A STUDY ON SEMICLOSED SUBSPACES AND SEMICLOSED OPERATORS IN HILBERT SPACES

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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DECLARATION

I hereby **declare** that the research thesis entitled **"A Study on Semiclosed Subspaces and Semiclosed Operators in Hilbert Spaces"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this research thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to **certify** that the research thesis entitled **"A Study on Semiclosed Subspaces and Semiclosed Operators in Hilbert Spaces"** submitted by **S. Balaji**, (Register Number MA08F01) as the record of the research work carried out by him, is *accepted as the research thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. P. Sam Johnson
Research Guide

Chairman - DRPC

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To forget good turns is not good
Good it is over wrong not to brood.

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ABSTRACT

Semiclosed subspaces (para-closed subspaces, in the terminology of C. Fioas) of Hilbert spaces have been considered for a long time, as a more flexible substitute of closed subspaces of Hilbert spaces. What is even more interesting is that the notion of semiclosed subspace coincides with that of a Hilbert space continuously embedded in $\mathcal{H}$. It is proved that the collection of all Hilbert spaces continuously embedded in a given Hilbert space $\mathcal{H}$ is in a bijective correspondence with the convex cone of all bounded positive self-adjoint operators in $\mathcal{H}$.

For two bounded operators A and B in $\mathcal{H}$ with the kernel condition $N(A) \subseteq N(B)$, the quotient $[B/A]$ defined in Izumino (1989), by $Ax \to Bx$, $x \in \mathcal{H}$. A quotient of bounded operators so defined is what was introduced by Kaufman (1978), as a "semiclosed operator", and several characterizations of it are given. It is proved that the family of quotients contains all closed operators and is itself closed under "sum" and "product". A merit for the quotient representation of a semiclosed operator is to make use of the theory of bounded operators. In the thesis, semiclosed subspaces and semiclosed operators in Hilbert spaces have been studied extensively.

Keywords : Semiclosed subspace ; operator range ; invariant subspace ; semiclosed operator ; quotient of bounded operators ; closed range ; Hyers-Ulam stability.

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Chapter 1

Preliminaries

1.1 General introduction

Many of the problems that arise in analysis (e.g., Mathematical physics, differential equations and partial differential equations etc.,) require solving operator equations of the form

$$Tx = y, \tag{1.1.1}$$

where $T : \mathcal{H}_1 \to \mathcal{H}_2$ is an operator possibly unbounded, $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces and $y \in \mathcal{H}_2$.

The theory of unbounded operators was stimulated by attempts in the late 1920s to put quantum mechanics on a rigorous mathematical foundation. The systematic development of the theory is due to John von Neumann and M. H. Stone. One of the motivations was quantum mechanics, which had been discovered in 1926 in two rather distinct forms by Erwin Schrödinger and Werner Heisenberg. It was von Neumann's insight that the natural language of quantum mechanics was that of self-adjoint operators on Hilbert space. This notion permeates modern physics. For a detailed discussion about these notions, we refer to Dunford and Schwartz (1988a,b); Kato (1976); Reed and Simon (1975, 1980); von Neumann (1950). Some examples of unbounded operators which are part of partial differential equations can be found in Goldberg (2006); Groetsch (2007).

The basic difference between bounded operators and unbounded operators is the domain on which they are defined. Domains of unbounded operators are proper subspaces of Hilbert spaces. Because of this fact, many aspects of the theory of unbounded operators are somewhat counterintuitive. For example, the algebraic rules for sums and products break down. Further, many of the important properties of the operators are very sensitive to the choice of domain (for example spectrum). Hence, one has to be careful while dealing with unbounded operators. Many of the techniques of bounded operators may fail to hold in the case of unbounded operators. In some cases, the techniques of bounded operators work. Hence to solve operator equations involving unbounded operators, one has to develop new techniques with the help of techniques of bounded operators. Through out the thesis, all operators are assumed to be linear but not necessarily bounded.

Among the unbounded operators, there is a class of operators called closed operators (operator whose graph is a closed subspace) behaves almost like bounded operators. Even though many important theorems which hold for bounded operators on Banach spaces also hold for closed operators, there are certain limitations for closed operators. The sum and product of two closed operators need not be closed, and therefore, we often need to be careful to consider the sum and product of closed operators.

The above mentioned limitations are due to lack of closedness of sum of two closed subspaces. Here the sum of graphs of two closed operators need not be closed. This demands the class of subspaces closed under "sum" which generalize the class of closed subspaces. W. E. Kaufman called this as a "semiclosed subspace". The new class of operators (operator whose graph is a semiclosed subspace) is called semiclosed operators. We will show that the collection of semiclosed operators is closed under sum and product. Moreover it is the smallest collection which contains all closed operators, their sum and product.

Consider the linear evolution equation as a system of equations which can be written as an abstract Cauchy problem

$$u' = Au(t); \quad u(0) = x; \quad t \in [0, T_x)$$

where A is a linear operator on an appropirately chosen Banach space X, $x \in X$ and $0 < T_x < \infty$. Various approaches like semigroup theory and transform approach are available for the linear evolution equation to study the relationships between its solution and characteristic equation. The major advantage of the transform approach is that the spectral assumption on the operator A can be kept at minimum. The transform approach allows us to study the abstract Cauchy problem for a wide variety of operators A, including sums (finite or infinite), products, or limits of closed operators which are not necessarily closed or closable, operators which are not densely defined and those where the point spectrum $\sigma_p(A)$ covers the whole complex plane. To be able to treat such operators, a class of unbounded operators mentioned above is considered.

In the thesis, semiclosed subspaces and semiclosed operators in complex Hilbert spaces (not necessarily separable) are discussed in detail.

1.2 Linear operators

Definition 1.2.1 (Linear operator). *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. A function T that sends every vector x of $\mathcal{H}_1$ into a vector $y = Tx$ of $\mathcal{H}_2$ is called a linear operator on $\mathcal{H}_1$ to $\mathcal{H}_2$ if T preserves linear relations, that is, if*

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for any scalars $\alpha, \beta \in \mathbb{C}$ (the complex numbers) and for every $x, y \in \mathcal{H}_1$.

We call $\mathcal{H}_1$, the domain space and $\mathcal{H}_2$, the range space of T. If $\mathcal{H}_2 = \mathcal{H}_1$ we say simply that T is a linear operator in $\mathcal{H}_1$. Throughout the thesis, we mean operator by linear operator.

In general, the operator may not be defined on the whole space $\mathcal{H}_1$ and it may be defined on a proper subspace of $\mathcal{H}_1$. In that case we denote the **domain of definition** (simply **domain**) by $D(T)$. We denote the set of all operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_1) = \mathcal{L}(\mathcal{H}_1)$. Every $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ gives rise to two important subspaces namely, the **null space** $N(T)$, defined by $N(T) := \{x \in D(T) : Tx = 0\}$ and the **range space** $R(T)$ defined as $R(T) := \{Tx : x \in D(T)\}$.

Definition 1.2.2 (Inverse)**.** *Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. If T is one-to-one, then the* ***inverse*** *of T is the operator $T^{-1}\colon R(T) \to \mathcal{H}_1$ defined by*

$$T^{-1}(Tx) = x \text{ for all } x \in D(T).$$

It can be seen that $TT^{-1}y = y$ for all $y \in R(T)$. If T is not one-to-one, we denote T^{-1} for $(T|_{N(T)^\perp})^{-1}$.

Definition 1.2.3 (Restriction and extension)**.** *If $T_1, T_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, then T_2 is an extension of T_1 (T_1 is a restriction of T_2) if $D(T_1) \subseteq D(T_2)$ and $T_1x = T_2x$ whenever $x \in D(T_1)$. In symbol $T_2 \supseteq T_1$.*

Definition 1.2.4 (Sum and product of operators)**.** *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $\alpha \in \mathbb{C}\backslash\{0\}$. Then*

(a) $T_1 + T_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ with domain $D(T_1 + T_2) = D(T_1) \cap D(T_2)$ and

$$(T_1 + T_2)x = T_1x + T_2x \quad \text{for all} \quad x \in D(T_1 + T_2).$$

(b) $ST_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3)$ with domain $D(ST_1) := \{x \in D(T_1) : T_1x \in D(S)\}$ and

$$(ST_1)x = S(T_1x) \quad \text{for all} \quad x \in D(ST_1).$$

(c) $\alpha T_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ with domain $D(\alpha T_1) = D(T_1)$ and

$$(\alpha T_1)x = \alpha T_1x \quad \text{for all} \quad x \in D(T_1).$$

Remark 1.2.5. *It may be possible that even though S and T are densely defined but $D(S + T)$ may be the zero space. Similarly, the domain of the product of two densely defined operators may be the zero space. For constructing such operators, one can refer to Chernoff (1983).*

In the following theorem, we collect some of the algebraic rules for operators.

Theorem 1.2.6 (Algebraic rules)**.** *Let* $T_i \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)\,(i = 1, 2, 3)$ *and* $S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ *and* $U \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1)$. *Then the following statements hold:*

1. $T_1 + T_2 = T_2 + T_1$.

2. $(T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$.

3. $\mathbf{0}T_1 \subseteq \mathbf{0}$.

4. $(ST_1)U = S(T_1U)$.

5. $(T_1 + T_2)U = T_1U + T_2U$.

6. $S(T_1 + T_2) \supseteq ST_1 + ST_2$ *(equality holds if* S *is everywhere defined).*

Definition 1.2.7 (Bounded operator)**.** *An operator* $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ *is called* ***bounded*** *if there exists* $c > 0$ *such that*

$$\|Tx\| \le c\,\|x\| \quad \text{for all } x \in \mathcal{H}_1.$$

In this case, the quantity

$$\|T\| := \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in D(T), x \neq 0\right\} < \infty,$$

is called the ***norm*** *of* T.

The set of all bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. We denote $\mathcal{B}(\mathcal{H}_1)$ for $\mathcal{H}_1 = \mathcal{H}_2$.

Example 1.2.8 (Multiplication operator)**.** *Let* $\mathcal{H} := L_2[a, b]$ *and* x *be a complex-valued function which is continuous on* $[a, b]$. *Define* $T : \mathcal{H} \to \mathcal{H}$ *by*

$$(Tf)(t) = x(t)f(t) \quad \text{for all } f \in \mathcal{H}.$$

It can be shown that $T \in \mathcal{B}(\mathcal{H})$ *and* $\|T\| = \max_{a \le t \le b} |x(t)|$.

Theorem 1.2.9 (Riesz representation theorem)**.** *If f is a bounded functional on a Hilbert space $\mathcal{H}$, then there exists some y in $\mathcal{H}$ such that for every $x \in \mathcal{H}$ we have $f(x) = \langle x, y\rangle$. Moreover, $\|f\| = \|y\|$.*

Theorem 1.2.10 (Open mapping theorem)**.** *Let X be a Banach space, Y a normed space and $T \in \mathcal{B}(X, Y)$, then either $R(T)$ is of first category in Y or $R(T) = Y$.*

Theorem 1.2.11 (Closed graph theorem)**.** *Let X and Y be Banach spaces and $T : X_0 \subset X \to Y$ be a closed operator. Then T is bounded iff X_0 is a closed suspace of X.*

Theorem 1.2.12 (Uniform boundedness principle)**.** *Let X be Banach, Y a normed space and $\mathcal{A} \subseteq \mathcal{B}(X, Y)$. If $\mathcal{A}$ is pointwise bounded, then $\mathcal{A}$ is uniformly bounded.*

Definition 1.2.13 (Unbounded operator)**.** *An operator which is not a bounded operator is called an **unbounded** operator. That is, if $\|T\| = \infty$, then it is called an **unbounded operator**.*

Example 1.2.14. *Let $\mathcal{H} := \ell_2$ and*

$$D(T) = \Big\{(x_1, x_2, \dots) \in \mathcal{H} : (x_1, 2x_2, 3x_3 \dots) \in \mathcal{H}\Big\}.$$

Define

$$T(x_1, x_2, x_3, \dots) = (x_1, 2x_2, 3x_3 \dots) \quad \text{for all } (x_1, x_2, \dots) \in D(T).$$

*If $\{e_n : n \in \mathbb{N}\}$, where $e_n(m) = \delta_{nm}$, the **Kronecker delta** function, then $Te_n = ne_n$. Hence the operator is unbounded.*

Definition 1.2.15. *An operator $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ with domain $D(T)$ is said to be **densely defined** if $\overline{D(T)} = \mathcal{H}_1$.*

Example 1.2.16. *The operator defined in Example 1.2.14 is densely defined since $D(T)$ contains c_{00}, the space of all complex sequences having at most finitely many nonzero terms, $\overline{c_{00}} = \ell_2$.*

Definition 1.2.17 (Adjoint operator)**.** *For any densely defined operator $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, there exists a unique operator T^* such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for all } x \in D(T) \text{ and } \quad y \in D(T^*).$$

*This operator is known as the **Hilbert adjoint** or simply the **adjoint** of T.*
In this case,

$$D(T^*) := \Big\{y \in \mathcal{H}_2 : x \mapsto \langle Tx, y\rangle \quad \text{for all } x \in D(T), \quad \text{is continuous}\Big\}.$$

Equivalently,

$$D(T^*) := \Big\{y \in \mathcal{H}_2 : \text{ for some } y^* \in \mathcal{H}_1,\ \langle Tx, y\rangle = \langle x, y^*\rangle \text{ for all } x \in D(T)\Big\}$$

and in this case, $T^*y = y^*$ for all $y \in D(T^*)$.

Remark 1.2.18. *Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Then T^* exists if and only if T is densely defined. Even though T is a densely defined operator, $D(T^*)$ may be the zero space.*

Proposition 1.2.19 (Properties of adjoint)**.** *Let $T, T_1, T_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$.*

1. *If T is densely defined, then $(cT)^* = \bar{c}T^*, c \neq 0$.*
2. *If T_1 and T_2 are densely defined such that T_1+T_2 is densely defined, then $(T_1+T_2)^* \supseteq T_1^* + T_2^*$*
3. *If T_1 and T_2 are densely defined such that $D(T_1T_2)$ is dense, then $(T_1T_2)^* \supseteq T_2^*T_1^*$*
4. *If T is one-to-one and $R(T)$ is dense in $\mathcal{H}_2$, then $(T^*)^{-1} = (T^{-1})^*$.*
5. *If T is densely defined such that $T \subset S$, then $S^* \subset T^*$.*

Definition 1.2.20. *An operator $T : \mathcal{H} \to \mathcal{H}$ is **symmetric** if T is densely defined and $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all x, y in $D(T)$ and **self adjoint** if $T = T^*$.*

Definition 1.2.21 (Positive operator)**.** *An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be **positive** if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.*

The set of all positive operators in $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})^+$.

Definition 1.2.22 (Square root)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be* ***square root*** *of T if $S^2 = T$.*

Proposition 1.2.23. *Every operator $T \in \mathcal{B}(\mathcal{H})^+$ has a unique square root $T^{1/2}$ in $\mathcal{B}(\mathcal{H})^+$, which commutes with every operator in $\mathcal{B}(\mathcal{H})$ that commutes with T.*

Theorem 1.2.24. *(Riesz and Sz.-Nagy (1955)) For each self-adjoint operator $T \in \mathcal{B}(\mathcal{H})^+$, there exists a unique self-adjoint $S \in \mathcal{B}(\mathcal{H})^+$ such that $S^2 = T$ and S is strong limit of the sequence given by the recursive relation*

$$T_0 = 0, T_n = [(I - T) - T_{n-1}^2], n \geq 1.$$

Remark 1.2.25. *If $T \in \mathcal{B}(\mathcal{H})^+$, then*

1. $\|T^{1/2}\|^2 = \|T\| = \|T^2\|^{1/2}$.

2. $N(T^{1/2}) = N(T) = N(T^2)$ *and* $\overline{R(T^{1/2})} = \overline{R(T)} = \overline{R(T^2)}$.

1.3 Orthogonal projections

Projection operators are the simplest non scalar operators. They play a significant role in Operator Theory. These projections enable us to decompose the operator into sum of restriction operators which have the same property as the original operator and easy to work with. The geometric properties of the subspaces can be studied through the algebraic properties of the projection operators.

Definition 1.3.1 (Projection)**.** *An operator $P \in \mathcal{L}(\mathcal{H})$ is said to be a* ***projection*** *if and only if $P^2 = P$.*

If P is a projection, then $I - P$ is a projection and $\mathcal{H} = R(P) \oplus N(P)$.

Combining the above observations one may call P, the projection onto $R(P)$ parallel to $N(P)$ (or along $N(P)$).

If a projection is bounded, then $I - P$ is also bounded. Hence $R(P) = N(I - P)$ is always closed.

Example 1.3.2. *Let $\mathcal{H} = \ell_2$ and $n \in \mathbb{N}$. Let $P_n : \mathcal{H} \to \mathcal{H}$ be given by*

$$P_n(x_1, x_2, x_3, \dots) = (x_1, x_2, \dots, x_n, 0, 0 \dots) \quad \text{for all} \quad (x_1, x_2, x_3, \dots) \in \mathcal{H}.$$

Then for each n, P_n is a projection and

$$R(P_n) = \{(x_1, x_2, \dots, x_n, 0, 0 \dots) : (x_1, x_2, \dots) \in \mathcal{H}\} \quad \text{and} \quad N(P_n) = R(P_n)^{\perp}.$$

It may be possible that a projection be unbounded. By the application of closed graph theorem, a projection P is unbounded only in case, its domain is a proper subspace of a Hilbert space as in the following example.

Example 1.3.3. *Let $\mathcal{H}$ be a separable Hilbert space and $\{e_n, f_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{H}$. Let $g_n = e_n + \frac{1}{n} f_n$. Let $G := \overline{span\{e_n : n \in \mathbb{N}\}}$ and $F := \overline{span\{g_n : n \in \mathbb{N}\}}$. Then $\mathcal{H} = \overline{G \oplus F}$. Let P be the operator with*

$$D(P) = G \oplus F \text{ such that } P(g + f) = g.$$

Clearly P is projection and cannot be bounded by the closed graph theorem. Note that both $R(P) = G$ and $N(P) = F$ are closed.

Among all projection operators, the self-adjoint projections have more significance as the following results reveal this fact.

Definition 1.3.4 (Orthogonal projection)**.** *A bounded projection $P : \mathcal{H} \to \mathcal{H}$ is said to be **orthogonal** if $R(P) = N(P)^{\perp}$.*

Example 1.3.5. *The projection given in example 1.3.2 is an orthogonal projection.*

Example 1.3.6. *Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be given by*

$$P(x_1, x_2) = (x_1, x_1) \quad \text{for all} \quad (x_1, x_2) \in \mathbb{R}^2$$

is a bounded projection but not orthogonal since $N(P) = \{(0, x) : x \in \mathbb{R}\}$ and $R(P) = \{(x, x) : x \in \mathbb{R}\}$ are not orthogonal to each other.

Example 1.3.7. *Let $\mathcal{H} := L_2[0,1]$ and E be a measurable subset of $[0,1]$. Define P on $\mathcal{H}$ by*

$$(Pf)(t) = f(t)\chi_E(t) \quad \text{for all} \quad f \in \mathcal{H},$$

here χ_E denotes the characteristic function on E. Then P is a bounded projection with

$$N(P) = \{g \in \mathcal{H} : g(t) = 0 \quad \text{for all } \ t \in E\}$$

and

$$R(P) = \{f \in \mathcal{H} : f(t) = 0 \quad \text{for all } \ t \notin E\}.$$

Hence P is an orthogonal projection.

Given any closed subspace M of $\mathcal{H}$ there exists an orthogonal projection such that $R(P) = M$. To see this, let $x \in \mathcal{H}$. Then by the projection theorem, $x = u + v$, where $u \in M$ and $v \in M^{\perp}$. Now define $Px = u$. Clearly P is a projection such that

$$R(P) = M \text{ and } N(P) = M^{\perp}.$$

Since the range of an orthogonal projection is closed, and $I-P$ is an orthogonal projection. Thus, orthogonal projection decomposes a Hilbert space

$$\mathcal{H} = R(P) \oplus^{\perp} N(P).$$

Hence there is a one-to-one correspondence between the closed subspaces of Hilbert space and orthogonal projections on the Hilbert space.

Theorem 1.3.8. *The following statements are equivalent:*

(i) P *is an orthogonal projection.*

(ii) P *is normal.*

(iii) P *is self-adjoint.*

(iv) $\|P\| = 1$.

If $P : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection with $R(P) = M$, then we denote it by P_M.

Theorem 1.3.9 (Algebraic rules)**.** *Let M_1 and M_2 be closed subspaces of $\mathcal{H}$ and $P_i = P_{M_i}$, $i = 1, 2$. Then*

1. *$P = P_1 - P_2$ is a projection if and only if $M_2 \subseteq M_1$. Then $R(P) = M_1 \cap M_2^{\perp}$.*

2. *$P = P_1 + P_2$ is a projection if and only if $P_1P_2 = 0$. Then $R(P) = M_1 \oplus M_2$.*

3. *$P = P_1P_2$ is a projection if and only if $P_1P_2 = P_2P_1$. Then $R(P) = M_1 \cap M_2$.*

Theorem 1.3.10. *If $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint and $T^2 = T$. Then T is an orthogonal projection onto $M = \{x \in \mathcal{H} : Tx = x\}$.*

Definition 1.3.11. *Two orthogonal projections P and Q are said to be* ***mutually orthogonal*** *if $PQ = 0$.*

Theorem 1.3.12. *Let M and N be closed subspaces of $\mathcal{H}$. Let $P = P_M$ and $Q = Q_N$. Then the following statements are equivalent:*

1. *$M \perp N$.*

2. *$PQ = 0$.*

3. *$QP = 0$.*

4. *$P|_N = 0$.*

5. *$Q|_M = 0$.*

We can define an ordering for orthogonal projections as follows. Let $P_i = P_{M_i}$, $i = 1, 2$. Then $P_1 \leq P_2 \Leftrightarrow M_1 \subseteq M_2$. This " $\leq$ " is a partial order on the class of orthogonal projections.

Theorem 1.3.13. *Let $P = P_M$ and $Q = P_N$. Then the following statements are equivalent:*

1. *$P \leq Q$.*

2. $\|Px\| \leq \|Qx\| \quad$ *for all* $\quad x \in \mathcal{H}$*.*

3. $M \subseteq N$*.*

4. $QP = P$*.*

5. $PQ = P$*.*

1.4 Partial isometries

An operator $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $D(U) = \mathcal{H}_1$ is called an **isometry** if

$$\|Ux\| = \|x\| \quad \text{for all } \ x \in \mathcal{H}_1.$$

If U is an isometry and $R(U) = \mathcal{H}_2$, then U is called a **unitary** operator.

An isometry is a distance preserving operator. A necessary and sufficient condition for an operator to be an isometry is that $U^*U = I$.

Example 1.4.1 (Bilateral shift)**.** *Let* $\mathcal{H} := \ell_2(\mathbb{Z})$*. Define* $V : \mathcal{H} \to \mathcal{H}$ *by*

$$V(\ldots, x_{-1}, \boxed{x_0}, x_1, \ldots) = (\ldots, x_{-2}, \boxed{x_{-1}}, x_0, \ldots).$$

Here the box $\Box$ *indicates the zeroth position in the sequence. It can be shown that* V *is a unitary operator and* $V^{-1} : \mathcal{H} \to \mathcal{H}$ *is given by*

$$V^{-1}(\ldots, x_{-1}, \boxed{x_0}, x_1, \ldots) = (\ldots, x_0, \boxed{x_1}, x_2, \ldots).$$

Proposition 1.4.2. *For an operator* $U : \mathcal{H}_1 \to \mathcal{H}_2$*, the following are equivalent:*

1. $\|Ux\|^2 = \|x\|^2$*.*

2. $\langle U^*Ux, x\rangle = \langle x, x\rangle$*.*

3. $\langle U^*Ux, y\rangle = \langle x, y\rangle$

4. $U^*U = 1$*.*

Remark 1.4.3. *It is worth to note that* $UU^* = I$ *is not equivalent to* $U^*U = I$*. The former condition is satisfied in case* U^* *is an isometry. In that case,* U *is called* ***co-isometry***.

Theorem 1.4.4. *Let* $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ *be such that* $D(U) = \mathcal{H}_1$*. Then the following assertions are equivalent:*

1. U *is unitary.*

2. $R(U) = \mathcal{H}_2$ *and* $\langle Uf, Ug \rangle = \langle f, g \rangle$ *for all* $f, g \in \mathcal{H}_1$.

3. $U^*U = I$ *and* $UU^* = I$*., i.e* $U^* = U^{-1}$.

4. U^* *is unitary.*

Instead of looking for an operator to be an isometry on the whole space, it is sometimes convenient to consider an operator that acts isometrically on a subspace of a Hilbert space, that is, $\|Ux\| = \|x\|$ for all x in that subspace.

Definition 1.4.5 (Partial isometry)**.** *A* ***partial isometry*** *is an operator that is isometric onto the orthogonal complement of its null space.*
That is, an operator $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ *is a partial isometry if* $U|_{N(U)^\perp} : N(U)^\perp \to \mathcal{H}_2$*, is an isometry.*

In this case $N(U)^\perp$ *is called the* ***initial subspace*** *and* $R(U)$ *is called the* ***final space***.

Remark 1.4.6. *Unitary operators, isometries and projections are examples for partial isometry. Partial isometries are bounded and in fact the norm of a nonzero partial isometry is one.*

Proposition 1.4.7. *If* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *is a partial isometry, then*

$$U = VP,$$

where $V : N(U)^\perp \to \mathcal{H}_2$ *is an isometry and* $P : \mathcal{H}_1 \to \mathcal{H}_1$ *is an orthogonal projection on to* $N(U)^\perp$.

Theorem 1.4.8. *Let $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that $D(U) = \mathcal{H}_1$. Then the following assertions are equivalent:*

1. *U is a partial isometry with initial space M and final space N.*

2. *$R(U) = N$ and $\langle Uf, Ug\rangle = \langle P_M f, g\rangle$ for all $f, g \in \mathcal{H}_1$.*

3. *$U^*U = P_M$ and $UU^* = P_N$.*

4. *U^* is a partial isometry with initial space N and final space M.*

Definition 1.4.9 (Polar decomposition)**.** *If an operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is the product of a partial isometry $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and a positive operator $Q \in \mathcal{B}(\mathcal{H}_1)$, and if $N(U) = N(Q)$, then the representation $A = UQ$ is called the* ***polar decomposition*** *of A.*

Proposition 1.4.10. *For any $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, there exists a partial isometry $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with initial space $N(A)^\perp$ and final space $\overline{R(A)}$ such that*

$$A = U(A^*A)^{1/2}.$$

and $N(U) = N((A^*A)^{1/2})$. Moreover, if $A = ZQ$, where Q is a positive operator in $\mathcal{B}(\mathcal{H})$ and Z is a partial isometry in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with $N(Z) = N(Q)$, then $Q = (A^*A)^{1/2}$ and $Z = U$.

Theorem 1.4.11. *If $A = UQ$ is the polar decomposition of A, then*

1. *$U^*A = Q$.*

2. *U is isometry if and only if A is one-to-one.*

3. *U is co-isometry if and only if $R(A)$ is dense.*

1.5 Graphs of operators and graph norm

Let $\mathcal{H}_1, \mathcal{H}_2$ be complex Hilbert spaces. An inner product on $\mathcal{H}_1 \times \mathcal{H}_2$ is given by

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle_{\mathcal{H}_1} + \langle y_1, y_2\rangle_{\mathcal{H}_2} \quad (x_i, y_i) \in \mathcal{H}_1 \times \mathcal{H}_2,\ i = 1, 2.$$

The norm induced by the inner product is given by

$$\|(u,v)\| := \sqrt{\|u\|_{\mathcal{H}_1}^2 + \|v\|_{\mathcal{H}_2}^2}.$$

Under this norm, $\mathcal{H}_1 \times \mathcal{H}_2$ is a Hilbert space.

Definition 1.5.1 (Graph)**.** *Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. The **graph** of T is denoted by $G(T)$ and is defined by $G(T) := \Big\{(x, Tx) : x \in D(T)\Big\} \subseteq \mathcal{H}_1 \times \mathcal{H}_2$.*

Note 1.5.2. *It is easy to see that for an operator T its graph $G(T)$ is a linear subspace of $\mathcal{H}_1 \times \mathcal{H}_2$.*

Definition 1.5.3 (Graph norm)**.** *For any $x \in D(T)$, $\|x\|_T := \sqrt{\|x\|^2 + \|Tx\|^2}$ defines a norm and is called the **graph norm**.*

Definition 1.5.4 (Closed operator)**.** *Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. If $G(T)$ is closed in $\mathcal{H}_1 \times \mathcal{H}_2$, then T is called the **closed operator**. Equivalently, T is a closed operator if $\{x_n\} \subseteq D(T)$ such that $x_n \to x$ and $Tx_n \to y$ for some $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, then $x \in D(T)$ and $Tx = y$.*

We denote all closed operators by $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$.

Remark 1.5.5. *Let $T \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$. Then the space $\mathcal{H}_T := (D(T), \|\cdot\|_T)$, that is, $D(T)$ equipped with graph norm is a Hilbert space and $T : \mathcal{H}_T \to \mathcal{H}_2$ is bounded.*

Example 1.5.6 (Closed and non-closed operators)**.** *Let $\mathcal{H} := L_2[0,1]$,*

$$D_1(T) = C^1([0,1])$$

and

$$D_2(T) = \Big\{x \in C([0,1]) : x \text{ is absolutely continuous and } x' \in \mathcal{H}\Big\}.$$

Consider the operator $T : \mathcal{H} \to \mathcal{H}$ defined by $T(x) = x'$. The operator T with domain $D_2(T)$ is closed. We show that T with domain D_1 is not closed. To verify this, let

$$y(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{2} \\ 1, & \frac{1}{2} \le t \le 1 \end{cases} \qquad \text{and} \qquad y_n(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{2} - \frac{1}{n}, \\ n(t - \frac{1}{2} + \frac{1}{n}), & \frac{1}{2} - \frac{1}{n} < t \le \frac{1}{2} \\ 1, & \frac{1}{2} < t \le 1 \end{cases}$$

for $n = 2, 3, \ldots$. *Then it is clear that* $y_n \to y$ *in* $\mathcal{H}$. *For* $t \in [0,1]$, *define*

$$x(t) = \int_0^t y(s)ds \qquad \text{and} \qquad x_n(t) = \int_0^t y_n(s)ds$$

Then $x'(t) = y(t)$ *for all* $t \in [0,1]$. *Also,*

$$\|x_n(t) - x(t)\| \leq \int_0^t |y_n(s) - y(s)|ds \leq \|y_n - y\|_1 \leq \|y_n - y\|_2.$$

Hence $\{x_n\}$ *converges to* x *uniformly on* $[0,1]$. *Thus* $\{x_n\} \subseteq D_1(T)$, $x_n \to x$ *and* $x'_n = y_n \to y$ *in* $\mathcal{H}$. *However* x *is not differentiable at* $\frac{1}{2}$. *Hence* $x \notin D_1(T)$. *Therefore* T *with domain* D_1 *is not closed.*

Example 1.5.7 (Non-closed operator). *Let* $\mathcal{H} := L_2(0,1)$. *Define* $T : \mathcal{H} \to \mathcal{H}$ *by*

$$(Tf)(x) = xf(1) \quad \text{for all } f \in D(T) = C([0,1]) \text{ and } x \in [0,1].$$

Note that $D(T)$ *is dense in* $\mathcal{H}$. *We show that this operator is not closed. To verify this, let*

$$f_n(x) = 1, \quad x \in [0,1] \qquad \text{and} \qquad h_n(x) = \begin{cases} 1, & 0 \leq x \leq \frac{n-1}{n}, \\ n(1-x), & \frac{n-1}{n} \leq x \leq 1. \end{cases}$$

Clearly, $\lim_{n\to\infty} f_n(x) = 1 = \lim_{n\to\infty} h_n(x)$ *for all* $x \in [0,1]$. *Observe that* $f_n(1) = 1$ *and* $h_n(1) = 0$. *Hence* $(Tf_n)(x) = x$ *and* $(Th_n)(x) = 0$. *Even the limits* $\lim_{n\to\infty} Tf_n$ *and* $\lim_{n\to\infty} Th_n$ *exist, they are not equal. Hence* T *is not closed.*

Definition 1.5.8 (Continuous embedding). *Let* $(\mathcal{H}, \langle .,. \rangle)$ *be a Hilbert space and* M *be a subspace of* $\mathcal{H}$ *with the inner product* $\langle .,. \rangle_M$. *Then* M *is said to be* ***continuously embedded*** *in* $\mathcal{H}$ *if the inclusion map,*

$$J : (M, \langle .,. \rangle_M) \to (\mathcal{H}, \langle .,. \rangle)$$

defined by

$$J(x) = x$$

is continuous. That is, there exists a constant $b > 0$ *such that* $\|x\| \leq b\|x\|_M$ *for every* $x \in M$. *It is denoted by* $M \hookrightarrow \mathcal{H}$.

Remark 1.5.9. *If M is closed in $\mathcal{H}$, then the two norms are equivalent.*

Example 1.5.10. *(Evans (2010)) Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, Lipschitz domain, and let $1 \leq p \leq n$ and $p^* = \frac{np}{n-p}$. Then the Sobolev space $W^{1,p}(\Omega; \mathbb{R})$ is continuously embedded in the space $L^{p^*}(\Omega; \mathbb{R})$. In symbol, $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.*

Chapter 2

Semiclosed Subspaces

The well known procedure of making a closed operator to bounded is to renorm its domain with graph norm. One can also see that the domain of a closed operator becomes a Hilbert space with respect to the graph norm. Instead of considering only domains of closed operators, it is motivated to consider a subspace M of a Hilbert space $\mathcal{H}$ and look for some inner product on M which makes M a Hilbert space. Moreover, the new inner product in M is stronger than the one in $\mathcal{H}$. This leads to the definition of "semiclosed subspace". Semiclosed subspaces possess many special features that distinguish them from arbitrary subspaces and they can be considered as a more flexible substitute of closed subspaces of Hilbert spaces.

The chapter is started with a number of characterizations of semiclosed subspaces and examples. The ranges of members of the set of bounded positive self-adjoint operators $\mathcal{B}(\mathcal{H})^+$ can alone characterize all semiclosed subspaces of $\mathcal{H}$. The collection of semiclosed subspaces in $\mathcal{H}$ is in bijective correspondence with $\mathcal{B}(\mathcal{H})^+$. It is proved that the collection of semiclosed subspaces forms a lattice but it is not in the case of closed subspaces. For each such Hilbert inner product on a semiclosed subspace M, there corresponds a topology on M. It is proved that all such inner products are equivalent – making an unique topology on M.

2.1 Characterizations of semiclosed subspaces

Definition 2.1.1. *(Kaufman (1979)) A subspace M (need not be closed) of a Hilbert space $(\mathcal{H}, \langle ., . \rangle)$ is called semiclosed if there exists a Hilbert inner product $\langle ., . \rangle_*$ on M such that $(M, \langle ., . \rangle_*)$ is continuously embedded in $(\mathcal{H}, \langle ., . \rangle)$. That is, there exists an inner product $\langle ., . \rangle_*$ on M such that $(M, \langle ., . \rangle_*)$ is Hilbert and there exists $k > 0$ with*

$$\langle x, x \rangle \leq k \ \langle x, x \rangle_* \quad \textit{for all } x \in M.$$

It is known that every subspace of $\mathcal{H}$ is closed if and only if $\mathcal{H}$ is of finite dimension. In order to avoid trivial considerations we assume only infinite dimensional complex Hilbert spaces throughout the thesis. Every closed subspace is semiclosed by considering $\langle ., . \rangle_*$ the restriction of $\langle ., . \rangle$ to $M \times M$. Dixmier (1949) calls a semiclosed subspace a "Julia variety" ; "paraclosed subspace" by Foiaş (1972).

An operator range (Fillmore and Williams (1971)) in a Hilbert space $\mathcal{H}$ is a subspace of $\mathcal{H}$ that is the range of some bounded operator in $\mathcal{H}$. Semiclosed subspaces are characterized by operator ranges. Indeed, if M is a semiclosed subspace of $\mathcal{H}$. Then there exists an inner product $\langle ., . \rangle_*$ on M such that $(M, \langle ., . \rangle_*)$ is Hilbert and there exists $k > 0$ with

$$\langle x, x \rangle \leq k \langle x, x \rangle_* \text{ for all } x \in M.$$

The inclusion map $J : (M, \langle ., . \rangle_*) \to \mathcal{H}$ is bounded. Now consider the polar decomposition of J^*, $J^* = U(JJ^*)^{1/2}$, then $U : \mathcal{H} \to (M, \langle ., . \rangle_*)$ is a partial isometry with final space $cl(R(J^*)) = N(J)^{\perp} = M$. Considering U from $\mathcal{H}$ into $\mathcal{H}$, we have $R(U) = M$. Boundedness of U follows from

$$\langle Ux, Ux \rangle \leq k \langle Ux, Ux \rangle_* \leq k \langle x, x \rangle \text{ for all } x \in \mathcal{H}.$$

Hence M is an operator range.

Conversely, if M is an operator range. Then $M = R(A)$ for some $A \in \mathcal{B}(\mathcal{H})$. By closed graph theorem, A is closed. Now consider the operator $\tilde{A} = A|_{N(A)^{\perp}}$ which is an injective closed operator whose range equals M. The inverse of $\tilde{A}$ is a closed operator with domain M. Define

$$\langle x, y\rangle_* = \langle x, y\rangle + \langle \tilde{A}^{-1}x, \tilde{A}^{-1}y\rangle \text{ for } x, y \in M.$$

Then $(M, \langle ., .\rangle_*)$ is a Hilbert space and

$$\langle x, x\rangle \leq \langle x, x\rangle + \langle \tilde{A}^{-1}x, \tilde{A}^{-1}x\rangle = \langle x, x\rangle_* \text{ for all } x \in M.$$

Hence M is a semiclosed subspace.

From the above discussions one can observe that the ranges of members of $\mathcal{B}(\mathcal{H})^+$ can alone characterize all semiclosed subspaces of $\mathcal{H}$. Operator ranges have been studied by many authors, as shown in the semi-expository paper of Fillmore and Williams (1971).

In the theory of Hilbert spaces, there is a one-to-one correspondence between closed subspaces and orthogonal projections. Since all closed subspaces are semiclosed subspaces and the latter is nothing but the generalization of closed subspaces, it is quite natural to ask, is there any connection between semiclosed subspaces and some class of operators in Hilbert space. The following theorem answers the question partially and serves as characterizations of semiclosed subspaces.

Theorem 2.1.2. *(Fillmore and Williams (1971)) Let M be a subspace of a Hilbert space $\mathcal{H}$. The following are equivalent:*

1. *M is a semiclosed subspace of $\mathcal{H}$.*
2. *M is the range of a bounded operator in $\mathcal{H}$.*
3. *M is the range of a closed operator in $\mathcal{H}$.*
4. *M is the domain of a closed operator in $\mathcal{H}$.*
5. *There is a sequence $\{\mathcal{H}_n : n \geq 0\}$ of closed mutually orthogonal subspaces of $\mathcal{H}$ such that*
$$M = \left\{ \sum_{n=0}^{\infty} x_n : x_n \in \mathcal{H}_n \text{ and } \sum_{n=0}^{\infty} (2^n \|x_n\|)^2 < \infty \right\}.$$

The following examples show that the collection of semiclosed subspaces is strictly bigger than the collection of all closed subspaces and strictly smaller than the collection of all subspaces. In short the collection is strictly between all subspaces and all closed subspaces.

Example 2.1.3. *Consider the subspace*

$$M = \left\{(x_1, x_2, \ldots) : \sum_{n=1}^{\infty}(n|x_n|)^2 < \infty\right\}$$

of the space ℓ_2 of square-summable sequences. As M contains all sequences with finite support and $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ is not in M, it is a proper dense subspace of ℓ_2. So, M is not closed but it is semiclosed because M is the range of the bounded operator $A : \ell_2 \to \ell_2$ defined by $A(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, \ldots)$.

Example 2.1.4. *The subspace $L_p[0,1]$ $(2 < p < \infty)$ is not a semiclosed subspace of $L_2[0,1]$ because $L_p[0,1]$ cannot be the range of any bounded operator on $L_2[0,1]$. Since $L_p[0,1]$ is not isomorphic to a Hilbert space, $L_p[0,1]$ is not the range of any bounded operator on $L_2[0,1]$ by the following result.*

Theorem 2.1.5. *(Murray (1937)) If A is a continuous linear injection of a Banach space X in a Hilbert space $\mathcal{H}$, and if $R(A)$ is the range of a bounded operator on $\mathcal{H}$, then X is isomorphic to a Hilbert space.*

As every semiclosed subspace in a Hilbert space $\mathcal{H}$ is the range of some bounded operator in $\mathcal{H}$, every semiclosed subspace is necessarily a Borel set, in fact, an F_σ-set and every proper semicloscentral university tiruvarured subspace is necessarily of first category by an application of the open mapping theorem ; these conditions are not sufficient, we refer to Fillmore and Williams (1971). The following propositions give necessary conditions for a subspace of $\mathcal{H}$ to be semiclosed.

Proposition 2.1.6. *All semiclosed subspaces are F_σ-sets.*

Proof. Let M be a semiclosed subspace of $\mathcal{H}$. By theorem 2.1.2, $M = R(A)$ for some $A \in \mathcal{B}(\mathcal{H})$. Let $\mathbb{B}$ be a closed unit ball in $\mathcal{H}$. Then $\mathbb{B}$ is weakly compact and convex, so that

$A(\mathbb{B})$ is weakly closed and convex and is therefore a norm closed. If $\mathbb{B}_n = \{x : \|x\| \leq n\}$, then $\mathcal{H} = \bigcup_{n=0}^{\infty} \mathbb{B}_n$ and consequently

$$M = R(A) = \bigcup_{n=0}^{\infty} A(\mathbb{B}_n).$$

Therefore M is the countable union of closed sets. Hence M is a F_σ-set. □

Proposition 2.1.7. *All proper semiclosed subspaces are of first category.*

Proof. Let M be a proper semiclosed subspace of $\mathcal{H}$. Equivalently $M = R(A)$ for some $A \in \mathcal{B}(\mathcal{H})$ and $R(A) \neq \mathcal{H}$. By an application of the open mapping theorem, $R(A)$ is either of first category or A is onto. Hence M is of first category. □

Theorem 2.1.8. *(Douglas (1966)) Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{G}$ be Hilbert spaces and $A \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{G})$. Then the following conditions are equivalent:*

1. *The equation $AX = B$ has a solution in $\mathcal{B}(\mathcal{K}, \mathcal{H})$.*

2. *$R(B) \subseteq R(A)$.*

3. *A^* majorizes B^*, that is, there exists $k > 0$ such that*

$$\|B^*x\| \ \leq \ k\|A^*x\| \text{for all} x \in \mathcal{H},$$

in which case the least such k is $\|A^{-1}B\|$. Moreover, there is a positive number m such that $\|A^{-1}y\| \leq \ m\|B^{-1}y\|$ for all $y \in R(B)$.

The Douglas theorem 2.1.8 is fundamental for the discussion of semiclosed subspaces, by making use of the theory of bounded operators.

2.2 Lattice structure

The collection of all subspaces of a Hilbert space forms a lattice with respect to the operations "vector sum" and "intersection". The sum of two orthogonal subspaces of a Hilbert space $\mathcal{H}$ is closed. Orthogonality may be a strong assumption, but it is sufficient to ensure the conclusion. If M is a finite dimensional subspace and N is a closed subspace of $\mathcal{H}$, then the vector sum $M+N$ is necessarily closed. If no additional assumptions are made, then the sum of closed subspaces of a Hilbert space need not be closed in general (Ben-Israel and Greville, 2003, page 331). Therefore the set of all closed subspaces may not form a lattice (with respect to vector sum and intersection).

In this section we show that the set $SCS(\mathcal{H})$ of semiclosed subspaces of a Hilbert space $\mathcal{H}$, forms a complete lattice with the operations "join" and "meet" defined as usual to be "vector sum" and "intersection" respectively. Moreover, $SCS(\mathcal{H})$ is the smallest lattice containing all closed subspaces of $\mathcal{H}$.

Proposition 2.2.1. *The intersection of two semiclosed subspaces of $\mathcal{H}$ is again a semiclosed subspace.*

Proof. Let M_1 and M_2 be semiclosed subspaces of $\mathcal{H}$. Then there are two Hilbert inner products $\langle .,. \rangle_1$ and $\langle .,. \rangle_2$ such that each $(M_i, \langle .,. \rangle_i)$ $(i = 1, 2)$ is continuously embedded in $\mathcal{H}$. Define $\langle .,. \rangle_*$ on $M_1 \cap M_2$ by

$$\langle x, y \rangle_* = \langle x, y \rangle_1 + \langle x, y \rangle_2.$$

Then $\langle .,. \rangle_*$ is a Hilbert inner product and is stronger than the usual inner product on $\mathcal{H}$. Hence $M_1 \cap M_2$ is a semiclosed subspace. $\square$

Let A be a bounded operator in $\mathcal{H}$. The range of A can be given uniquely a Hilbert space structure, with norm $\|.\|_A$ as follows. Moreover, A becomes a coisometry from $\mathcal{H}$ to $(R(A), \|.\|_A)$. In fact, since A gives rise to a bijection from $N(A)^\perp = cl(R(A^*))$ is a Hilbert space, the inner product $\langle .,. \rangle_A$ on $R(A)$, defined by

$$\langle Aa, Ab \rangle_A = \langle Pa, Pb \rangle \text{ for } a, b \in \mathcal{H},$$

where P is the orthogonal projection to $N(A)^\perp$, makes $R(A)$ a Hilbert space and the uniqueness is obvious. Since $Aa = APa$ and $\|Pa\| \leq \|a\|$, norm $\|u\|_A$ admits the description

$$\|u\|_A = \min\{\|a\| : Aa = u\} \text{ for } u \in R(A)$$

and the following inequality holds

$$\|u\| \leq \|A\| \ \|u\|_A \text{ for } u \in R(A).$$

Hence for each $u \in R(A)$ there is uniquely $a \in cl(R(A^*))$ such that

$$Aa = u \text{ and } \|a\| = \|u\|_A.$$

The space $R(A)$ equipped with the Hilbert space structure $\|.\|_A$ is denoted by $\mathcal{M}(A)$:

$$\mathcal{M}(A) \equiv (R(A), \|.\|_A).$$

$\mathcal{M}(A)$ is called **de Branges space** induced by A.

Conversely, suppose that a subspace M of a Hilbert space $\mathcal{H}$ is equipped with a Hilbert space structure $\|.\|_*$ such that $(M, \langle ., . \rangle_*)$ is continuously embedded in $\mathcal{H}$. Then there is uniquely a positive operator A on $\mathcal{H}$ such that $(M, \|.\|_*) = \mathcal{M}(A)$.

Theorem 2.2.2. *(Ando (1990)) For $A_1, A_2 \in \mathcal{B}(\mathcal{H})$, let $A = (A_1A_1^* + A_2A_2^*)^{1/2}$. Then $\|u_1 + u_2\|_A^2 \leq \|u_1\|_{A_1}^2 + \|u_2\|_{A_2}^2$, for $u_1 \in R(A_1), u_2 \in R(A_2)$, and for any $u \in R(A)$, there are uniquely $u_1 \in R(A_1), u_2 \in R(A_2)$ such that $u = u_1 + u_2$ and*

$$\|u_1 + u_2\|_A^2 = \|u_1\|_{A_1}^2 + \|u_2\|_{A_2}^2.$$

Proposition 2.2.3. *The sum of two semiclosed subspaces of $\mathcal{H}$ is again a semiclosed subspace.*

Proof. Let M_1 and M_2 be semiclosed subspaces of $\mathcal{H}$. Then there are positive operators A_1, A_2 in $\mathcal{H}$ such that

$$(M_1, \|.\|_1) = \mathcal{M}(A_1) \text{ and } (M_2, \|.\|_2) = \mathcal{M}(A_2).$$

Moreover, there are positive numbers k_1 and k_2 such that

$$\|u_1\| \leq k_1\|u_1\|_1 \text{ and } \|u_2\| \leq k_2\|u_2\|_2 \text{ for all } u_1 \in M_1, u_2 \in M_2.$$

Let $A = (A_1A_1^* + A_2A_2^*)^{1/2}$. Then by theorem 2.2.2, for any $u \in R(A)$, there are uniquely $u_1 \in R(A_1), u_2 \in R(A_2)$ such that $u = u_1 + u_2$ and

$$\|u\|_A^2 = \|u_1 + u_2\|_A^2 = \|u_1\|_1^2 + \|u_2\|_2^2.$$

Hence $\|.\|_A$ is a Hilbert inner product on $M_1 + M_2$ such that $(M_1 + M_2, \langle ., . \rangle_A)$ is continuously embedded in $(\mathcal{H}, \langle ., . \rangle)$ because for any $u \in M_1 + M_2$,

$$\|u\|^2 \leq \|u_1\|^2 + \|u_2\|^2 \leq k^2(\|u_1\|_1^2 + \|u_2\|_2^2) = k^2\|u\|_A^2,$$

where $k = \max\{k_1, k_2\}$. Thus $M_1 + M_2$ is a semiclosed subspace of $\mathcal{H}$. □

2.3 Semiclosed subspaces and positive operators

By an application of Riesz representation theorem for Hilbert spaces, a substantially simpler proof is given than those in MacNerney (1959) which reveals that semiclosed subspaces can alone be characterized by the ranges of members of $\mathcal{B}(\mathcal{H})^+$. For a fixed inner product $\langle ., . \rangle_*$ on M, some properties for the operator A corresponding to inner product $\langle ., . \rangle_*$ are discussed below. Unless and otherwise specified, $\mathcal{H}$ and M have Hilbert inner products $\langle ., . \rangle$ and $\langle ., . \rangle_*$ respectively. We denote by $\mathbb{C}$ the set of complex numbers and the conjugate of $\langle x, y \rangle$ by $\overline{\langle x, y \rangle}$.

Theorem 2.3.1. *Let M be a semiclosed subspace of a Hilbert space $\mathcal{H}$. For each Hilbert inner product $\langle ., . \rangle_*$ on M such that $(M, \langle ., . \rangle_*)$ is continuously embedded in $\mathcal{H}$, there is a unique $A \in \mathcal{B}(\mathcal{H})^+$ such that*

$$\langle x, y \rangle = \langle x, Ay \rangle_* \quad \text{for all } x \in M, y \in \mathcal{H}.$$

Moreover, if the square root $A^{1/2}$ of A is considered, then $R(A^{1/2}) = M$ and

$$\langle x, y \rangle_* = \langle A^{-1/2}x, A^{-1/2}y \rangle \quad \text{for all } x, y \in M.$$

Proof. Given that $(M, \langle ., . \rangle_*)$ is a Hilbert space and there exists $k > 0$ such that

$$\langle x, x \rangle \leq k\langle x, x \rangle_* \text{ for all } x \in M.$$

Let $y \in \mathcal{H}$. Define $f_y : \mathcal{H} \to \mathbb{C}$ by $f_y(x) = \langle x, y \rangle$. The restriction of f_y to M is bounded on $(M, \langle ., . \rangle_*)$ because for $x \in M$,

$$|f_y(x)| \leq \|x\| \ \|y\| \leq k\|x\|_* \ \|y\|.$$

By Riesz representation theorem for Hilbert spaces, there exists a unique $z \in M$ so that

$$f_y(x) = \langle x, z \rangle_* \text{ for all } x \in M.$$

Define $A : (\mathcal{H}, \langle ., . \rangle) \to (M, \langle ., . \rangle_*)$ by $Ay = z$. Then $\langle x, y \rangle = \langle x, Ay \rangle_*$ for all $x \in M, y \in \mathcal{H}$. Clearly $A(\mathcal{H}) \subset M$ and the uniqueness of A follows from the Riesz representation theorem. For each $x \in \mathcal{H}$,

$$\begin{aligned}
\|Ax\| &\leq k\|Ax\|_* \ \ (\because \ \langle ., . \rangle_* \text{ is stronger than} \langle ., . \rangle) \\
&= k \sup_{z \in M} \{|\langle z, Ax \rangle_*| : \|z\|_* = 1\} (\text{by Hahn Banach Theorem}) \\
&= k \sup_{z \in M} \{|\langle z, x \rangle| : \|z\|_* = 1\} \\
&\leq k\|z\|\|x\| \\
&\leq kc\|z\|_*\|x\| = K\|x\|, where K = kc.
\end{aligned}$$

hence A is bounded. From the relation

$$\langle Ax, y \rangle = \langle Ax, Ay \rangle_* = \overline{\langle Ay, Ax \rangle_*} = \overline{\langle Ay, x \rangle} = \langle x, Ay \rangle,$$

we get $\langle x, Ay \rangle = \langle Ax, y \rangle$ for all $x, y \in \mathcal{H}$. The positiveness of the operator A comes from $\langle Ax, x \rangle = \langle Ax, Ax \rangle_* = \|Ax\|_*^2 \geq 0$ for all $x \in \mathcal{H}$. □

Proposition 2.3.2. *Let M be a semiclosed subspace of $\mathcal{H}$ and A_M be the restriction of A to M with the Hilbert norm $\|.\|_*$. Then $A_M : M \to M$ is a bounded positive self-adjoint operator on M.*

Proof. The boundedness of A_M comes from the following:

$$\begin{aligned}
\|A_M x\|_* &= \sup\{|\langle z, A_M x \rangle_*| : \|z\|_* = 1\} \\
&= \sup\{|\langle z, x \rangle| : \|z\|_* = 1\} \leq \|x\| \leq k\|x\|_*.
\end{aligned}$$

For each $x, y \in M$,

$$\langle A_M x, y\rangle_* = \overline{\langle y, A_M x\rangle_*} = \overline{\langle y, x\rangle} = \langle x, y\rangle = \langle x, A_M y\rangle_*$$

hence we get $\langle x, A_M y\rangle_* = \langle A_M x, y\rangle_*$ for all $x, y \in M$. A_M is positive because for $x \in M$, $\langle A_M x, x\rangle_* = \|x\|^2 \geq 0$. □

In addition to the notational conventions mentioned in the previous chapter, we denote the positive square roots of A and A_M by $A^{1/2}$ and $A_M^{1/2}$ respectively. To differentiate the convergence of a sequence in M with respect to the new norm $\|.\|_*$, we denote $\lim_* x_n = x$ for the sequence $\{x_n\}$ in M converging to x with respect to the norm $\|.\|_*$. The closure of a subset N of M corresponding to the norm $\|.\|_*$ is denoted by $\mathrm{cl}_*(N)$ whereas $\mathrm{cl}(N)$ denotes the (strong) closure of N with respect to the usual inner product on $\mathcal{H}$.

Proposition 2.3.3. *For $x \in M$, $\|x\| = \|A_M^{1/2} x\|_*$ and $A^{1/2}$ agrees with $A_M^{1/2}$ on M.*

Proof. Let $x \in M$. Then $\|x\|^2 = \langle x, A_M x\rangle_* = \langle A_M^{1/2} x, A_M^{1/2} x\rangle_* = \|A_M^{1/2} x\|_*^2$. We next claim that $A^{1/2}$ and $A_M^{1/2}$ are same at every point of M. If $x \in M$, then $A_M^{1/2} x = \lim_* A_n x$. As $\|.\|_*$ is stronger than $\|.\|$, we get $\lim A_n x = A_M^{1/2} x$, hence $A^{1/2} x = A_M^{1/2} x$, where $\{A_n\}_{n\geq 1}$ are as used in the theorem 1.2.24. □

Theorem 2.3.4. *Let M be a semiclosed subspace of $\mathcal{H}$ and A be the operator corresponding to the inner product $\langle .,.\rangle_*$. Then A from $\mathcal{H}$ to the Hilbert space $(M, \langle .,.\rangle_*)$ is compact then A from $\mathcal{H}$ to $\mathcal{H}$ is compact.*

Proof. Suppose $A : \mathcal{H} \to M$ is compact. Then for a bounded sequence $\{x_n\}$ in $\mathcal{H}$, $\{Ax_n\}$ has a convergent subsequence in M. Since the norm $\|.\|_*$ is stronger than the usual norm $\|.\|$ on $\mathcal{H}$, $\{Ax_n\}$ has a convergent subsequence in $\mathcal{H}$.

□

2.4 Metric between semiclosed subspaces

Let M be a semiclosed subspace of a Hilbert space $\mathcal{H}$. Then there exists an inner product $\langle ., . \rangle_*$ on M such that $(M, \langle ., . \rangle_*)$ is Hilbert and there exists $k > 0$ with

$$\langle x, x \rangle \leq k \ \langle x, x \rangle_* \quad \text{for all } x \in M.$$

For each such inner product $\langle ., . \rangle_*$, there corresponds a topology on M. Interestingly all such inner products are equivalent which is shown by the following theorem. Hence the topology on M induced by the inner product $\langle ., . \rangle_*$, as in the definition, is unique. This topology coincides with the relative topology inherited from $\mathcal{H}$ if the subspace M is closed in $\mathcal{H}$.

Theorem 2.4.1. *Let M be a semiclosed subspace of $\mathcal{H}$. Then all inner products $\langle ., . \rangle_*$ as in the definition 2.1.1 generate the same topology on M.*

Proof. Suppose M has two such Hilbert inner products $\langle ., . \rangle_1$ and $\langle ., . \rangle_2$ respectively. Then by theorem 2.3.1, there are $A, B \in \mathcal{B}(\mathcal{H})^+$ such that

$$\langle x, y \rangle_1 = \langle A^{-1/2}x, A^{-1/2}y \rangle \text{ and } \langle x, y \rangle_2 = \langle B^{-1/2}x, B^{-1/2}y \rangle \text{ for all } x, y \in M$$

and $R(A^{1/2}) = R(B^{1/2}) = M$. As $R(B^{1/2}) \subset R(A^{1/2})$, by theorem 2.1.8 there exists $m > 0$ such that $\|A^{-1/2}y\| \leq m\|B^{-1/2}y\|$ for all $y \in M$. This implies

$$\|y\|_1 \leq m\|y\|_2 \text{ for all } y \in M.$$

In a similar way, we can show that $\|.\|_1$ is stronger than $\|.\|_2$. Hence the two norms are equivalent. Thus each semiclosed subspace of $\mathcal{H}$ has a unique topology. □

Definition 2.4.2. *Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle ., . \rangle$ and let $A_1, A_2 \in \mathcal{B}(\mathcal{H})^+$. We say that A_1 and A_2 are **operator-inner product equivalent** if the inner products $\langle ., . \rangle_1$ and $\langle ., . \rangle_2$ generated from A_1 and A_2 respectively are equivalent, where the inner product $\langle ., . \rangle_i$ $(i = 1, 2)$ is defined by $\langle x, y \rangle_i = \langle x, A_i y \rangle$ for $x, y \in \mathcal{H}$.*

The "operator-inner product equivalent" is an equivalence relation which partitions the set $\mathcal{B}(\mathcal{H})^+$. The operators in an equivalence class correspond to all equivalent inner products asssociated with a semiclosed subspace of $\mathcal{H}$. Hence each semiclosed subspace can be associated with an equivalence class. We denote the equivalences classes by operators. However, in practice there is little risk of error by thinking of elements in the set of equivalence classes as operators rather than equivalence classes of operators.

Proposition 2.4.3. *Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle .,. \rangle$ and let $A_1, A_2 \in \mathcal{B}(\mathcal{H})^+$. Then A_1 and A_2 are operator-inner product equivalent iff there are positive numbers a and b such that $aA_1 \leq A_2 \leq bA_1$.*

Proof. Let A_1 and A_2 be operator-inner product equivalent. Then the inner products

$$\langle x, y \rangle_1 = \langle x, A_1 y \rangle \text{ and } \langle x, y \rangle_2 = \langle x, A_2 y \rangle, \text{ for all } x, y \in \mathcal{H}$$

are equivalent. Therefore there exist positive numbers a and b such that

$$a\langle x, x \rangle_1 \leq \langle x, x \rangle_2 \leq b\langle x, x \rangle_1, \text{ for all } x \in \mathcal{H}.$$

This implies

$$a\langle A_1 x, x \rangle \leq \langle A_2 x, x \rangle \leq b\langle A_1 x, x \rangle, \text{ for all } x \in \mathcal{H}.$$

Hence $aA_1 \leq A_2 \leq bA_1$.

Suppose $aA_1 \leq A_2 \leq bA_1$ for some $a > 0$ and $b > 0$. Then

$$\langle aA_1 x, x \rangle \leq \langle A_2 x, x \rangle \leq \langle bA_1 x, x \rangle, \text{ for all } x \in \mathcal{H}.$$

This implies

$$a\langle x, x \rangle_1 \leq \langle x, x \rangle_2 \leq b\langle x, x \rangle_1, \text{ for all } x \in \mathcal{H}.$$

Hence A_1 and A_2 are operator-inner product equivalent. □

For any semiclosed subspace M, we choose a Hilbert inner product $\langle .,. \rangle_*$ from the set of all Hilbert inner products on M, and let α be a choice function to choose a Hilbert inner product for each semiclosed subspace. We shall now define a metric for semiclosed subspaces of a Hilbert space $\mathcal{H}$. Let M_1 and M_2 be semiclosed subspaces of $\mathcal{H}$. Then by the correspondence α, there exist unique positive bounded operators A_1 and A_2 which correspond to M_1 and M_2 respectively. The real valued function d_α defined by

$$d_\alpha(M_1, M_2) = \|A_1 - A_2\|$$

is a metric on $SCS(\mathcal{H})$. Especially, it coincides with a gap metric (Kato, 1976, page 197) on the set of all closed subspaces. Indeed, if M is a closed subspace of $\mathcal{H}$, then $(M, \|.\|)$ is isometrically isomorphic to de Branges space $\mathcal{M}(P_M)$ onto M. Because

$$\|u\|_{P_M}^2 = \|u\|^2 \text{ for all } u \in M = N(P_M)^\perp,$$

the metric between closed subspaces M_1 and M_2 is given by $d_\alpha(M_1, M_2) = \|P_{M_1} - P_{M_2}\|$ which stands for the gap metric.

Theorem 2.4.4. *Let α and β be distinct choice functions to choose a Hilbert inner product for each semiclosed subspace of $\mathcal{H}$. The metrics d_α and d_β as defined above are equivalent.*

2.5 Conclusion

Semiclosed subspaces of Hilbert spaces have been considered for a long time, as a more flexible substitute of closed subspaces of Hilbert spaces. The ranges of members of the set of bounded positive self-adjoint operators $\mathcal{B}(\mathcal{H})^+$ can alone characterize all semiclosed subspaces of $\mathcal{H}$. The collection of semiclosed subspaces in $\mathcal{H}$ is in bijective correspondence with $\mathcal{B}(\mathcal{H})^+$. This feature is put into a more concrete perspective using the connection between range inclusions, majorizations and factorizations of bounded operators. A metric through the bounded positive self-adjoint operators is defined in the class of semiclosed subspaces of $\mathcal{H}$.

Chapter 3

Dense and Invariant Semiclosed Subspaces

Every proper dense subspace of a Hilbert space is never closed in the strong topology. But semiclosed subspaces possess many special features that distinguish them from arbitrary subspaces and they can be considered as a more flexible substitute of closed subspaces of Hilbert spaces. Some properties of operators associated with proper dense semiclosed subspaces are discussed and some results on invariant semiclosed subspaces are proved in the chapter.

For example, the subspace

$$M = \left\{ (x_1, x_2, \ldots) : \sum_{n=1}^{\infty} (n|x_n|)^2 < \infty \right\}$$

of ℓ_2 is a proper dense semiclosed subspace of ℓ_2.

3.1 Properties of semiclosed subspaces

We first discuss some properties of semiclosed subspaces of a Hilbert space $\mathcal{H}$. For a semiclosed subspace M of $\mathcal{H}$, by theorem 2.1.2 there corresponds $A \in \mathcal{C}(\mathcal{H})$ with $M = D(A)$. A well-known procedure of making a closed operator to a bounded operator by renorming its domain with the graph norm, is helpful to define an inner product on M

by

$$\langle x, y\rangle_* = \langle Ax, Ay\rangle + \langle x, y\rangle, \text{ for } x, y \in D(A).$$

Then $(M, \langle ., .\rangle_*)$ is a Hilbert space and the inner product $\langle ., .\rangle_*$ is stronger than the usual inner product. By considering the Hilbert space $(M, \langle ., .\rangle_*)$, we have a relation between the semiclosed subspaces of M and the semiclosed subspaces of $\mathcal{H}$.

Proposition 3.1.1. *If L is a semiclosed subspace of M and M is a semiclosed subspace of N, then L is a semiclosed subspace of N.*

Proof. Since M is a semiclosed subspace of N, there exists $\langle ., .\rangle_M$ on M such that $(M, \langle ., .\rangle_M)$ is complete and there exists $b > 0$,

$$\langle x, x\rangle_N \leq b\langle x, x\rangle_M \text{ for all } x \in M.$$

If L is a semiclosed subspace of $(M, \langle ., .\rangle_M)$, then there exists $\langle ., .\rangle_L$ on L such that $(L, \langle ., .\rangle_L)$ is complete and there exists $c > 0$ such that

$$\langle x, x\rangle_M \leq c\langle x, x\rangle_L \text{ for all } x \in L.$$

Now for any $x \in L$, we have $\langle x, x\rangle_N \leq b\langle x, x\rangle_M \leq bc\langle x, x\rangle_L$ for all $x \in L$. Also $(L, \langle ., .\rangle_L)$ is complete. Hence L is a semiclosed subspace of N. □

3.2 Dense semiclosed subspaces

Theorem 3.2.1. *Let M be a dense semiclosed subspace of a Hilbert space $\mathcal{H}$ and A be the operator corresponding to the inner product $\langle ., .\rangle_*$. Then A has dense range in $\mathcal{H}$ and A is injective.*

Proof. Suppose $y_0 \in M$ such that $\langle Ax, y_0\rangle_* = 0$ for each $x \in \mathcal{H}$. Then

$$0 = \langle Ay_0, y_0\rangle_* = \langle y_0, y_0\rangle,$$

so $y_0 = 0$, hence $R(A)$ is dense in M with respect to $\langle ., .\rangle_*$. As $M = \text{cl}_*(R(A)) \subset \text{cl}(R(A))$ and M is dense in $\mathcal{H}$, we get

$$\mathcal{H} = \text{cl}(M) \subset \text{cl}(R(A)) \subset \mathcal{H},$$

hence $R(A)$ is dense in $\mathcal{H}$.

Suppose that for some $x \in \mathcal{H}$ with $Ax = 0$. Then for each $y \in \mathcal{H}$,

$$\langle x, Ay \rangle = \langle Ax, y \rangle = 0,$$

so x is in the orthogonal complement of $R(A)$. The denseness of $R(A)$ in $\mathcal{H}$ gives that $x = 0$. Hence A is injective. $\square$

Theorem 3.2.2. *Let M be a dense semiclosed subspace of a Hilbert space $\mathcal{H}$ and A be the operator corresponding to the inner product $\langle ., . \rangle_*$. Then $R(A^{1/2}) = M$.*

Proof. Suppose $x \in \mathcal{H}$. Since M is dense in $\mathcal{H}$, there exists a sequence $\{x_n\}$ in M such that $\lim x_n = x$. By proposition 2.3.3, for each n, $\|x_n\| = \|A_M^{1/2} x_n\|_*$. The boundedness of A gives that $A^{1/2}$ is bounded from $\mathcal{H}$ to $\mathcal{H}$. Then

$$\|x\| = \lim \|x_n\| = \lim \|A^{1/2} x_n\|_* = \|A^{1/2} x\|_*.$$

Thus if $x \in \mathcal{H}$, we get $\|x\| = \|A^{1/2} x\|_*$.

Suppose $\{A^{1/2} x_n\}$ converges in M to y. Since $\|A^{1/2} x_n\| = \|x_n\|$, we have that $\{x_n\}$ is Cauchy in $\mathcal{H}$. Let $\lim_* x_n = x$. Then $A^{1/2} x = y$ since

$$\|A^{1/2} x - y\| \leq \|A^{1/2} x - A^{1/2} x_n\| + \|A^{1/2} x_n - y\|$$

and given $\varepsilon > 0$, there exists an n so that the right side of the inequality is less than ε. Hence $A^{1/2} x = y$. From the theorem 3.2.1, A^{-1} exists.

Suppose $A^{1/2} x = 0$. Then $Ax = A^{1/2} A^{1/2} x = 0$ and so $x = 0$, hence $A^{1/2}$ is injective. To show the range of $A^{1/2}$ is dense in M. Suppose there exists $y \in M$ so that

$$\langle A^{1/2} x, y \rangle_* = 0$$

for all $x \in \mathcal{H}$. Then if we let $x = A^{1/2} y$,

$$\langle A^{1/2} A^{1/2} y, y \rangle_* = \langle A^{1/2} y, A^{1/2} y \rangle_* = \|A^{1/2} y\|_* = 0.$$

This implies that $A^{1/2} y = 0$, so $y = 0$. Thus $R(A^{1/2})$ is a dense and closed subspace of M, hence it is equal to M. $\square$

Lemma 3.2.3. *(MacNerney (1959)) If A is a positive self-adjoint bounded operator on a Hilbert space $\mathcal{H}$ and $z \in \mathcal{H}$, then $z \in A^{1/2}(\mathcal{H})$ iff there exists $b > 0$ such that*

$$|\langle x, z\rangle|^2 \leq b\langle x, Ax\rangle$$

for each $x \in \mathcal{H}$. Moreover, $\|A^{-1/2}z\|^2$ is the least of b.

Theorem 3.2.4. *Let M be a proper dense semiclosed subspace of $\mathcal{H}$. Then there exists a semiclosed subspace L of M such that L is dense in $\mathcal{H}$.*

Proof. The semiclosedness of M gives a Hilbert inner product $\langle ., .\rangle_*$ on M such that for some $k > 0$, $\langle x, x\rangle \leq k\langle x, x\rangle_*$ for each $x \in M$. Then by theorems 2.3.1, 3.2.1 and 3.2.2, there exists $B \in \mathcal{B}(\mathcal{H})^+$ such that

$$\langle x, y\rangle = \langle x, By\rangle_*$$

for all $x \in M, y \in \mathcal{H}$ and $M = R(B^{1/2})$. As M is proper in $\mathcal{H}$, choose u_0 in $\mathcal{H}$ not in M such that $\|u_0\| = 1$.

Let P be the orthogonal projection onto span$\{u_0\}$ from $\mathcal{H}$ defined by

$$P(x) = x - \langle x, u_0\rangle u_0.$$

Let $A = B^{1/2}PB^{1/2}$, $L = R(A^{1/2})$ and $\langle ., .\rangle_1$ be the inner product on L defined by

$$\langle A^{1/2}x, A^{1/2}y\rangle_1 = \langle x, y\rangle$$

for each $x, y \in \mathcal{H}$. Clearly $\langle ., .\rangle_1$ is a Hilbert inner product on L. For each $x \in \mathcal{H}$,

$$\begin{aligned} |\langle x, z\rangle|^2 &= \|z\|_1^2 \langle x, Ax\rangle \\ &\leq \|z\|_1^2 \langle x, Bx\rangle \end{aligned}$$

so by lemma 3.2.3 $\|z\| \leq \|z\|_1$, which proves that $(L, \langle ., .\rangle_1)$ is a semiclosed subspace of $(M, \langle ., .\rangle_*)$. We next claim that L is dense in $\mathcal{H}$. Let z belong to $\mathcal{H}$ such that

$$\langle y, A^{1/2}z\rangle = 0, \text{ for all } z \in \mathcal{H}.$$

Then for each $x \in \mathcal{H}$, $\langle PB^{1/2}y, x\rangle = \langle y, B^{1/2}Px\rangle = 0$ so that $PB^{1/2}y = 0$, hence

$$B^{1/2} = \langle B^{1/2}y, u_0\rangle u_0.$$

As u_0 is not in M, $B^{1/2}y = 0$, so that $y = 0$. Thus L is dense in $\mathcal{H}$.

$\square$

Theorem 3.2.5. *Let M be a proper dense semiclosed subspace of $\mathcal{H}$. Then there exists a proper subspace N of $\mathcal{H}$ such that M is a semiclosed subspace of N.*

Proof. The proper semiclosedness of M gives the operator A and the element u_0 as in the proof of the above theorem. Define $B : \mathcal{H} \to \mathcal{H}$ by

$$Bx = Ax + \langle x, u_0\rangle u_0, \ x \in \mathcal{H}.$$

Clearly $B \in \mathcal{B}(\mathcal{H})^+$ injective and therefore $B^{-1/2}$ exists.
Let $N = R(B^{1/2})$ and $\langle ., .\rangle_2$ be the inner product on N defined by

$$\langle B^{1/2}x, B^{1/2}y\rangle_2 = \langle x, y\rangle$$

for each $x, y \in \mathcal{H}$. Clearly $\langle ., .\rangle_2$ is a Hilbert inner product on N. Let $z \in M$. For each $x \in \mathcal{H}$,

$$\begin{aligned} |\langle x, z\rangle|^2 &= |\langle Ax, z\rangle_*|^2 \leq \|Ax\|_*^2 \ \|z\|_*^2 \\ &\leq \|z\|_*^2 \ \langle x, Ax\rangle \leq \|z\|_*^2 \ \langle x, Bx\rangle \end{aligned}$$

so by lemma 3.2.3 $\|z\|_2 \leq \|z\|_*$. Let $z \in N$. Then for each $x \in \mathcal{H}$,

$$\begin{aligned} |\langle x, z\rangle|^2 &= |\langle B^{1/2}x, B^{1/2}z\rangle_2|^2 \\ &= |\langle Bx, z\rangle_2|^2 \leq \|z\|_2^2 \ \|Bx\|_2^2 \leq \|z\|_2^2 \ \langle x, Bx\rangle \\ &\leq \|z\|_2^2 \ (\langle x, Ax\rangle + \langle x, x\rangle) \\ &= \|z\|_2^2 \ (k+1)\langle x, x\rangle \end{aligned}$$

so $\|z\| \leq (k+1)^{1/2}\|z\|_2$. We proved that $(M, \langle ., .\rangle_*)$ is a semiclosed subspace of $(N, \langle ., .\rangle_2)$ which is a semiclosed subspace of $\mathcal{H}$.

We next claim that N is a proper subspace of $\mathcal{H}$. Suppose $B^{1/2}(\mathcal{H}) = \mathcal{H}$. Then $\mathcal{B}(\mathcal{H}) = \mathcal{H}$. Therefore there exists $x_0 \in \mathcal{H}$ such that $Bx_0 = u_0$. Then

$$A^{1/2}(A^{1/2})x_0 = Ax_0 = [1 - \langle x_0, u_0 \rangle]u_0.$$

Since $u_0 \notin M = R(A^{1/2})$, $A^{1/2}x_0 = 0$ so that $x_0 = 0$ implies that $u_0 = 0$, which is a contradiction to $u_0 \neq 0$, hence $B^{1/2}(\mathcal{H}) \neq \mathcal{H}$. Note that $A \leq B$ because $\langle x, Ax \rangle \leq \langle x, Bx \rangle$ for all $x \in \mathcal{H}$. □

Corollary 3.2.6. *Let L and N be semiclosed subspaces of a Hilbert space $\mathcal{H}$ such that the dimension of the quotient space N/L is at least one. If L is not a closed subspace of $\mathcal{H}$, then there exists a semiclosed subspace M of $\mathcal{H}$ such that $L \subsetneq M \subsetneq N$.*

Proof. Suppose L is not dense in N. We are done – take M to be the closure of L. Suppose L is dense in N. Then by theorem 3.2.4, there exists a proper subspace M of $\mathcal{H}$ such that M is a semiclosed subspace of N. As N is a semiclosed subspace of $\mathcal{H}$, by proposition 3.1.1, M is a semiclosed subspace of $\mathcal{H}$ such that $L \subsetneq M \subsetneq N$. □

Corollary 3.2.7. *If M is a non-closed semiclosed subspace of $\mathcal{H}$, then*

$$dim(cl(M)/M) = \infty.$$

Corollary 3.2.8. *Let M be a semiclosed subspace of a Hilbert space $\mathcal{H}$ and m, n be any natural numbers. Then there are semiclosed subspaces L and N of $\mathcal{H}$ such that L is a semiclosed subspace of M such that L is dense in $\mathcal{H}$ with $dim(L^{\perp_M}) = m$, and M is a semiclosed subspace of N with $dim(M^{\perp_N}) = n$. Here $L^{\perp_M}$ denotes the orthogonal complement of L in the Hilbert space M.*

3.3 Invariant semiclosed subspaces

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space. Motivated by the invariant subspace problem, we derive some results for invariant semiclosed subspaces under bounded operators in $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$, we denote by $ISC(T)$ the set of all semiclosed subspaces M that are T-invariant: $Tx \in M$ for every $x \in M$.

Proposition 3.3.1. *The set $ISC(\mathcal{T})$ is a lattice with respect to "vector sum" and "intersection".*

Proof. The proof follows from the general fact that intersection and vector sum of two semiclosed subspaces are again semiclosed subspaces. □

Lemma 3.3.2. *If $T \in \mathcal{B}(\mathcal{H})$, and if M is a T-invariant semiclosed subspace, then T is bounded, as an operator on the Hilbert space M.*

Proof. By the closed graph theorem, we only have to check the graph of T is closed in the Hilbert space $M \times M$. Let a sequence $\{(x_n, Tx_n)\}$ in $M \times M$ converge to $(y, z) \in M \times M$. Then $\{x_n\}$ and $\{Tx_n\}$ converge to y and z respectively in M ; therefore $\{x_n\}$ and $\{Tx_n\}$ converge to y and z respectively in $\mathcal{H}$. Since $T \in \mathcal{B}(\mathcal{H})$, we must have $z = Ty$, which proves the closedness of the graph of T in $M \times M$. □

Theorem 3.3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and let M_1, M_2 be T-invariant semiclosed subspaces such that the dimension of the quoient linear space M_2/M_1 exceeds one. If M_1 is not closed in $\mathcal{H}$, then there exists $M \in ISC(T)$ with the property that $M_1 \subsetneq M \subsetneq M_2$.*

Proof. Suppose M_1 is not dense in $\mathcal{H}$. We can take M to be the closure of M_1. The proof is done. Suppose M_1 is dense in $\mathcal{H}$. Then by theorem 2.1.2, $M_1 = R(A)$ for some $A \in \mathcal{B}(\mathcal{H})$. Moreover, by lemma 3.3.2, T is bounded as an operator on the Hilbert space M_1. It is also bounded as an operator on the Hilbert space $\mathcal{H}$. Therefore, by Donoghue's theorem (Donoghue (1967)), the operator T maps $R(\phi(A))$ into intself for every Lowner function ϕ. Using a description of $R(A^\alpha)$, $0 < \alpha < 1$, in terms of the spectral decomposition of A, these semiclosed subspaces are properly contained in $\mathcal{H}$ and properly contain M_1. Thus, we obtain a continuum of required T-invariant semiclosed subspaces. □

3.4 Conclusion

A nice interesting theory is avaiable for a proper dense semiclosed subspace but not for proper dense closed subspaces of Hilbert spaces because every proper dense subspace of a Hilbert space is never closed in the strong topology. Some properties of operators associated with proper dense semiclosed subspaces are discussed through the thoery of bounded operators and few results on invariant semiclosed subspaces are proved in the end of the chapter.

Chapter 4

Semiclosed Operators

Closed operators in $\mathcal{H}$ are defined through closed subspaces of the product Hilbert space $\mathcal{H}\times\mathcal{H}$. In the second chapter, it is shown that semiclosed subspaces are the generalization of closed subspaces. Combining these two facts, the notion of closed operator also can be generalized to semiclosed operators as follows.

The chapter started with simple characterizations of semiclosed operator followed by some important properties of the same. It is shown that the family of semiclosed operators in $\mathcal{H}$ is closed under sum, product and inversion. Moreover they behave nicely with respect to restriction and strong limits on semiclosed subspaces of $\mathcal{H}$. Semiclosed operators also can be defined using its graph in terms of semiclosed subspaces. The results presented in this chapter are from Kaufman (1979) article. Moreover it is shown that a bounded below linear operator with range as a semiclosed subspace is a semiclosed operator.

4.1 Simple characterizations

Definition 4.1.1. *(Kaufman (1979)) An operator T on a Hilbert space $(\mathcal{H}, \langle ., . \rangle)$ is said to be **semiclosed** if its graph*

$$G(T) = \{(x, Tx) : x \in D(T)\}$$

is a semiclosed subspace of the Hilbert space $\mathcal{H} \times \mathcal{H}$.

As closed subspaces are semiclosed subspaces, all closed operators are semiclosed operators.

Example 4.1.2. *Let $\mathcal{H} = L_2[0,1]$ and $e_n = e^{in\theta}$, $n \in \mathbb{Z}$. Consider $Y = [e_n]$, $n \geq 0$ and $Z = [e_{-n} + ne_n]$, $n \geq 1$ which are closed subspaces of $\mathcal{H}$. Let $M = Y + Z$. Then M is a semiclosed subspace which is not closed. Let P and Q are projections onto the subspaces Y and Z respectively. Clearly P and Q are closed operators but its sum is a semiclosed operator but not closed.*

One among the following characterizations of semiclosed operator T says that the domain $D(T)$ of T is a semiclosed subspace. We denote D_T, the domain of T with a Hilbert inner product $\langle ., . \rangle_T$ in the sense of considering semiclosedness of $D(T)$.

Theorem 4.1.3. *(Simple characterizations) Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be an operator on a Hilbert space $(\mathcal{H}, \langle ., . \rangle)$. Then the following are equivalent:*

1. *T is a semiclosed operator in $\mathcal{H}$.*
2. *$D(T)$ is a semiclosed subspace and $T \in \mathcal{B}(D_T, \mathcal{H})$.*
3. *There is a member B of $\mathcal{B}(\mathcal{H})$ such that $R(B) = D(T)$ and $TB \in \mathcal{B}(\mathcal{H})$*

Proof. $\underline{(1) \Rightarrow (2)}$: Suppose T is a semiclosed operator in $\mathcal{H}$. Let $\langle ., . \rangle_*$ be a Hilbert space inner product on $G(T)$ such that $(G(T), \langle ., . \rangle_*)$ is continuously embedded in $\mathcal{H} \times \mathcal{H}$. Define

$$\langle x, y \rangle_T = \langle (x, Tx), (y, Ty) \rangle_*, \text{ for } \ x, y \in D(T).$$

Clearly $\langle x, y \rangle_T$ is an inner product for $D(T)$ such that $(D(T), \langle x, y \rangle_T)$ is complete. Now the semiclosedness of $D(T)$ and $T \in \mathcal{B}(D_T, \mathcal{H})$ come from the following inequality.

$$\|x\|^2 + \|Tx\|^2 \leq b^2 \ \|x\|_T^2, \text{ for each } x \in D(T).$$

$\underline{(2) \Rightarrow (3)}$: Let $\langle ., . \rangle_T$ denote an inner product for the semiclosed subspace $D(T)$. By theorem 2.3.1, there exists $B \in \mathcal{B}(\mathcal{H})^+$ such that $R(B) = D(T)$ and

$$\langle x, y \rangle_T = \langle B^{-1}x, B^{-1}y \rangle \quad \text{ for all } x, y \in D(T).$$

Now $TB \in \mathcal{B}(\mathcal{H})$ follows from the fact that $B \in \mathcal{B}(\mathcal{H}, D_T)$ and $T \in \mathcal{B}(D_T, \mathcal{H})$.

(3) $\Rightarrow$ (1) : Suppose $D(T)$ is a semiclosed subspace and $T \in \mathcal{B}(D_T, \mathcal{H})$. Now define $\langle ., . \rangle_*$ on $G(T)$ as follows

$$\langle (x, Tx), (y, Ty) \rangle_* = \langle x, y \rangle_T \ + \ \langle Tx, Ty \rangle, \text{ for } x, y \in D_T.$$

Then $\langle (x, Tx), (x, Tx) \rangle \leq k \langle (x, Tx), (x, Tx) \rangle_*$ for some $k > 0$ and $(G(T), \langle ., . \rangle_*)$ is a complete space. Indeed, for any Cauchy sequence $\{(x_n, Tx_n)\}$ in $(G(T), \langle ., . \rangle_*)$, the sequences $\{x_n\}$ and $\{Tx_n\}$ are Cauchy sequences in D_T and $\mathcal{H}$ respectively. Completeness of D_T and $T \in \mathcal{B}(D_T, \mathcal{H})$ imply that $x_n \to x$ in D_T and $Tx_n \to Tx$ in $\mathcal{H}$, hence $\{(x_n, Tx_n)\}$ converges to (x, Tx) in $(G(T), \langle ., . \rangle_*)$. Therefore $G(T)$ is a semiclosed subspace of $\mathcal{H} \times \mathcal{H}$. $\square$

4.2 Properties of semiclosed operators

In the section, some well-known properties for closed operators generalize to semiclosed operators are discussed. It is known that the null space of "bounded and closed operators" are closed subspaces. But this is not true in case of semiclosed operator as we shall see in the section.

Theorem 4.2.1. *(Properties of semiclosed operators) Let T be a semiclosed operator in $\mathcal{H}$ with domain $D(T)$. Then the following statements hold:*

1. *The nullspace of T is a closed subspace of D_T.*

2. *Then $R(T)$ is a semiclosed subspace.*

3. *If $D(T)$ is a closed subspace, then T is a closed operator.*

4. *The restriction of T to a semiclosed subspace is also a semiclosed operator.*

5. *Let M be a semiclosed subspace of $\mathcal{H}$ such that $M \subset D(T)$. Then $T(M)$ is a semiclosed subspace of $\mathcal{H}$.*

6. *If T is one-to-one, then T^{-1} is a semiclosed operator.*

Proof. Let $\{x_n\}$ be a sequence in $N(T)$ such that $\{x_n\}$ converges x in D_T. By the theorem 4.1.3, we have $T \in \mathcal{B}(D_T, \mathcal{H})$. Therefore $Tx = \lim_{n\to\infty} Tx_n = 0$ implies $x \in N(T)$. Hence $N(T)$ is closed in D_T.

By theorem 4.1.3, there is member B of $\mathcal{B}(H)$ such that $R(B) = D(T)$ and $TB \in \mathcal{B}(\mathcal{H})$. Clearly $R(T) = R(TB)$. By theorem 2.1.2, $R(TB)$ is a semiclosed subspace. Hence $R(T)$ is also a semiclosed subspace.

Suppose $D(T)$ is a closed subspace of $\mathcal{H}$. Using theorem 4.1.3, we can show that $D(T)$ and D_T are isometrically isomorphic. Hence T is a bounded operator with closed domain. Thus T is a closed operator.

For the proof of $4, 5$ and 6 we refer Kaufman (1979). □

Remark 4.2.2. *It can be observed that the null space of a semiclosed operator need not be closed in $\mathcal{H}$. Also, the first statement in the above theorem says the null space of a semiclosed operator is always a semiclosed subspace.*

For any semiclosed operator T on a Hilbert space $\mathcal{H}$, by theorem 4.2.1 $R(T)$ is always a semiclosed subspace. The converse need not be true. That is, if T is an operator with $R(T)$ is a semiclosed subspace, then it need not be a semiclosed operator. From the following theorem, we can see that the converse is also true for a bounded below operator.

Theorem 4.2.3. *Let T be an operator in a Hilbert space $\mathcal{H}$ with domain $D(T)$. Suppose that $R(T)$ is a semiclosed subspace and there exists $m > 0$ such that*

$$\|Tx\| \geq m\|x\| \text{ for all } x \in D(T).$$

Then T is a semiclosed operator.

Proof. We first claim $D(T)$ is a semiclosed subspace. The space $R(T)$ can be given a Hilbert space structure, with the inner product $\langle ., . \rangle_*$ on $R(T)$ as in the definition of semiclosed subspace of $R(T)$. Define

$$\|x\|_T = \|Tx\|_* \text{ for all } x \in D(T).$$

The above norm on $D(T)$ is well defined because T is one-to-one. For $x \in D(T)$,

$$\|x\|_T \;=\; \|Tx\|_* \geq \; k\|Tx\| \geq \; k\; m\; \|x\|,$$

where k comes from the semiclosedness of $R(T)$. Now we prove that $(D(T), \|.\|_T)$ is complete. Let $\{x_n\}$ be a Cauchy sequence in $D_T = (D(T), \|.\|_T)$. By the definition of $\|.\|_T$, $\{Tx_n\}$ is also a Cauchy sequence in $(R(T), \|.\|_*)$. Since $(R(T), \|.\|_*)$ is complete, there exists $x_0 \in D(T)$ such that $\|Tx_n - Tx_0\|_*$ converges to 0 as n, m tend to infinity. Again by the definition of $\|.\|_T$, $\{x_n\}$ converges to x_0 in D_T. Therefore $D(T)$ is a semiclosed subspace. Using the following inequality

$$\|Tx\| \leq \; k\; \|Tx\|_* \;=\; k\; \|x\|_T \text{ for all } \; x \in D_T$$

we can conclude that $T \in \mathcal{B}(D_T, \mathcal{H})$. Thus by theorem 4.1.3, T is semiclosed. $\square$

The following theorem is crux of this chapter which differentiate the class of closed operators from the class of closed operators. It is proved that the set $SCO(\mathcal{H})$ of all semiclosed operators is closed under sum and product.

Theorem 4.2.4. *(Kaufman (1979)) Let T_1 and T_2 be two semiclosed operators in $\mathcal{H}$. Then each of $T_1 + T_2$ and $T_1 T_2$ is also a semiclosed operator.*

Proof. Let S_1 and S_2 denote the domains of T_1 and T_2 respectively. By a characterization of semiclosed operator, S_1 and S_2 are semiclosed subspaces. Therefore $S_1 \cap S_2$ is a semiclosed subspace of $\mathcal{H}$. Now the restriction of T_1 and T_2 to $S_1 \cap S_2$ is also a semiclosed operator. Hence each of T_1 and T_2 is in the linear space $\mathcal{B}(S_1 \cap S_2, \mathcal{H})$. Therefore $T_1 + T_2 \in \mathcal{B}(S_1 \cap S_2, \mathcal{H})$. Again by theorem 4.1.3, $T_1 + T_2$ is a semiclosed operator.

Let $S_3 = S_1 \cap T_2(S_2)$ which is a semiclosed subspace. By a property of semiclosed operator, $T_2^{-1}(S_3)$ is a semiclosed subspace of $\mathcal{H}$. Let T_3 denote the restriction of T_2 to $T_2^{-1}(S_3)$ which is also a semiclosed operator. Then $T_1\; T_2 = T_1\; T_3$ with $T_1 \in \mathcal{B}(S_1, \mathcal{H})$ and $T_3 \in \mathcal{B}(T_2^{-1}(S_3), S_1)$. It follows that $T_1\; T_2 \in \mathcal{B}(T_2^{-1}(S_3), S_1)$ and hence $T_1 T_2$ is also semiclosed. $\square$

There are characterizations for a linear operator to be a closed operator. Closed operators in a Hilbert space are characterized (shown below) as quotients AB^{-1}, where A and B are bounded operators in $\mathcal{H}$ such that the vector sum $R(A^*) + R(B^*)$ is closed.

Lemma 4.2.5. *(Kaufman (1978)) Suppose that $A, B \in \mathcal{B}(\mathcal{H})$, $A(B^{-1}B) = A$ and $T = AB^{-1}$. Then T is a closed operator in $\mathcal{H}$ only in case $A^*(\mathcal{H}) + B^*(\mathcal{H})$ is a closed subspace of $\mathcal{H}$.*

In the above mentioned characterization given by Kaufman (1978) for closed operators, if we relax the last condition that the vector sum $R(A^*) + R(B^*)$ is closed, we arrive the class of semiclosed operators.

Theorem 4.2.6. *(Kaufman (1978)) If $A, B \in \mathcal{B}(\mathcal{H})$ and A is invertible then AB^{-1} is a closed operator in $\mathcal{H}$.*

Theorem 4.2.7. *(Sequential characterization) Suppose that M is a semiclosed subspace of $\mathcal{H}$. Let $\{T_n\}$ be a sequence of semiclosed operators on $\mathcal{H}$ such that $M \subset D(T_n)$ for all n and $\{T_n\}$ converges strongly to T on M. Then T is a semiclosed operator in $\mathcal{H}$.*

Proof. Since each T_n is a semiclosed operator and $D(T_n)$ contains a semiclosed subspace M for all n, T_n restricted to M is also a semiclosed operator on M for each n. Therefore $T_n \in \mathcal{B}(M, \mathcal{H})$ for all n. Since T is the strong limit of T_n on M, By uniform boundedness principle $T \in \mathcal{B}(M, \mathcal{H})$. Hence T is a semiclosed operator. □

Theorem 4.2.8. *T is a semiclosed operator if and only if $D(T)$ is a semiclosed subspace and there exists a sequence $\{T_n\}$ of bounded operators on $\mathcal{H}$ such that $\{T_n\}$ converges strongly to T on $D(T)$.*

Proof. Suppose $D(T)$ is a semiclosed subspace and there exists $\{T_n\} \in \mathcal{B}(\mathcal{H})$ such that $\{T_n x\}$ converges Tx for each $x \in D(T)$. Then T_n is a semiclosed operator in $\mathcal{H}$ for all n. Therefore for each n, the restriction of T_n to $D(T)$ is also a semiclosed operator. Now by theorem 4.2.7, T is a semiclosed operator. □

Remark 4.2.9. *By the theorems in this section, it is easy to see that the class of semi-closed operators is the smallest class of unbounded operators which contains all closed operators and is closed under sum and product.*

Remark 4.2.10. *From the theorem 4.2.4, the class of all semiclosed operators in $\mathcal{H}$ is closed under "sum" and "product". But it does not form a vector space because the zero vector is not unique in $SCO(\mathcal{H})$, as shown in the following example.*

Example 4.2.11. *Let $A\varphi(x) = x\varphi(x)$ and $B\varphi(x) = x^2\varphi(x)$ defined on $L^2(\mathbb{R})$ with the following domains*

$$D(A) = C_0^\infty(\mathbb{R}) \text{ and } D(B) = \{\varphi \in C^\infty(\mathbb{R}) : \mathcal{F}\varphi \in C_0^\infty(\mathbb{R})\}.$$

where $\mathcal{F}$ is a Fourier transform (as a unitary operator on $L^2(\mathbb{R})$). Then A and B are self-adjoint operators and $D(A + B) = \{0\}$.

4.3 Semiclosed extension

We know that a densely defined bounded operator with co-domain Banach can be uniquely extended to the whole space without affecting the norm. It is natural to ask the similar kind of questions in the case of semiclosed operators as follows.

- Is it possible to extend the semiclosed operator to the whole space?
- For a given operator, when will it have semiclosed extension?

In this section we answer these questions.

Proposition 4.3.1. *Let T be a semiclosed operator in a Hilbert space. Then T has semiclosed extension.*

Proof. Let P be the projection from $\mathcal{H}$ onto $\overline{D(T)}$. Then P is a semiclosed operator. Let $\tilde{T} = TP$ which is the product of two semiclosed operator. Hence $\tilde{T}$ is a semiclosed operator with domain $\mathcal{H}$. □

Definition 4.3.2. *(Kaufman (1979)) Let T and S be operators in $\mathcal{H}$. We say T is* ***continuous with respect*** *to S if $D(T) \subset (DS)$ and there exists a non-negative number b such that $\langle Tx, Tx\rangle \ \leq b\ \langle Sx, Sx\rangle$ for all $x \in D(T)$.*

$T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is bounded if and only if T is continuous with respect to I, the identity operator on $\mathcal{H}$.

Lemma 4.3.3. *For an operator T in a Hilbert space $\mathcal{H}$, the following are equivalent:*

1. T is continuous with respect to the inverse of a bounded operator in $\mathcal{H}$.

2. T is continuous with respect to a self-adjoint operator.

3. T is continuous with respect to a closed operator.

4. T is continuous with respect to a semiclosed operator.

Proof. (1) $\Rightarrow$ (2) : Suppose T is continuous with respect to inverse of some bounded operator in $\mathcal{H}$, say $B \in \mathcal{B}(\mathcal{H})$. Then B^{-1} is a closed operator which in turn a semiclosed operator. Now consider the projection $B^{*-1}B^*$ from $\mathcal{H}$ onto the closure of $R(B)$ and $B^{-1}B^{*-1}B^*$ is a semiclosed extension of B^{-1}.

Let $C = B^{-1}(B^{*-1}B^*)$ be a semiclosed operator with dense domain. Then there exists $A, G \in \mathcal{B}(\mathcal{H})$ such that $C = AG^{-1}$ where G^{-1} is a self-adjoint operator whose domain equals domain of C. By assumption there exists $b > 0$ such that

$$\begin{aligned} \langle Tx, Tx\rangle &\leq b\ \langle B^{-1}x, B^{-1}x\rangle = b\ \langle B^{-1}(B^{*-1}B^*)x, B^{-1}(B^{*-1}B^*)x\rangle \\ &= b\ \langle Cx, Cx\rangle \leq\ b\ \langle AG^{-1}x, AG^{-1}x\rangle \\ &\leq b\|A\|\ \|G^{-1}\| \end{aligned}$$

for all $x \in D(T)$. Hence T is continuous with respect to a self-adjoint operator.

(2) $\Rightarrow$ (3) : From the above proof, we can see that T is continuous with respect to a densely defined self-adjoint operator. Since densely defined self-adjoint operator are closed, we have T is continuous with respect to a closed operator.

The implication (3) $\Rightarrow$ (4) follows from the fact that every closed operator is a semiclosed operator.

(4) $\Rightarrow$ (1) : Let T be continuous with respect to a semiclosed operator, say S. Then $S = AB^{-1}$ for some $A, B \in \mathcal{B}(H)$. Then for $x \in D(T)$,

$$\begin{aligned} \|Tx\| &\leq b\ \|Sx\| = b\ \|AB^{-1}x\| \\ &\leq b\ \|A\|\ \|B^{-1}x\| \end{aligned}$$

concludes that T is continuous with respect to the inverse of a bounded operator. □

Theorem 4.3.4. *Let T be an operator in $\mathcal{H}$. Then T has semiclosed extension if and only if T is continuous with respect to a semiclosed operator.*

Proof. Suppose that T has a semiclosed extension, say $\tilde{T}$. Let $\tilde{T} = AB^{-1}$ be the decomposition of $\tilde{T}$, where $A, B \in \mathcal{B}(\mathcal{H})$. By assumption $\tilde{T}|_{D(T)} = T$, that is, $Tx = AB^{-1}x$, $x \in D(T)$. Then

$$\|Tx\| \leq k\ \|B^{-1}x\|,\ x \in D(T).$$

Therefore T is continuous with respect to inverse of a bounded operator. By lemma 4.3.3, we can conclude that T is continuous with respect to a semiclosed operator.

Conversely, suppose that T is continuous with respect to a semiclosed operator, say C and $C = AB^{-1}$ for some $A, B \in \mathcal{B}(\mathcal{H})$. Then there exists $b > 0$ such that

$$\|Tx\| \leq b\ \|Cx\| \leq b\ \|A\|\|B^{-1}x\|,\ x \in D(T).$$

Let $z \in \mathcal{H}$ such that $Bz \in D(T)$, that is, $z \in D(TB)$. Then $\|TBz\| \leq b\ \|A\|\ \|z\|$. Therefore TB is a bounded operator from $D(TB)$ into $\mathcal{H}$ which can be extended to $\mathcal{H}$ and denote the extension by E. For $x \in D(T)$,

$$Tx = TBB^{-1}x = EB^{-1}x$$

shows that EB^{-1} is an extension of T. Since $E \in \mathcal{B}(\mathcal{H})$ and B^{-1} is closed, E and B^{-1} are semiclosed operators. Hence EB^{-1} is a semiclosed extension of T. □

4.4 Conclusion

For two bounded operators A and B in $\mathcal{H}$ with the kernel condition $N(A) \subseteq N(B)$, the quotient $[B/A]$ defined in Izumino (1989), by $Ax \to Bx$, $x \in \mathcal{H}$. A quotient of bounded operators so defined is what was introduced by Kaufman (1978), as a "semiclosed operator", and several characterizations of it are given. It is proved that the family of quotients contains all closed operators and is itself closed under "sum" and "product". A merit for the quotient representation of a semiclosed operator is to make use of the theory of bounded operators. Using the quotient representation of a semiclosed operator, a topology is defined and the topological properties are studied in the set of those operators in Hirasawa (2007). One of the future plans is to define topology and study the topological properties in the class of semiclosed operators in Hilbert spaces.

Chapter 5

Semiclosed Operators with Closed Range

The operator equation $Tx = y$ is solvable if and only if $y \in R(T)$. If $y \notin R(T)$ and $R(T)$ is closed, then we can find a *unique* solution of smallest norm $\tilde{x} \in X$ such that

$$\|T\tilde{x} - y\| = \inf\ \Big\{\|Tx - y\| : x \in X\Big\}.$$

The solution $\tilde{x}$ is called the generalized solution corresponding to y. The problem of finding the generalized solution is well-posed (in sense of Hadamard (1923)) if and only if $R(T)$ is closed. Thus, the knowledge whether the range of an operator T is closed or not is particularly important in solving the operator equation. Many characterizations are available for bounded and closed operators between Banach spaces to have closed range [we refer, Schechter (2002), Goldberg (2006)].

The Banach's closed range theorem (Goldberg (2006)) says that if X and Y are Banach spaces and if $T \in \mathcal{B}(X, Y)$, then $R(T)$ is closed in Y if and only if $R(T^*)$ is closed in X^*. Given a differential operator T defined on some subspace of $L_p(\Omega)$, one may be interested in determining the family of functions $y \in L_p(\Omega)$ for which $Tf = y$ has a solution. It is well known that if $T \in \mathcal{B}(X, Y)$ has closed range, then the space of such y is the orthogonal complement of the solutions to the homogeneous equation $Tg = 0$. There are many important applications of "closed range unbounded operators" in the spectral study of differential operators and also in the context of perturbation theory (Kulkarni et al. (2008), Goldberg (2006)). One can find many characterizations for bounded operators and closed operators to have closed range in Schechter (2002), Goldberg (2006), Ganesa Moorthy and Johnson (2004).

In the chapter, we define Hyers-Ulam stability of an operator between Frechet spaces which is an extension of the result to closed operators between Hilbert spaces (Miura et al. (2003)). In general, the composition of two continuous linear operators in Frechet spaces having Hyers-Ulam stability, need not to have the same ; some results in the direction are given. Necessary and sufficient conditions for a semiclosed operator to have closed range are derived which merely depend on the properties of semiclosed subspaces. Similar kind of results for closed operators can be found in Goldberg (2006).

5.1 Hyers-Ulam-Rassias stability

Ulam (1964) gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability problem of functional equations: "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism?" In 1941, Hyers gave an answer to the problem by considering approximately mappings as follows. Let X and Y be real Banach spaces. If there exists an $\varepsilon \geq 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$, then there exists the unique additive mappings $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \varepsilon.$$

Rassias (1978) provided a generalization of Hyers's theorem which allows the Cauchy difference to be unbounded. Since then several mathematicians were attracted to the result of Rassias and investigated a number of stability problems of functional equations. This stability phenomenon that was introduced and proved by Rassias in his 1978 paper is called *Hyers-Ulam-Rassias stability.*

The notion of the Hyers-Ulam stability of a mapping between two normed spaces was introduced in Miura et al. (2003). Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces and T be a (not necessarily linear) mapping from X into Y. We say that T has the Hyers-Ulam stability if there exists $K > 0$ with the following property : For any $v \in R(T)$, $\varepsilon \geq 0$ and $u \in X$ with $\|Tu - v\|_Y \leq \varepsilon$, there exists a $u_0 \in X$ such that $Tu_0 = v$ and

$$\|u - u_0\|_X \leq K\varepsilon.$$

In other words, if T has the Hyers-Ulam stability, then to each "ε-approximate solution" u of the equation $Tx = v$ there corresponds an exact solution u_0 of the equation in the $K\varepsilon$-neighbourhood of u.

The linearity of T implies the following condition : For any $u \in X$ and $\varepsilon \geq 0$ with $\|Tu\|_Y \leq \varepsilon$, there exists a $u_0 \in X$ such that

$$Tu_0 = 0 \text{ and } \|u - u_0\|_X \leq K\varepsilon.$$

The above condition is equivalent to : For given $u \in X$, there is a $u_0 \in X$ such that

$$Tu = Tu_0 \text{ and } \|u_0\|_X \leq K\|Tu\|_Y.$$

We call such $K > 0$ a HUS constant for T, and denote by K_T the infimum of all HUS constants for T. If, in addition, K_T becomes a HUS constant for T, then we call it the HUS constant for T. Miura et al. (2003) have given a necessary and sufficient condition for the existence of the best HUS constant. The existence of the best HUS constants for the weighted composition operators and the first order linear differential operators are shown in Hatori et al. (2004).

We define the Hyers-Ulam stability of an operator between Frechet spaces. A Frechet space is a complete metrizable topological vector space [Rudin (1973)].

5.2 Compositions

Let X and Y be Frechet spaces and $T : X \to Y$ be a linear operator. T has the Hyers-Ulam stability if for a given open neighbourhood U of 0 in X there is an open neighbourhood V of 0 in Y such that for a given $x \in X$ with $Tx \in V$ there is a $y \in U$ satisfying $Tx = Ty$.

We first give a necessary and sufficient condition in order that T have the Hyers-Ulam stability.

Theorem 5.2.1. *Let X and Y be Frechet spaces and $T : X \to Y$ be a continuous linear operator. Then the following statements are equivalent:*

1. *T has the Hyers-Ulam stability*

2. *T has closed range.*

Proof. Suppose $R(T)$ is closed in Y. We denote $N = N(T)$ and $\widetilde{X} = X/N$. Define $\widetilde{T} : \widetilde{X} \to R(T)$ by

$$\widetilde{T}(x + N) = Tx.$$

Then $\widetilde{T}$ is a one-to-one continuous linear operator from $\widetilde{X}$ onto $R(T)$ and by the open mapping theorem, $\widetilde{T}^{-1}$ is continuous. Let $\pi : X \to \widetilde{X}$ be the quotient mapping. Now fix an open neighborhood U of 0 in X. Then $\pi(U) = U + N = \widetilde{U}$ (say) is an open neighborhood of $0 + N$ in $\widetilde{X}$. Then there is an open neighborhood V of 0 in Y such that

$$R(T) \cap V \subseteq \widetilde{T}(\widetilde{U}) = \widetilde{T}(\pi(U)) = T(U).$$

Thus, for a given $x \in X$ with $Tx \in V$, there is a $y \in U$ such that $Tx = Ty$. This proves that T has the Hyers-Ulam stability.

Conversely assume that T has the Hyers-Ulam stability. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of balanced open neighborhoods of 0 which form a local base at 0 in X such that

$$U_{n+1} + U_{n+1} \subseteq U_n, \text{ for every } n.$$

For each U_n, let us find an open neighborhood V_n of 0 in Y such that if $Tx \in V_n$ for some $x \in X$, then $Tx = Ty$ for some $y \in U_n$. Without loss of generality, we assume that $\{V_n : n = 1, 2, \cdots\}$ is a local base at 0 in Y such that $V_{n+1} + V_{n+1} \subseteq V_n$ for every n.

Let $y_0 \in \overline{R(T)}$. Find a sequence $\{x'_n\}$ in X such that $Tx'_n \to y_0$ as $n \to \infty$, and $Tx'_{n+1} - Tx'_n \in V_n$ for every n. For every n, find $x_n \in U_n$ such that

$$Tx_n = Tx'_{n+1} - Tx'_n \in V_n.$$

Then, as $m \to \infty$,

$$\begin{aligned}\sum_{n=1}^{m} Tx_n &= (Tx_2' - Tx_1') + \cdots + (Tx_{m+1}' - Tx_m') \\ &= Tx_{m+1}' - Tx_1' \to y_0 - Tx_1'.\end{aligned}$$

Thus $\sum_{n=1}^{\infty} Tx_n$ converges to $y_0 - Tx_1'$. Also for $m < n$, we have

$$\begin{aligned}x_m + \cdots + x_n &\in U_m + U_{m+1} + \cdots + U_{n-1} + U_n \\ &\subseteq U_m + U_{m+1} + \cdots + U_{n-1} + U_{n-1} \\ &\subseteq U_m + U_{m+1} + \cdots + U_{n-2} + U_{n-2} \\ &\subseteq \cdots\cdots\cdots\cdots\cdots\cdots \\ &\subseteq U_m + U_m \subseteq U_{m-1}.\end{aligned}$$

This proves that $\sum_{n=1}^{\infty} x_n$ converges to x_0, say, in the Frechet space X, and hence $\sum_{n=1}^{\infty} Tx_n$ converges to Tx_0 in Y. Therefore $Tx_0 = y_0 - Tx_1' = \sum_{n=1}^{\infty} Tx_n$ so that $y_0 = Tx_0 + Tx_1' \in R(T)$. This proves that $R(T)$ is closed in Y. □

The following theorem gives a particular version of the Hyers-Ulam stability of a bounded operator between Banach spaces which was proved in Takagi et al. (2003). However, our proof enjoys the standard technique for the proof of the open mapping theorem.

Theorem 5.2.2. *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded operator. Then the following statements are equivalent:*

1. *T has the Hyers-Ulam stability.*

2. *T has closed range.*

Proof. Suppose $R(T)$ is closed in Y. Define $\widetilde{T} : \widetilde{X} \to R(T)$ by $\widetilde{T}(x+N) = Tx$, for $x \in X$. Then $\widetilde{T}$ is a well defined one-to-one bounded operator from $\widetilde{X}$ onto $R(T)$. Therefore, by the open mapping theorem, there exists a constant $K' > 0$ such that

$$\|x + N\| \le K' \|\widetilde{T}(x + N)\| = \|Tx\|$$

for every $x \in X$. Take $K = K^{'} + 1$. Then for given $x \in X$, if $Tx \neq 0$, then there is an element $z \in N$ such that

$$\begin{aligned} \|x + z\| &\leq \|x + N\| + \|Tx\| \\ &\leq K^{'}\|Tx\| + \|Tx\| \\ &= K\|Tx\|. \end{aligned}$$

In this case, we take $y = x + z$ so that $\|y\| \leq K\|Tx\|$. If $Tx = 0$, then we take $y = 0$ so that $\|y\| \leq K\|Tx\|$. Thus T has the Hyers-Ulam stability with a HUS constant K.

Conversely assume that T has a HUS constant K. Fix $y_0 \in \overline{R(T)}$, the closure of $R(T)$ in Y. Then there is a sequence $\{x_n\}$ in X such that $\|x_n\| \leq K\|Tx_n\|$ and for every $n = 1, 2, 3, \ldots$, $\|(y_0 - Tx_1 - Tx_2 - \cdots - Tx_{n-1}) - Tx_n\| \leq \frac{1}{2^{n+2}}$. Then

$$\begin{aligned} \frac{1}{K}\|x_n\| &\leq \|y_0 - Tx_1 - Tx_2 - \cdots - Tx_n\| + \|y_0 - T_{x_1} - T_{x_2} - \cdots - T_{x_{n-1}}\| \\ &\leq \frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} \leq \frac{1}{2^n}. \end{aligned}$$

Therefore, the series $\sum_{n=1}^{\infty} x_n$ converges to x_0, say, in X and the series $\sum_{n=1}^{\infty} Tx_n$ converges to y_0. Since T is continuous, $\sum_{n=1}^{\infty} Tx_n$ converges to $T(\sum_{n=1}^{\infty} x_n) = Tx_0$. Therefore $y_0 = Tx_0 \in R(T)$. This proves that $R(T)$ is closed in Y. □

Corollary 5.2.3. *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Then T has the Hyers-Ulam stability if and only if for a given bounded sequence $\{y_n\}$ in $R(T)$ there is a bounded sequence $\{x_n\}$ in X such that $Tx_n = y_n$ for every n.*

Proof. Suppose T has the Hyers-Ulam stability. Then there is a constant $K > 0$ such that for a given $y \in R(T)$, there is an element $x \in X$ such that

$$\|x\| \leq K\|y\| \text{ and } Tx = y.$$

Consider a bounded sequence $\{y_n\}$ in $R(T)$. To each y_n, there is an element $x_n \in X$ such that $Tx_n = y_n$ and $\|x_n\| \leq K\|y_n\|$. Then $\{x_n\}$ is a bounded sequence in X such that $Tx_n = y_n$, for every n.

To prove the converse part, assume that T does not have a Hyers-Ulam stability constant. Then, for every given n, there is an element y_n in X such that

$$n\|y_n\| < \|x\|, \text{ for any } x \in X \text{ with } Tx = y_n$$

and such that $\|y_n\| = 1$. Therefore, if there is an element x_n such that $Tx_n = y_n$, then $\|x_n\| > n$. Thus, there is no bounded sequence $\{x_n\}$ in X such that $Tx_n = y_n$ for all n, when $\{y_n\}$ is a bounded sequence in $R(T)$. □

Let X, Y and Z be Frechet spaces. Let $S : X \to Y$ and $T : Y \to Z$ be linear operators each of which having a HUS constant. It is not ture in general that the composition $TS : X \to Z$ has the Hyers-Ulam stability. The following example shows that even for continuous linear operators between Frechet spaces the composition of operators with HUS constants need not have the Hyers-Ulam stability.

Example 5.2.4. *Suppose $X = Y = Z = \ell_2$ with the usual norm on this Hilbert space. Define $S : X \to Y$ and $T : Y \to Z$ by*

$$S(x_1, x_2, x_3, x_4, \cdots) = (x_1, 0, x_2, 0, x_3, 0, x_4, 0, \cdots)$$

and

$$T(x_1, x_2, x_3, x_4, \cdots) = (x_1 + x_2, x_3/3 + x_4, x_5/5 + x_6, \cdots).$$

Then S and T have HUS constants. But TS does not have a HUS constant because $R(TS)$ is not closed in Z.

We provide a necessary and sufficient condition which gives that the composition of operators with HUS constants is again an operator with HUS constant.

Theorem 5.2.5. *Suppose $S : X \to Y$ and $T : Y \to Z$ are continuous linear operators between Frechet spaces such that S and T have the Hyers-Ulam stability. Then TS has the Hyers-Ulam stability if and only if $R(S) + N(T)$ is closed in Y.*

Proof. Let $Y' = R(S) + N(T)$. Then $R(S)$, $N(T)$ and Y' are Frechet spaces under the subspace topologies. Define $P : R(S) \times N(T) \to Y'$ by

$$P(x, y) = x + y,$$

for $x \in R(S)$ and $y \in N(T)$. Then P is a continuous linear mapping when the domain is endowed with the product topology and the coordinatewise algebraic operations. Then, by the open mapping theorem, for a given open neighborhood V_1 of 0 in $R(S)$ and V_2 of 0 in $N(T)$, there is an open neighborhood V_3 of 0 in Y' such that $V_1 + V_2 \supseteq V_3$. We shall use this observation in the following part.

Fix an open neighborhood U of 0 in X and find, by theorem 5.2.1, an open neighborhood V of 0 in Y such that if $x_1 \in X$ and $Sx_1 \in V$, then there is a $x_2 \in U$ such that $Sx_1 = Sx_2$. Find an open neighborhood V' of 0 in Y such that

$$V' \cap Y' \subseteq [V \cap R(S)] + [V \cap N(T)].$$

For this neighborhood V', we find, by theorem 5.2.1, an open neighborhood W of 0 in Z such that if $y_1 \in Y$ and $Ty_1 \in W$, then $Ty_1 = Ty_2$ for some $y_2 \in V'$.

Now if $x_1 \in X$ and $T(Sx_1) \in W$, then there is a $y_2 \in V'$ such that $T(Sx_1) = Ty_2$. Then $y_2 - Sx_1 \in N(T)$ and $y_2 = Sx_1 + (y_2 - Sx_1) \in R(S) + N(T) = Y'$ which implies that

$$y_2 \in V' \cap Y' \subseteq [V \cap R(S)] + [V \cap N(T)].$$

Therefore, there are y_3 and y_4 in Y such that $y_3 \in V \cap R(S)$ and $y_4 \in V \cap N(T)$ and $y_2 = y_3 + y_4$. Since $y_3 \in V \cap R(S)$, there is a $x_2 \in U$ such that $Sx_2 = y_3$. Therefore

$$\begin{aligned} T(Sx_1) &= Ty_2 = T(y_3 + y_4) \\ &= T(Sx_2) + Ty_4 \\ &= T(Sx_2) + 0 = T(Sx_2). \end{aligned}$$

Thus, for a given $x_1 \in X$ with $(TS)x_1 \in W$, there is a $x_2 \in U$ such that

$$(TS)x_1 = (TS)x_2.$$

This proves that $R(TS)$ is closed in Z. The proof for the other way implication comes from

$$R(S) + N(T) = T^{-1}[T(S(X))]$$

and $T(S(X))$ is closed in Z. □

5.3 Semiclosed operators with closed range

Definition 5.3.1. *(Kato (1976)) A linear operator is said to be normally solvable if it is a closed operator with closed range.*

The following theorem gives the characterizations of a semiclosed operator with respect to decomposition of normally solvable operators.

Theorem 5.3.2. *(Kaufman (1979))(Decompositions) Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be an operator in a Hilbert space $(\mathcal{H}, \langle ., . \rangle)$. Then the following are equivalent:*

1. T is a semiclosed operator in $\mathcal{H}$.
2. There exist $A, B \in \mathcal{B}(\mathcal{H})$ such that $T = AB^{-1}$.
3. There exist a bounded operator C on $\mathcal{H}$ and a positive normally solvable operator D on $\mathcal{H}$ with domain $D(T)$ such that $T = CD$.
4. There are two normally solvable operators F and G such that $T = F + G$.

Proof. (1) $\Rightarrow$ (2) : Suppose T is a semiclosed operator. Then by the theorem 4.1.3, $D(T)$ is a semiclosed subspace and $T \in \mathcal{B}(D_T, \mathcal{H})$. Hence there exists a positive operator $B \in \mathcal{B}(\mathcal{H})$ such that $B(\mathcal{H}) = D(T)$ and

$$\langle x, y \rangle_T = \langle B^{-1}x, B^{-1}y \rangle \text{ for all } x, y \in D(T).$$

Let $A = TB$. Since $B \in \mathcal{B}(\mathcal{H}, D_T)$ and $T \in \mathcal{B}(D_T, \mathcal{H})$, we have $A \in \mathcal{B}(\mathcal{H})$. Hence $T = AB^{-1}$ where $A, B \in \mathcal{B}(\mathcal{H})$.

(2) $\Rightarrow$ (3) : Suppose $T = AB^{-1}$ for some $A, B \in \mathcal{B}(\mathcal{H})$ where B is positive. Take $C = A$ and $D = B^{-1}$. As the inverse of injective bounded operator is closed, B^{-1} is closed.

(3) $\Rightarrow$ (4) : Suppose $T = AB^{-1}$ for some $A, B \in \mathcal{B}(\mathcal{H})$ where B is a positive operator. Let λ be a complex number in the resolvent set of A, so that $A - \lambda$ is invertible and by the theorem 4.2.6, $(A-\lambda)B^{-1}$ is a closed operator in $\mathcal{H}$. Let $E = (A-\lambda)B^{-1}$. Moreover, the range of E is the closed subspace $EB(\mathcal{H})$ of $\mathcal{H}$, which shows that that E^{-1} is bounded in $\mathcal{H}$.

Let P denote the projection from $\mathcal{H}$ onto $EB(H)$ and let $F = E^{-1}P$ be the extension of E^{-1} to $\mathcal{H}$ and G denote $\lambda^{-1}B$. Then

$$F^{-1} = (A - \lambda)B^{-1} \text{ and } G^{-1} = \lambda B^{-1},$$

so that $F(\mathcal{H}) = G(\mathcal{H}) = B(\mathcal{H}) = D(T)$ and $F^{-1} + G^{-1} = T$.

(4) $\Rightarrow$ (1) : Suppose there are two normally solvable operators F and G such that

$$T = F + G.$$

Then the semiclosedness of T follows at once from (1) of theorem 4.2.4 and the fact that closed operators are semiclosed operators. □

Theorem 5.3.3. *Let M, N be two subspaces of a Hilbert space $\mathcal{H}$ such that $M + N$ is closed in $\mathcal{H}$. The subspaces M and N are closed in $\mathcal{H}$ if and only if M and N are semiclosed in $\mathcal{H}$.*

Proof. Suppose M and N are semiclosed subspaces of $\mathcal{H}$. Then there exist norms $\|.\|_1$ and $\|.\|_2$ respectively on M and N such that $(M, \|.\|_1)$ and $(N, \|.\|_2)$ are Hilbert and for some positive numbers b and c,

$$\|x\| \leq b\|x\|_1 \text{ for all } x \in M$$

and

$$\|y\| \leq c\|y\|_1 \text{ for all } y \in N.$$

By proposition 2.2.3, $M + N$ is a semiclosed subspace with respect to the the norm

$$\|u\|_T^2 = \|u_1 + u_2\|_T^2 = \|u_1\|_1^2 + \|u_2\|_2^2.$$

Let $\{y_n\}$ be a sequence in M such that $\{y_n\}$ converges to y in $\mathcal{H}$. We show that $y \in M$. Consider $\{y_n\}$ as a sequence in $M+N$. Since $M+N$ is closed in $\mathcal{H}$, the topology generated by the norm $\|.\|_T$ coincides with the induced topology on $M+N$, we have $\{y_n\}$ converges to y in $(M+N, \|.\|_T)$. Thus $\{y_n\}$ is a Cauchy sequence in $(M+N, \|.\|_T)$.

Now by the definition of the $\|.\|_T$, $\{y_n\}$ is Cauchy in the Hilbert space $(M, \|.\|_1)$ which is Hilbert, $\{y_n\}$ converges to some element $y_0 \in M$ with respect to $\|.\|_1$. Hence $\{y_n\}$ converges to y_0 in $(M, \|.\|)$. Therefore $y = y_0 \in M$. Hence M is closed in $\mathcal{H}$. Replacing M and $\|.\|_1$ by N and $\|.\|_2$ in the above argument, we can show that N is also closed in $\mathcal{H}$.

The proof for the other way follows easily from the fact that all closed subspaces are semiclosed. □

Theorem 5.3.4. *Let T be a semiclosed operator on a Hilbert space $\mathcal{H}$. Then $R(T)$ is closed if and only if $R(T)+N$ is closed for some closed subspace N of $\mathcal{H}$.*

Proof. The one way implication is clear. Suppose there exists a closed subspace N such that $R(T)+N$ is closed in $\mathcal{H}$. Since T is a semiclosed operator in $\mathcal{H}$, by (2) of theorem 4.2.1, its range $R(T)$ is always a semiclosed subspace. Since N is closed, it is semiclosed as well. Therefore we have two semiclosed subspaces $R(T)$ and N such that $R(T)+N$ is closed. By theorem 5.3.3, $R(T)$ is closed in $\mathcal{H}$. □

5.4 Conclusion

The operator A in the abstract Cauchy problem given in the first chapter is closable. Even though, we might not want to switch to its closure because much of the basic information on the underlying problem is contained in the domain of A. This information would be lost by considering the larger domain and is often difficult to describe the domain of the closure of A. To be able to treat linear evolution equation for all operators appearing in applications, the notion of semiclosed operators is used. Hyers-Ulam stability of an operator between Frechet spaces is defined which is an extension of the result to closed operators between Hilbert spaces. Characterizations for semiclosed operators are given in terms of decomposition of normally solvable operators. One of the future plans is to define the notion of the Hyers-Ulam stability and the existence of the best HUS constants for semiclosed operators in Hilbert spaces.

Bibliography

Ando, T. (1990.). De branges spaces and analytic operator functions. Lecture Notes, Hokkaido University, Sapporo.

Ben-Israel, A. and Greville, T. N. E. (2003). *Generalized inverses.* CMS Books in Mathematics. Springer-Verlag, New York.

Chernoff, P. R. (1983). A semibounded closed symmetric operator whose square has trivial domain. *Proc. Amer. Math. Soc.*, 89(2):289–290.

Dixmier, J. (1949). Étude sur les variétés et les opérateurs de Julia, avec quelques applications. *Bull. Soc. Math. France*, 77:11–101.

Donoghue, Jr., W. F. (1967). The interpolation of quadratic norms. *Acta Math.*, 118:251–270.

Douglas, R. G. (1966). On majorization, factorization, and range inclusion of operators on Hilbert space. *Proc. Amer. Math. Soc.*, 17:413–415.

Dunford, N. and Schwartz, J. T. (1988a). *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons Inc., New York.

Dunford, N. and Schwartz, J. T. (1988b). *Linear operators. Part II.* Wiley Classics Library. John Wiley & Sons Inc., New York.

Evans, L. C. (2010). *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, second edition.

Fillmore, P. A. and Williams, J. P. (1971). On operator ranges. *Advances in Math.*, 7:254–281.

Foiaş, C. (1972). Invariant para-closed subspaces. *Indiana Univ. Math. J.*, 21:887–906.

Ganesa Moorthy, C. and Johnson, P. S. (2004). Composition of closed range operators. *J. Anal.*, 12:165–169.

Goldberg, S. (2006). *Unbounded linear operators.* Dover Publications Inc., Mineola, New York.

Groetsch, C. W. (2007). *Stable approximate evaluation of unbounded operators*, volume 1894 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin.

Hadamard, J. (1923). *Lectures on the Cauchy Problem in Linear Partial Differential equations.* Yale University Press, New Haven.

Hatori, O., Kobayashi, K., Miura, T., Takagi, H., and Takahasi, S.-E. (2004). On the best constant of Hyers-Ulam stability. *J. Nonlinear Convex Anal.*, 5(3):387–393.

Hirasawa, G. (2007). A topology for semiclosed operators in a Hilbert space. *Acta Sci. Math. (Szeged)*, 73(1-2):271–282.

Izumino, S. (1989). Quotients of bounded operators. *Proc. Amer. Math. Soc.*, 106(2):427–435.

Kato, T. (1976). *Perturbation theory for linear operators.* Springer-Verlag, Berlin.

Kaufman, W. E. (1978). Representing a closed operator as a quotient of continuous operators. *Proc. Amer. Math. Soc.*, 72(3):531–534.

Kaufman, W. E. (1979). Semiclosed operators in Hilbert space. *Proc. Amer. Math. Soc.*, 76(1):67–73.

Kulkarni, S. H., Nair, M. T., and Ramesh, G. (2008). Some properties of unbounded operators with closed range. *Proc. Indian Acad. Sci. Math. Sci.*, 118(4):613–625.

MacNerney, J. S. (1959). Investigation concerning positive definite continued fractions. *Duke Math. J.*, 26:663–677.

Miura, T., Miyajima, S., and Takahasi, S.-E. (2003). Hyers-Ulam stability of linear differential operator with constant coefficients. *Math. Nachr.*, 258:90–96.

Murray, F. J. (1937). On complementary manifolds and projections in spaces L_p and l_p. *Trans. Amer. Math. Soc.*, 41(1):138–152.

Rassias, T. M. (1978). On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.*, 72(2):297–300.

Reed, M. and Simon, B. (1975). *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press, New York.

Reed, M. and Simon, B. (1980). *Methods of modern mathematical physics. I.* Academic Press Inc., New York.

Riesz, F. and Sz.-Nagy, B. (1955). *Functional analysis*. Frederick Ungar Publishing Co., New York.

Rudin, W. (1973). *Functional analysis*. McGraw-Hill Book Co., New York. McGraw-Hill Series in Higher Mathematics.

Schechter, M. (2002). *Principles of functional analysis*, volume 36 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.

Takagi, H., Miura, T., and Takahasi, S.-E. (2003). Essential norms and stability constants of weighted composition operators on $C(X)$. *Bull. Korean Math. Soc.*, 40(4):583–591.

Ulam, S. M. (1964). *Problems in modern mathematics*. Science Editions John Wiley & Sons, Inc., New York.

von Neumann, J. (1950). *Functional Operators. II. The Geometry of Orthogonal Spaces*. Annals of Mathematics Studies, no. 22. Princeton University Press, New Jersy.

PUBLICATIONS

International Refereed Journals

- P. Sam Johnson and S. Balaji, Hyers-Ulam Stability of Linear Operators in Frechet Spaces, *Applied Mathematics and Information Sciences*, Vol.6, No.3, 2012, 525-528.
- P. Sam Johnson and S. Balaji, On Semiclosed Subspaces of Hilbert Spaces, *International Journal of Pure and Applied Mathematics*, Vol.79, No.2, 2012, 249-258.

Papers Presented in Conferences

- S. Balaji and P. Sam Johnson, "Topology on Semiclosed Operators", Presented in the *National Conference on Nonlinear Functional Analysis 2011 (NCNFA11)*, Manonmaniam Sundaranar University, Tirunelveli, 3-5 March 2011.
- S. Balaji and P. Sam Johnson, "On Semiclosed Operators with Closed Range", Presented in the *Twenty First International Conference of FIM on Interdisciplinary Mathematics, Statistics and Computational Techniques*, Panjab University, Chandigarh, 15-17 December 2012 (***Received R. S. Varma - Third Best Paper Award***).

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