$	$2.022885e-03$
	$E_{\alpha,\beta}$	$1.552694e-01$	$1.355981e-01$	$8.589792e-02$	$4.765525e-02$	$8.444161e-02$	$7.593863e-02$
	r	2	2	2	2	2	2

Table 4.3: Relative errors obtained from Adaptive method

β		$n=100$		$n=500$		$n=1000$	
		$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$
0	α	$1.169230e-01$	$7.260000e-02$	$8.784600e-02$	$6.600000e-02$	$7.986000e-02$	$6.600000e-02$
	$E_{\alpha,\beta}$	$2.020121e-01$	$1.850613e-01$	$1.929860e-01$	$1.821091e-01$	$1.876446e-01$	$1.820621e-01$
0.15	α	$1.286153e-01$	$7.260000e-02$	$8.784600e-02$	$6.600000e-02$	$7.986000e-02$	$6.600000e-02$
	$E_{\alpha,\beta}$	$1.976656e-01$	$1.775612e-01$	$1.812653e-01$	$1.731357e-01$	$1.812824e-01$	$1.734605e-01$
0.25	α	$1.556245e-01$	$7.986000e-02$	$9.663060e-02$	$7.260000e-02$	$8.784600e-02$	$6.600000e-02$
	$E_{\alpha,\beta}$	$1.945216e-01$	$1.709314e-01$	$1.754912e-01$	$1.678725e-01$	$1.722799e-01$	$1.655920e-01$
0.35	α	$1.883057e-01$	$8.784600e-02$	$1.169230e-01$	$7.260000e-02$	$9.663060e-02$	$7.260000e-02$
	$E_{\alpha,\beta}$	$1.871009e-01$	$1.682024e-01$	$1.732220e-01$	$1.548234e-01$	$1.641372e-01$	$1.587685e-01$

operator. We obtained an optimal order error estimate under a general Hölder type source condition. Numerical experiments confirms the theoretical results.

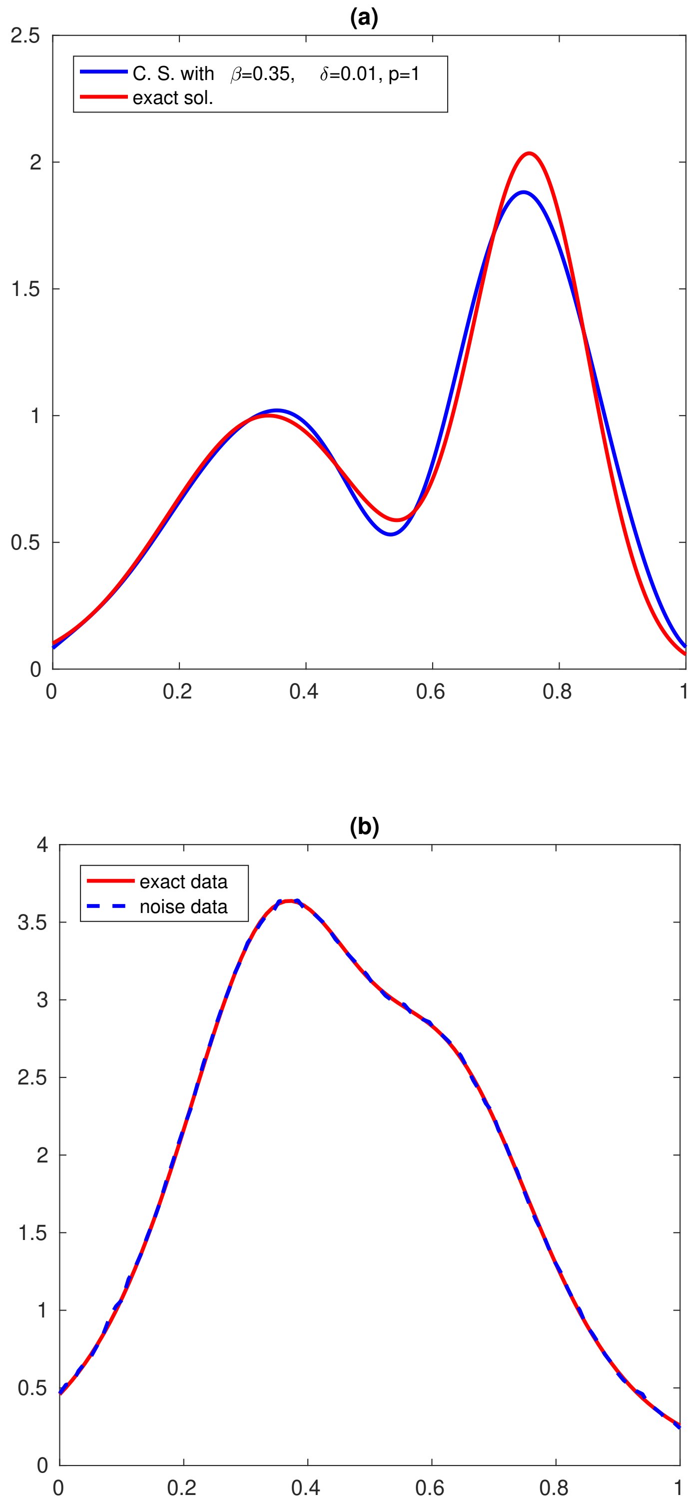

Figure 4.1: (a) Solution and (b) data of *Shaw* example (using discrepancy principle I) with $\beta = 0.35$, $\delta = 0.01$, $p = 1$ and $n = 1000$.

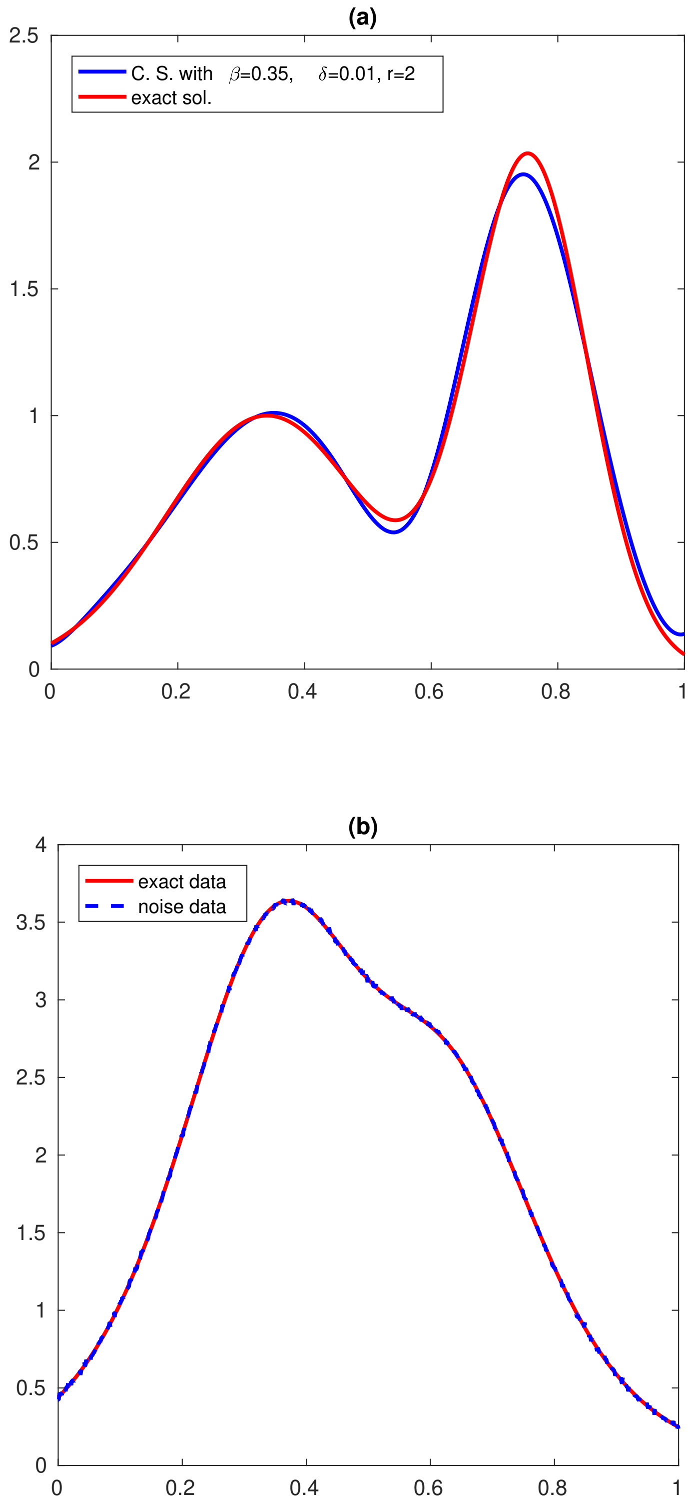

Figure 4.2: (a) Solution and (b) data of *Shaw* example (using discrepancy principle II) with $\beta = 0.35$, $\delta = 0.01$, $r = 2$ and $n = 1000$.

CHAPTER 5

WEIGHTED SIMPLIFIED REGULARIZATION METHOD: FINITE DIMENSIONAL REALIZATION

5.1 INTRODUCTION

In the previous chapter, we assume that the available data $Q\,y^\delta \in R(A^\beta)$, where Q is the orthogonal projection onto $\overline{R(A)}$. This, is a severe restriction, so we consider the finite dimensional realization of (4.1.3), namely, we consider the $w^\delta_{\alpha,\beta,h}$, the solution of

$$(A_h^{1-\beta} + \alpha\, I)\, x = A_h^{-\beta} P_h y^\delta, \tag{5.1.1}$$

where P_h is the orthogonal projection onto $R(P_h)$ and $A_h = P_h A P_h$.

Remark 5.1.1. *Note that, $P_h y^\delta \in R(A_h)$, i.e., $P_h y^\delta \in R(A_h^\beta)$ for $\beta \in [0, 1)$. So, $w^\delta_{\alpha,\beta,h}$ is well defined.*

One of the main constrain in regularization methods is the choice of the regularization parameter α. In this chapter, we consider the finite dimensional version of the adaptive parameter choice method considered by Pereverzev and Schock (2005) for choosing the regularization parameter α in (5.1.1).

The rest of the Chapter is organized as follows. In Section 5.1 we provide error estimates for $\|w^\delta_{\alpha,\beta,h} - w_{\alpha,\beta,h}\|$, $\|w_{\alpha,\beta,h} - w_{\alpha,\beta}\|$ and $\|w_{\alpha,\beta} - \hat{x}\|$, where $w_{\alpha,\beta,h}$ is the solution of (5.1.1) with y in the place of y^δ. In Section 5.2 we consider the finite dimensional version of the adaptive parameter choice strategy for weighted simplified regularization method. Numerical example is given in Section 5.3 and the conclusion in Section 5.4.

5.2 ERROR ESTIMATES

In this Section, we obtain the error estimates for $\|w^\delta_{\alpha,\beta,h} - w_{\alpha,\beta,h}\|$ and $\|w_{\alpha,\beta,h} - w_{\alpha,\beta}\|$ under the assumption (3.1.2) and the source condition given in (4.2.1)

If $\hat{x}$ satisfies (4.2.1), then by (i) in proposition 4.2.1 we have

$$\|w_{\alpha,\beta} - \hat{x}\| = O(\alpha^{\frac{\nu}{1-\beta}}). \tag{5.2.1}$$

For the results that follow, we impose the following conditions (cf. Plato and Vainikko (1990)). Let

$$\epsilon_h := \|A(I - P_h)\|$$

and assume that $\lim_{h\to 0}\epsilon_h = 0$. The above assumption is satisfied if $P_h \to I$ point-wise and if A is a compact operator. Let $A_h := P_h A P_h$. Then

$$\|A_h - A\| \leq \|P_h A(P_h - I)\| + \|(P_h - I)A\| \leq 2\epsilon_h.$$

In order to obtain an estimate for $\|w_{\alpha,\beta,h} - w_{\alpha,\beta}\|$, we shall make use of the following formula (Krasnosel skiĭ et al. (1976), Page 287);

$$\begin{aligned} B^z x &= \frac{\sin \pi z}{\pi} \int_0^\infty t^z \left[(B + tI)^{-1}x - \frac{\theta(t)}{t}x + \ldots + (-)^n \frac{\theta(t)}{t^n} B^{n-1}x\right] dt \\ &\quad + \frac{\sin \pi z}{\pi}\left[\frac{x}{z} - \frac{Bx}{z-1} + \ldots + (-1)^{n-1}\frac{B^{n-1}x}{z-n+1}\right], \ x \in X, \end{aligned}$$

where

$$\theta(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq 1 \\ 1 & \text{if} \quad 1 < t < \infty \end{cases}$$

for any positive self-adjoint operator B and for any complex number z such that $0 < Rez < n$. Taking $z = 1 - \beta, \ 0 \leq \beta < 1$, we have

$$B^{1-\beta}x = \frac{\sin\pi(1-\beta)}{\pi}\left[\frac{x}{1-\beta} + \int_0^\infty t^{1-\beta}(B+tI)^{-1}x dt - \int_1^\infty \frac{x}{t^\beta}dt\right].$$

Using the above formula, for any $Z \in X$, we have,

$$[A_h^{1-\beta} - A^{1-\beta}]Z = \frac{\sin\pi(1-\beta)}{\pi}\int_0^\infty t^{1-\beta}(A_h+tI)^{-1}(A-A_h)(A+tI)^{-1}Z dt. \quad (5.2.2)$$

Proposition 5.2.1. *Suppose y^δ satisfies (3.1.2) and $w_{\alpha,\beta,h}$ satisfies (5.1.1) with y in place of y^δ. Then, for $\nu < 1-\beta$ the following hold:*

(i) $\|w^\delta_{\alpha,\beta,h} - w_{\alpha,\beta,h}\| = O\left(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}}\right)$.
and

(ii) $\|w_{\alpha,\beta,h} - w_{\alpha,\beta}\| = O\left(\frac{\epsilon_h}{\alpha^{\frac{1}{1-\beta}}}\right)$.
In particular,

(iii) $\|w^\delta_{\alpha,\beta,h} - \hat{x}\| \leq c_1 \frac{\delta+\epsilon_h}{\alpha^{\frac{1}{1-\beta}}} + c_2\, \alpha^{\frac{\nu}{1-\beta}}$.

Proof. From (5.1.1), we have

$$\begin{aligned}\|w_{\alpha,\beta,h} - w^\delta_{\alpha,\beta,h}\| &= \|(A_h^{1-\beta}+\alpha)^{-1}\, A_h^{-\beta}\, P_h(y-y^\delta)\| \\ &\leq \delta \sup_{\lambda>0}\left|\frac{\lambda^{-\beta}}{(\lambda^{1-\beta}+\alpha)}\right| \\ &= O\left(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}}\right).\end{aligned}$$

Hence we proved (i). To Prove (ii), notice that

$$\begin{aligned}w_{\alpha,\beta,h} &= (A_h^{1-\beta} + \alpha\, I)^{-1}A_h^{-\beta}P_h y \\ &= (A_h^{1-\beta} + \alpha\, I)^{-1}A_h^{-\beta}P_h A\hat{x} \\ &= (A_h^{1-\beta} + \alpha\, I)^{-1}A_h^{1-\beta}\hat{x} + (A_h^{1-\beta} + \alpha\, I)^{-1}A_h^{-\beta}P_h A(I-P_h)\hat{x}, \\ w_{\alpha,\beta} &= (A^{1-\beta} + \alpha\, I)^{-1}A^{-\beta}y \\ &= (A^{1-\beta} + \alpha\, I)^{-1}A^{1-\beta}\hat{x}\end{aligned}$$

and hence

$$\begin{aligned} w_{\alpha,\beta,h} - w_{\alpha,\beta} &= [(A_h^{1-\beta} + \alpha\, I)^{-1} A_h^{1-\beta} - (A^{1-\beta} + \alpha\, I)^{-1} A^{1-\beta}]\hat{x} \\ &\quad + (A_h^{1-\beta} + \alpha\, I)^{-1} A_h^{-\beta} P_h A(I - P_h)\hat{x}. \end{aligned}$$

So

$$\|w_{\alpha,\beta,h} - w_{\alpha,\beta}\| \leq \|Ł\| + \|(A_h^{1-\beta} + \alpha\, I)^{-1} A_h^{-\beta} P_h A(I - P_h)\hat{x}\|, \tag{5.2.3}$$

where $Ł = [(A_h^{1-\beta} + \alpha\, I)^{-1} A_h^{1-\beta} - (A^{1-\beta} + \alpha\, I)^{-1} A^{1-\beta}]\hat{x}$.

Further, we have

$$\|(A_h^{1-\beta} + \alpha I)^{-1} A_h^{-\beta} P_h A(I - P_h)\hat{x}\| \leq \frac{\epsilon_h}{\alpha^{\frac{1}{1-\beta}}}\|\hat{x}\| \tag{5.2.4}$$

and

$$\begin{aligned} Ł &= [(A_h^{1-\beta} + \alpha I)^{-1} A_h^{1-\beta} - (A^{1-\beta} + \alpha I)^{-1} A^{1-\beta}]\hat{x} \\ &= (A_h^{1-\beta} + \alpha I)^{-1}[A_h^{1-\beta}(A^{1-\beta} + \alpha I) - (A_h^{1-\beta} + \alpha I)A^{1-\beta}](A^{1-\beta} + \alpha I)^{-1}\hat{x} \\ &= (A_h^{1-\beta} + \alpha I)^{-1}\alpha[A_h^{1-\beta} - A^{1-\beta}](A^{1-\beta} + \alpha I)^{-1}\hat{x} \\ &= (A_h^{1-\beta} + \alpha I)^{-1}[A_h^{1-\beta} - A^{1-\beta}]\alpha(A^{1-\beta} + \alpha I)^{-1}\hat{x}, \end{aligned}$$

so by (5.2.2), we have

$$\begin{aligned} \|Ł\| &= \|\frac{\sin\pi(1-\beta)}{\pi}\alpha(A_h^{1-\beta} + \alpha I)^{-1} \\ &\quad \times \int_0^\infty t^{1-\beta}(A_h + tI)^{-1}(A - A_h)(A + tI)^{-1}(A^{1-\beta} + \alpha I)^{-1}\hat{x}dt.\| \\ &\leq \frac{\sin\pi(1-\beta)}{\pi}\|\alpha(A_h^{1-\beta} + \alpha I)^{-1}\| \\ &\quad \times \int_0^\infty t^{1-\beta}\|(A_h + tI)^{-1}(A - A_h)(A + tI)^{-1}(A^{1-\beta} + \alpha I)^{-1}\hat{x}\|dt \\ &\leq \frac{\sin\pi(1-\beta)}{\pi}\|\alpha(A_h^{1-\beta} + \alpha I)^{-1}\| \\ &\quad \times [\int_0^1 t^{1-\beta}\|(A_h + tI)^{-1}(A - A_h)(A + tI)^{-1}(A^{1-\beta} + \alpha I)^{-1}\hat{x}\|dt \\ &\quad + \int_1^\infty t^{1-\beta}\|(A_h + tI)^{-1}(A - A_h)(A + tI)^{-1}(A^{1-\beta} + \alpha I)^{-1}\hat{x}\|dt] \end{aligned}$$

$$\begin{aligned}
\|Ł\| &\leq \frac{\sin\pi(1-\beta)}{\pi}[\int_0^1 t^{1-\beta}\|(A_h+tI)^{-1}\|\|A-A_h\|\|(A+tI)^{-1}(A^{1-\beta}+\alpha I)^{-1}A^{\nu}z\|dt \\
&+\int_1^\infty t^{1-\beta}\|(A_h+tI)^{-1}\|\|A-A_h\|\|(A+tI)^{-1}\|\|(A^{1-\beta}+\alpha I)^{-1}\hat{x}\|dt] \\
&\leq \frac{\sin\pi(1-\beta)}{\pi}[\int_0^1 t^{1-\beta}\frac{2\epsilon_h}{t}\|(A+tI)^{-1}A^{\nu}(A^{1-\beta}+\alpha I)^{-1}z\|dt \\
&+\int_1^\infty \frac{t^{1-\beta}2\epsilon_h}{t^2}\|(A^{1-\beta}+\alpha I)^{-1}A^{\nu}z\|dt]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Ł\| &\leq \frac{\sin\pi(1-\beta)}{\pi}\left[\int_0^1 t^{1-\beta}\frac{2\epsilon_h}{t^{2-\nu}}\|(A^{1-\beta}+\alpha I)^{-1}z\|dt\right. \\
&\left.+ 2\epsilon_h\|z\|\frac{\nu^{\nu}}{(1-\beta)(1-\beta-\nu)^{1-\nu}}\frac{1}{\alpha^{1-\frac{\nu}{1-\beta}}}\int_1^\infty \frac{1}{t^{1+\beta}}dt\right] \\
&\leq \frac{\sin\pi(1-\beta)}{\pi}\left[\frac{\|z\|}{\alpha}\int_0^1 \frac{2\epsilon_h}{t^{1-\nu-\beta}}dt\right. \\
&\left.+ 2\epsilon_h\|z\|\frac{\nu^{\nu}}{\beta(1-\beta)(1-\beta-\nu)^{1-\nu}}\frac{1}{\alpha^{1-\frac{\nu}{1-\beta}}}\right] \\
&\leq \frac{\sin\pi(1-\beta)}{\pi}\left[\frac{\|z\|}{\alpha}\frac{2\epsilon_h}{\nu}+2\epsilon_h\|z\|\frac{\nu^{\nu}}{\beta(1-\beta)(1-\beta-\nu)^{1-\nu}}\frac{1}{\alpha^{1-\frac{\nu}{1-\beta}}}\right] \\
&\leq 2\frac{\sin\pi(1-\beta)\|z\|}{\pi}\left[\frac{1}{\nu}+\frac{\nu^{\nu}}{\beta(1-\beta)(1-\beta-\nu)^{1-\nu}}\right]\frac{\epsilon_h}{\alpha^{\frac{1}{1-\beta}}}. \qquad (5.2.5)
\end{aligned}$$

Hence (ii) follows from (5.2.3), (5.2.4) (5.2.5) and the fact that $\max\{\frac{1}{\alpha}, \frac{1}{\alpha^{1-\frac{\nu}{1-\beta}}}\} \leq \frac{1}{\alpha^{\frac{1}{1-\beta}}}$.

The result (iii) is follows from (i), (ii) and (5.2.1).

□

5.3 ADAPTIVE SELECTION OF THE PARAMETER

Note that by (iii) of Proposition 5.2.1, we have

$$\|w^{\delta}_{\alpha,\beta,h}-\hat{x}\| \leq C\left(\frac{\delta+\epsilon_h}{\alpha^{\frac{1}{1-\beta}}}+\alpha^{\frac{\nu}{1-\beta}}\right), \qquad (5.3.1)$$

where

$$C = \max\{c_1, c_2\}.$$

Further, observe that the error $\frac{\delta+\epsilon_h}{\alpha^{\frac{1}{1-\beta}}} + \alpha^{\frac{\nu}{1-\beta}}$ in (5.3.1) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies, $\frac{\delta+\epsilon_h}{\alpha^{\frac{1}{1-\beta}}} = \alpha^{\frac{\nu}{1-\beta}}$. That is $\alpha_\delta = (\delta+\epsilon_h)^{\frac{1-\beta}{\nu+1}}$. Pereverzev and Schock (2005), introduced the adaptive selection of the parameter strategy, we modified adaptive method suitably for the situation for choosing the parameter α to obtain the optimal order in (5.3.1). Let $i \in \{0, 1, 2, \dots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$l := \max\left\{ i : \alpha_i^{\frac{1+\nu}{1-\beta}} \leq \delta + \epsilon_h \right\} < N \quad \text{and} \tag{5.3.2}$$

$$k := \max\left\{ i : \|w^\delta_{\alpha_i,\beta,h} - w^\delta_{\alpha_j,\beta,h}\| \leq 4C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}}, j = 0, 1, 2, \dots, i-1 \right\} \tag{5.3.3}$$

where $C = \max\{c_1, c_2\}$ where c_1, c_2 is as in Proposition 5.2.1. Now we have the following Theorem.

Theorem 5.3.1. *(cf. George and Nair (2008)) Assume that there exists $i \in \{0, 1, \dots, N\}$ such that $\alpha_i^{\frac{1+\nu}{1-\beta}} \leq \delta + \epsilon_h$. Let assumptions of Proposition 5.2.1 be fulfilled, and let l and k be as in (5.3.2) and (5.3.3) respectively. Then $l \leq k$; and*

$$\|w^\delta_{\alpha_k,\beta,h} - \hat{x}\| \leq 6\,C\mu^{\frac{\nu+1}{1-\beta}}\,(\delta+\epsilon_h)^{\frac{\nu}{\nu+1}}.$$

Proof. To prove $l \leq k$, it is enough to show that, for each $i \in \{1, 2, \dots, N\}$, $\alpha_i^{\frac{\nu}{1-\beta}} \leq \frac{\delta+\epsilon_h}{\alpha_i^{\frac{1}{1-\beta}}} \Longrightarrow \|w^\delta_{\alpha_i,\beta,h} - w^\delta_{\alpha_j,\beta,h}\| \leq 4\,C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}},\ \forall j = 0, 1, 2, \dots, i-1$. For $j < i$, we have

$$\begin{aligned}
\| w^\delta_{\alpha_i,\beta,h} - w^\delta_{\alpha_j,\beta,h} \| &\leq \| w^\delta_{\alpha_i,\beta,h} - \hat{x} \| + \| \hat{x} - w^\delta_{\alpha_j,\beta,h} \| \\
&\leq C\left(\alpha_i^{\frac{\nu}{1-\beta}} + \frac{\delta+\epsilon_h}{\alpha_i^{\frac{1}{1-\beta}}}\right) + C\left(\alpha_j^{\frac{\nu}{1-\beta}} + \frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}}\right) \\
&\leq 2\,C\,\alpha_i^{\frac{\nu}{1-\beta}} + 2\,C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}} \\
&\leq 4\,C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}}.
\end{aligned}$$

Thus the relation $l \leq k$ is proved. Further note that

$$\| \hat{x} - w^\delta_{\alpha_k,\beta,h} \| \leq \| \hat{x} - w^\delta_{\alpha_l,\beta,h} \| + \| w^\delta_{\alpha_l,\beta,h} - w^\delta_{\alpha_k,\beta,h} \|$$

where

$$\| \hat{x} - w^\delta_{\alpha_l,\beta,h} \| \leq C \left(\alpha_l^{\frac{\nu}{1-\beta}} + \frac{\delta + \epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}} \right) \leq 2\,C\,\frac{\delta + \epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}}.$$

Now since $l \leq k$, we have

$$\| w^\delta_{\alpha_k,\beta,h} - w^\delta_{\alpha_l,\beta,h} \| \leq 4\,C\,\frac{\delta + \epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}}.$$

Hence

$$\| \hat{x} - w^\delta_{\alpha_k,\beta,h} \| \leq 6\,C\,\frac{\delta + \epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}}$$

Again, since $\alpha_\delta^{\frac{\nu+1}{1-\beta}} = \delta + \epsilon_h \leq \alpha_{l+1}^{\frac{\nu+1}{1-\beta}} \leq \mu^{\frac{\nu+1}{1-\beta}} \alpha_l^{\frac{\nu+1}{1-\beta}}$, it follows that

$$\frac{\delta + \epsilon_h}{\alpha_\delta^{\frac{1}{1-\beta}}} \leq \frac{\delta + \epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}} \leq \mu^{\frac{\nu+1}{1-\beta}} \alpha_l^{\frac{\nu}{(1-\beta)}} \leq \mu^{\frac{\nu+1}{1-\beta}} \alpha_\delta^{\frac{\nu}{(1-\beta)}} \leq \mu^{\frac{\nu+1}{1-\beta}} (\delta + \epsilon_h)^{\frac{\nu}{\nu+1}}.$$

This completes the proof.

5.3.1 Implementation of adaptive choice rule

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 5.3.1 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \ldots, N$.

5.3.2 Algorithm

1. Set $i = 0$.
2. Solve $w^\delta_{\alpha_i,\beta,h}$ by using (5.1.1).
3. If $\|w^\delta_{\alpha_i,\beta,h} - w^\delta_{\alpha_j,\beta,h}\| > 4\,C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}}, j = 0, 1, 2, \ldots, i-1$, then take $k = i - 1$ and return $w_{\alpha_k,\beta,h}$.
4. Else set $i = i + 1$ and go to 2.

5.4 NUMERICAL EXAMPLES

In this section, we consider an academic example for the numerical discussion to validate our theoretical results. The discrete version of the operator A is taken from the Regularization Toolbox by Hansen (2007).

Relative errors $E_{\alpha,\beta,h} := \left(\frac{\|w^\delta_{\alpha,\beta,h} - \hat{x}\|}{\|\hat{x}\|}\right)$, and α are presented in the tables for different values of β, n (size of the mesh) and noise level δ.

Example 5.1 (cf. Phillips (1962)) Define the function

$$\phi(x) = \begin{cases} 1 + \cos\left(\frac{x\pi}{3}\right) & |x| < 3 \\ 0 & |x| \geq 3. \end{cases}$$

Consider the problem of solving integral equation

$$[Tx](s) := \int_{-6}^{6} k(s,t)\ x(t)dt = g(s), \quad -6 \leq s \leq 6,$$

where $k(s,t) = \phi(s-t)$, $g(s) = (6 - |s|)\left(1 + \frac{1}{2}\cos\left(\frac{s\pi}{3}\right)\right) + \frac{9}{2\pi}\sin\left(\frac{|s|\pi}{3}\right)$. We take $A = T^* T$.

The solution of this problem $\hat{x}(t)$ is given by $\hat{x}(t) = \phi(t)$. We have introduced the random noise level $\delta = 0.05$ and 0.01 in the exact data. Relative errors and α values are showcased in Tables 5.1 obtained using adaptive method for different values of β, n and δ. In figures, Fig: 5.1, Fig: 5.3, Fig: 5.5 and Fig: 5.7 contains the computed solution (C.S) and exact solution (exact sol.). Fig: 5.2, Fig: 5.4, Fig: 5.6 and Fig: 5.8 contains the exact data and noise data.

Table 5.1: Relative errors obtained from Adaptive method

β		$n = 100$		$n = 500$		$n = 1000$	
		$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.01$
	α	$1.393509e+00$	$4.884165e-01$	$1.266827e+00$	$3.335950e-01$	$1.151661e+00$	$3.032682e-01$
0	$E_{\alpha,\beta,h}$	$1.114813e-01$	$5.507954e-02$	$1.063084e-01$	$4.492278e-02$	$1.040632e-01$	$4.212422e-02$
	α	$1.686146e+00$	$4.440150e-01$	$1.532860e+00$	$3.032682e-01$	$1.393509e+00$	$2.756984e-01$
0.15	$E_{\alpha,\beta,h}$	$1.059385e-01$	$4.664234e-02$	$1.004385e-01$	$4.396330e-02$	$1.002486e-01$	$4.037034e-02$
	α	$1.854761e+00$	$4.036500e-01$	$1.686146e+00$	$2.756984e-01$	$1.266827e+00$	$2.506349e-01$
0.25	$E_{\alpha,\beta,h}$	$1.057641e-01$	$4.397211e-02$	$1.008134e-01$	$4.237829e-02$	$8.230723e-02$	$3.856134e-02$
	α	$2.040237e+00$	$3.669545e-01$	$1.854761e+00$	$2.506349e-01$	$1.151661e+00$	$2.278499e-01$
0.35	$E_{\alpha,\beta,h}$	$1.047137e-01$	$4.313658e-02$	$1.000185e-01$	$3.810415e-02$	$6.896331e-02$	$3.495328e-02$

5.5 CONCLUSION

In this Chapter, we considered weighted simplified regularization method for ill-posed equations in the finite dimensional subspaces of a Hilbert space involving positive self-adjoint operator. We obtained an optimal order error estimate under a general Hölder type source condition. Numerical experiments confirms the theoretical results.

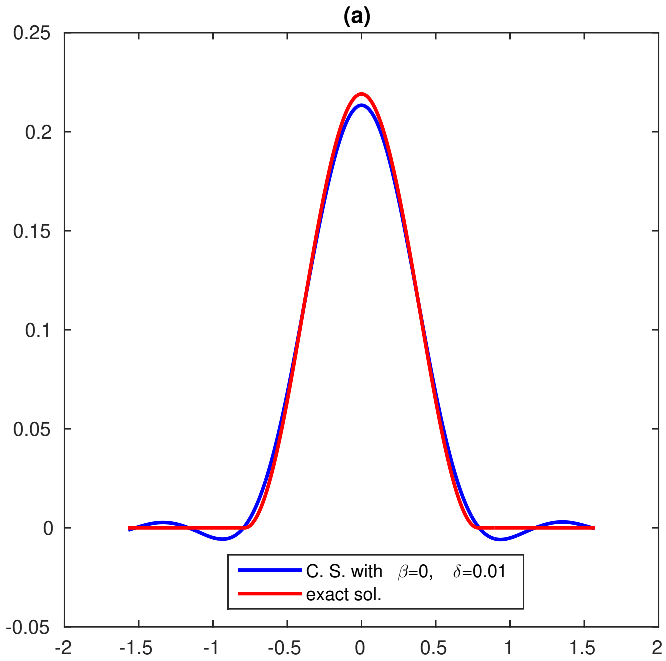

Figure 5.1: Solution of *Phillips* example with $\delta = 0.01$, $\beta = 0$ and $n = 1000$.

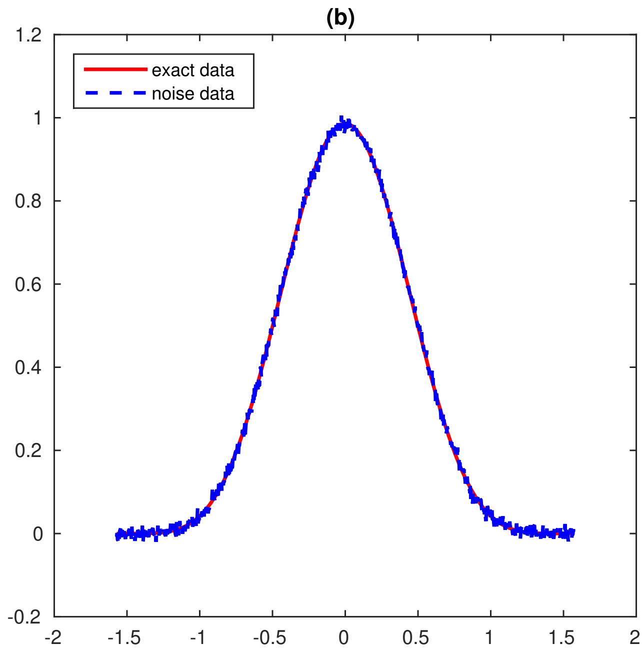

Figure 5.2: Data of *Phillips* example with $\delta = 0.01$, $\beta = 0$ and $n = 1000$.

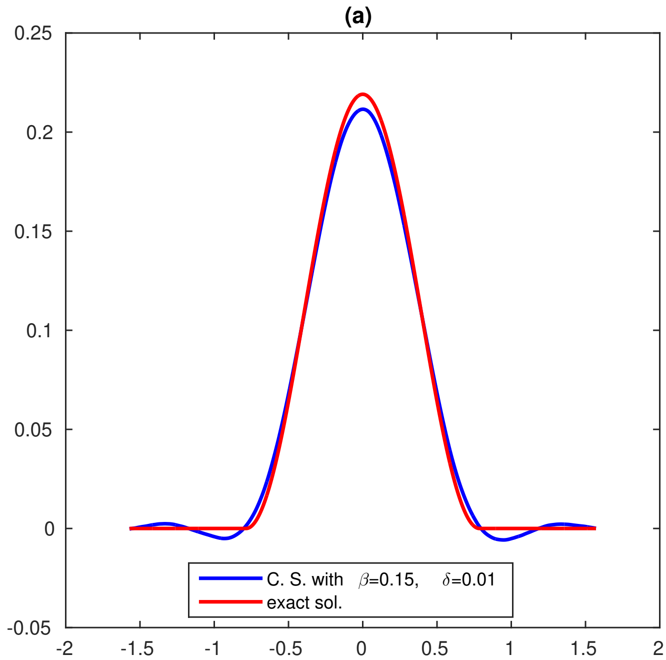

Figure 5.3: Solution of *Phillips* example with $\delta = 0.01$, $\beta = 0.15$ and $n = 1000$.

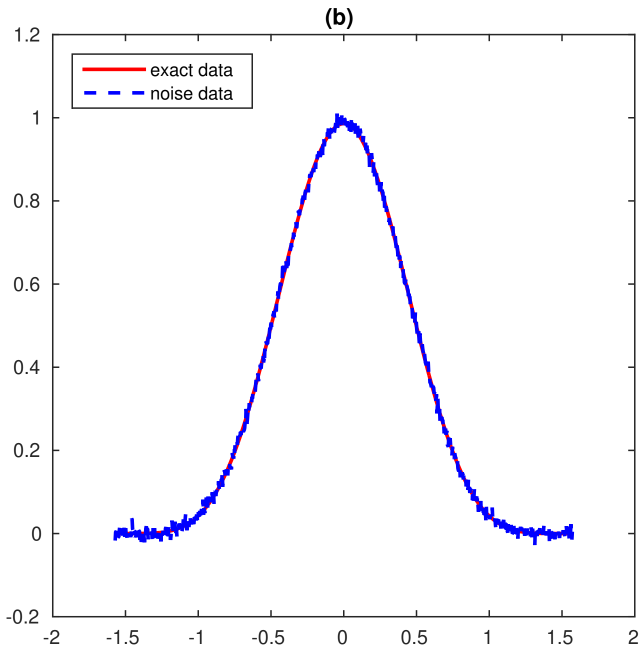

Figure 5.4: Data of *Phillips* example with $\delta = 0.01$, $\beta = 0.15$ and $n = 1000$.

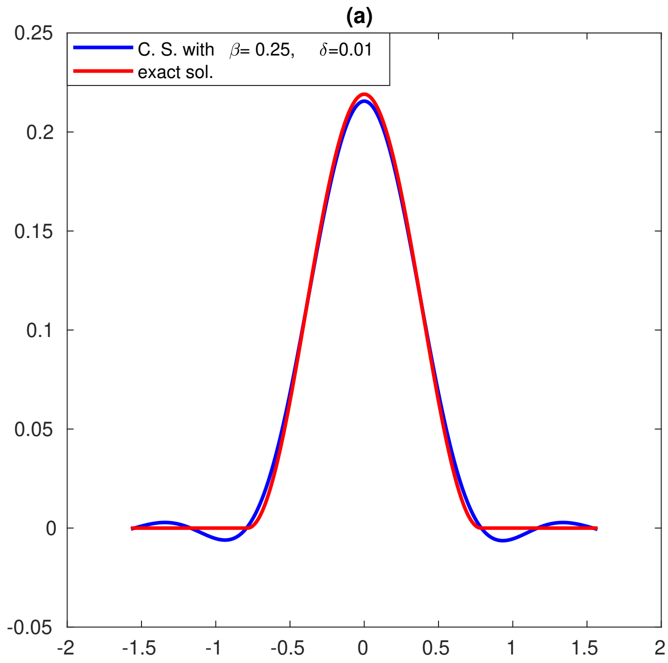

Figure 5.5: Solution of *Phillips* example with $\delta = 0.01$, $\beta = 0.25$ and $n = 1000$.

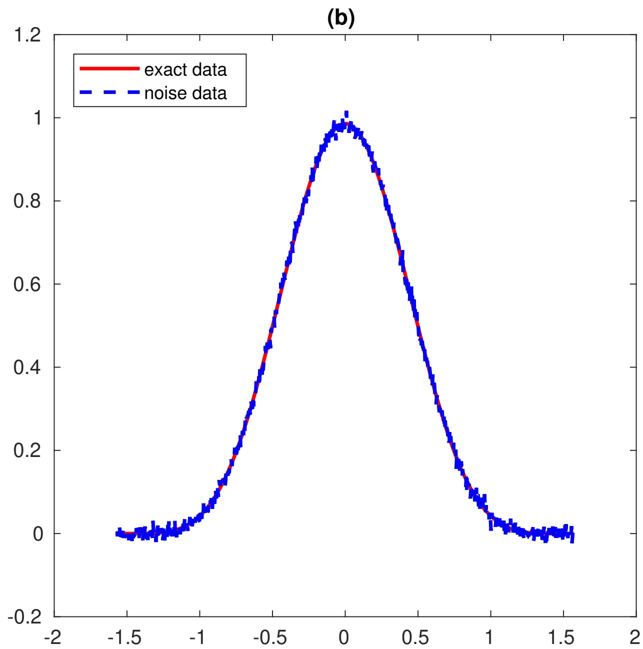

Figure 5.6: Data of *Phillips* example with $\delta = 0.01$, $\beta = 0.25$ and $n = 1000$.

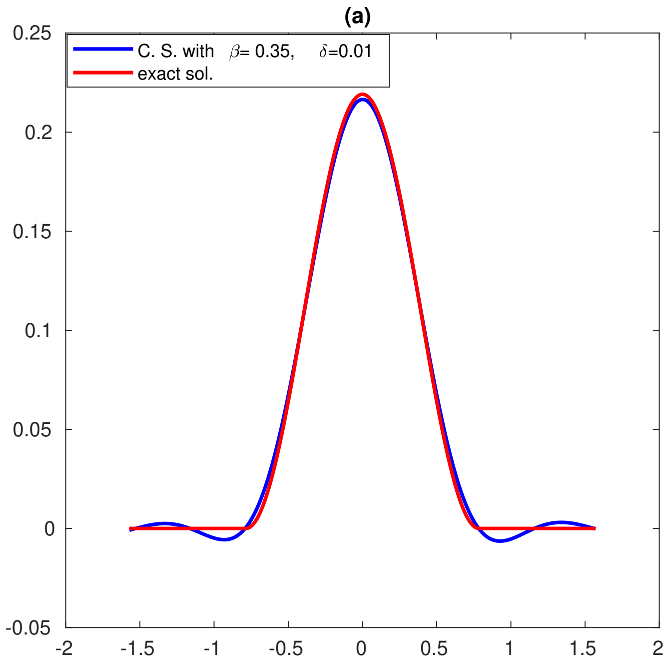

Figure 5.7: Solution of *Phillips* example with $\delta = 0.01$, $\beta = 0.35$ and $n = 1000$.

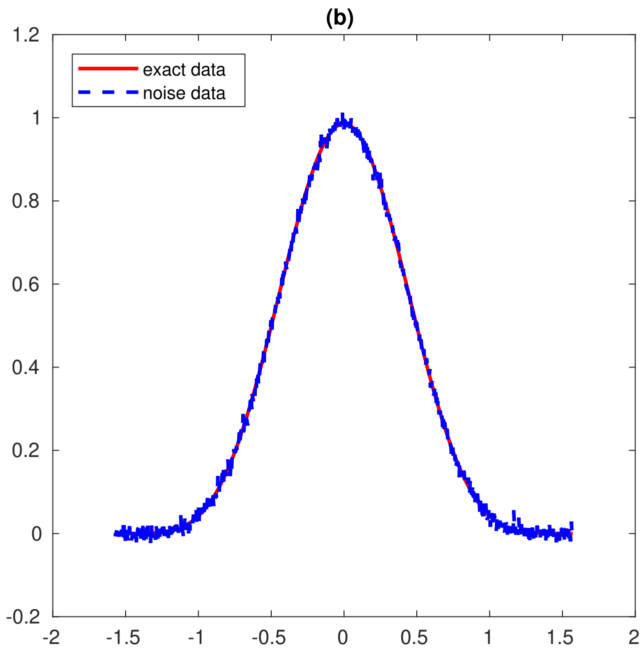

Figure 5.8: Data of *Phillips* example with $\delta = 0.01$, $\beta = 0.35$ and $n = 1000$.

CHAPTER 6

CONCLUSION AND FUTURE WORK

In Chapter 1, we have given a brief introduction to the ill-posed operator equations involving bounded linear operator between Hilbert spaces. Further, we defined the concepts and terms are used in this thesis.

In Chapter 2, we considered a derivative-free iterative method to find stable approximation for the solution of non-linear equation involving with monotone operator. We obtained an optimal order error estimate under a general Hölder-type source condition. Also we used the adaptive parameter choice strategy considered by Pereverzev and Schock (2005), for choosing the regularization parameter.

In Chapter 3, we considered Schock-type discrepancy principle for choosing the regularization parameter α for weighted Tikhonov regularization and we showed that weighted Tikhonov regularization gives better error estimate than Tikhonov regularization. We obtained an optimal order error estimate under a general Hölder-type source condition.

In Chapter 4, we considered weighted simplified regularization to find stable approximation for the solution of operator equation $Ax = y$, where $A : X \to X$ is a positive self-adjoint operator. We consider three discrepancy principle for choosing the regularization parameter α and we obtained optimal order under a general Hölder-type source condition.

In Chapter 5, we considered finite dimensional realization of weighted simplified regularization. We obtained an optimal order error estimate under a general Hölder type source condition.

The methods studied in this thesis for ill-posed operator equation, by no means, is exhaustive. During the study, we come across the following problems, where further research may be possible.

1) Can we consider Rule of Raus and Gfrerer (Gfrerer (1987); Raus (1984, 1985)) for choosing the regularization parameter in weighted Tikhonov regularization method?

2) We have studied weighted or fractional regularization method for linear operator equations. So, can we extend the weighted or fractional regularization method to non-linear ill-posed operator equations ?

3) Can we extend the fractional method to steepest descent and minimal error methods for linear and non-linear ill-posed problems ?

4) Finite dimensional realization of method considered in Chapter 3, is another problem, we would like to attend(We have obtained partial success in this direction).

It future, we are interested in studying the above problems.

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