STEEPEST DESCENT TYPE METHODS FOR NONLINEAR ILL-POSED OPERATOR EQUATIONS

Thesis

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by

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Dedicated to

My Parents

Muppidathy & Kannammal

DECLARATION

By the Ph.D. Research Scholar

I hereby **declare** that the research thesis entitled **"STEEPEST DESCENT TYPE METHODS FOR NONLINEAR ILL-POSED OPERATOR EQUATIONS"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this research thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to **certify** that the research thesis entitled **“STEEPEST DESCENT TYPE METHODS FOR NONLINEAR ILL-POSED OPERATOR EQUATIONS”** submitted by **Sabari M**, (Register Number MA14F05) as the record of the research work carried out by her, is *accepted as the research thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Prof. Santhosh George
Research Guide

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ABSTRACT

In this thesis, we consider steepest descent method and minimal error method for approximating a solution of the nonlinear ill-posed operator equation $F(x) = y$, where $F : D(F) \subseteq X \to Y$ is nonlinear Fréchet differentiable operator between the Hilbert spaces X and Y. In practical application, we have only noisy data y^δ with $\|y - y^\delta\| \leq \delta$. To our knowledge, convergence rate result for the steepest descent method and minimal error method with noisy data are not known. We provide error estimate for these methods with noisy data. We modified these methods with less computational cost. Error estimate for steepest descent method and minimal error method is not known under Hölder-type source condition. We provide an error estimate for these methods under Hölder-type source condition and also with noisy data. We also studied the regularized version of steepest descent method and regularization parameter in this regularized version is selected through the adaptive scheme of Pereverzev and Schock (2005).

Keywords: ***Ill-posed nonlinear equations, Steepest descent method, Minimal error method, Regularization method, Tikhonov regularization, Discrepancy principle, Balancing principle.***

Mathematics Subject Classification: 65J15, 65J20, 47H17, 47A52.

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Chapter 1

INTRODUCTION

American mathematician Keller (1976) introduced the general definition of inverse problem and his frequently quoted statement is "We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other. Direct problems have been studied widely for some time, while the other is newer and not so well understood and it is called the inverse problem." Inverse problems are some of the most important mathematical problems in science and mathematics because they tell us about parameters that we cannot directly observe. They have wide application in optics, radar, acoustics, communication theory, signal processing, medical imaging, oceanography, geophysics, computer vision, astronomy, remote sensing, machine learning, natural language processing, and many other fields. Inverse problem is a very active field of research in applied sciences. Many inverse problems in science and engineering have their mathematical formulation as an operator equation

$$F(x) = y, \tag{1.0.1}$$

where $F : X \to Y$ is a linear or nonlinear operator between suitable normed spaces, y is the observation and x is sought for the solution. Inverse problems most often do not fulfill Hadamard's postulates of well-posedness (see Section 1.1 below) i.e., the equation (1.0.1) might not have a solution in the strict sense, solution might not be unique and/or might not depend continuously on the data. Hence their mathematical analysis is subtle. Problems that are not well-posed in the sense of Hadamard [Hadamard (1953)] are termed ill-posed. Inverse problems are often ill-posed.

Throughout the thesis we will be using the following notations.

- Let $\langle .,. \rangle$ and $\|.\|$, respectively stand for inner product and norm.
- $D(F)$ denotes the domain of F.
- $R(F)$ denotes the range of F.
- Fréchet derivative of F is denoted by $F'(.)$ (see definition 1.1.2) and its adjoint by $F'(.)^*$.
- $B(x,r)$, $\overline{B(x,r)}$ stand, respectively for the open and closed balls in X, with center $x \in X$ and of radius $r > 0$.

1.1 Ill-posed problem

French mathematician Hadamard (1953) formulated the following conditions of well-posedness of mathematical problems. The problem of solving the operator equation (1.0.1) is said to be well-posed (according to Hadamard) if the following three conditions are fulfilled:

(a) for each $y \in Y$, there is a solution $x \in X$ of (1.0.1) (existence);

(b) the solution x is unique (uniqueness);

(c) the dependence of x upon F is continuous (stability).

It is evident from the definition that the well-posedness of (1.0.1) is intimately connected not only with the operator F, but also with the spaces X and Y and the topologies they carry, i.e., well-posedness is a property of the triple (F, X, Y). Clearly for (a) to hold, we must have $Y = F(X)$ that is the mapping $F : X \to Y$ must be surjective. Condition (b) is of course equivalent to injectivity of F and the condition (c) is merely another way of saying that, the inverse mapping of F is continuous. That means the operator F in (1.0.1) is bijective and F^{-1} is continuous. The operator equation (1.0.1) which is not well-posed is called ill-posed. Since the theory of linear ill-posed problems are well developed [Groetsch (1984); Engl et al. (1996); Nashed (1976)], we are interested in studying nonlinear ill-posed problems.

1.1.1 Nonlinear ill-posed problem

Let X and Y be linear spaces and F is a nonlinear operator from $D(F) \subseteq X$ into Y. As for linear case the theory is not so well developed in nonlinear case. If F is not surjective, then the operator equation (1.0.1) is not solvable. We use the concept of an x^* -minimum-norm solution $\hat{x}$. We need the following definitions in the sequel.

DEFINITION 1.1.1 (Engl et al. (1996))**.** *Let X and Y be Hilbert spaces and $F : D(F) \subseteq X \to Y$ is a nonlinear operator. We say that $\hat{x}$ is an x^* -minimum-norm solution of (1.0.1) if*

$$F(\hat{x}) = y$$

and

$$\|\hat{x} - x^*\| = \min\{\|x - x^*\| : F(x) = y\}.$$

An x^* -minimum-norm solution $\hat{x}$ need not exist. If at all it exists, it need not be unique.

DEFINITION 1.1.2. *Let F be an operator mapping from a Hilbert space X into Hilbert space Y. If there exists a bounded linear operator $L : X \to Y$ such that for $x_0 \in X$*

$$\lim_{h \to 0} \frac{\|F(x_0 + h) - F(x_0) - L(h)\|}{\|h\|} = 0,$$

then F is said to be a Fréchet-differentiable at x_0 and the bounded linear operator $F'(x_0) := L$ is called the first Fréchet-derivative of F at x_0.

If F is a nonlinear and Fréchet differentiable, then there are several possibilities:

(a) If $F'(x_0)$ is boundedly invertible at some x_0, then $F(x_0)$ is a local homeomorphism at this point by Inverse function theorem, but it may not be a global homeomorphism.

(b) If $F'(x_0)$ is not boundedly invertible does not imply that F is not a homeomorphism. For example, a homeomorphism $F(x_0)$ may have compact derivative, so its linearization yields an ill-posed problem. Otherwise $F(x_0)$ may be compact, so (1.0.1) is ill-posed, but $F'(x_0)$ may be a finite rank operator.

Hence for a nonlinear operator F, there are several possibilities of ill-posedness [Ramm (2005)] as seen above (see (a) and (b) above). Thus, for nonlinear case ill-posedness always means that the solution does not depend continuously on the data, i.e, $F'(.)$ is not boundedly invertible. We assume throughout the thesis that

- Equation (1.0.1) is ill-posed in the sense that the solution is not depending continuously on the given data y.
- Equation (1.0.1) has a solution $\hat{x}$.
- The available data $y^\delta \in Y$ such that $\|y - y^\delta\| \leq \delta$.

Next we give two examples of nonlinear ill-posed problems.

EXAMPLE 1.1.3 (Scherzer et al. (1993)). *Consider the problem of estimating c in*

$$\begin{aligned} -\Delta u + cu &= f \text{ in } \Omega, \\ u &= g \text{ in } \partial\Omega, \end{aligned} \tag{1.1.2}$$

where Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary or with Ω being a parallelepiped, $f \in L^2(\Omega)$ and $g \in H^{\frac{3}{2}}(\partial\Omega)$. Let $c_0 \in U$, where

$$U = \{c \in L^2 : c(x) \geq 0 \ a.e. \ on \ \Omega\}.$$

The nonlinear mapping $F : D(F) \subseteq L^2(\Omega) \to L^2(\Omega)$ is defined as the parameter to solution mapping

$$F(c) = u(c)$$

with $u(c)$ the solution of (1.1.2) and, with some $\epsilon > 0$,

$$D(F) := \{c \in L^2 : \|c - \tilde{c}\| \leq \epsilon \text{ for some } \tilde{c} \in U\}.$$

EXAMPLE 1.1.4. *(cf. Hoang and Ramm (2010)) Consider a nonlinear operator equation $F : H = L^2[0, 1] \to H$ defined by*

$$F(u) := \int_0^1 e^{-|x-y|} u(y)dy + (\arctan(u))^3.$$

The Fréchet derivative of F is

$$F'(u)w = \frac{3(arctan(u))^2}{1+u^2} w + \int_0^1 e^{-|x-y|} w(y)dy.$$

If $u(x)$ vanishes on a set of positive Lebesgue measure, then $F'(u)$ is not boundedly invertible. If $u \in C[0, 1]$ vanishes even at one point x_0, then $F'(u)$ is not boundedly invertible in H.

1.1.2 Regularization Method

Procedures that lead to stable approximations to an ill-posed problems are called regularization methods. Since (1.0.1) is ill-posed in general, the strong convergence and stability of approximate solutions can be proved only by applying some regularization procedure. The most widely used regularization methods for (1.0.1) with nonlinear F and approximate data y^δ are:

1. Tikhonov regularization method in which the solution x_α^δ of the equation

$$F'(x)^*(F(x) - y^\delta) + \alpha(x - x_0) = 0$$

is taken as the approximate solution of (1.0.1) [Tautenhahn and Jin (2003)].

2. If F is monotone operator and $X = Y$, in this case one consider Lavrentiev regularization method, in which the solution x_α^δ of the equation

$$F(x) + \alpha(x - x_0) = y^\delta$$

is taken as an approximate for $\hat{x}$ [Tautenhahn (2002)].

In our study we will be using Tikhonov regularization method for obtaining stable approximation for $\hat{x}$.

1.1.3 Source Conditions

To obtain error bounds on the distance $\|x_\alpha^\delta - \hat{x}\|$ one needs some additional smoothness assumptions on $\hat{x} - x_0$ with respect to the operator $F'(\hat{x})$ or $F'(x_0)$ are called the source condition. Various source conditions are used in the literature. For example Hölder-type source condition [Tautenhahn (2002),Tautenhahn (2004)], of the form $\hat{x} - x_0 \in R((F'(\hat{x})^*F'(\hat{x})^\nu), 0 < \nu \leq 1$, general source condition $\hat{x} - x_0 \in R(\phi((F'(\hat{x})^*F'(\hat{x}))))$, with index functions ϕ [Semenova (2010); Argyros et al. (2013); Nair and Mahale (2013); Argyros et al. (2014)] and the new variational source conditions [Hofmann et al. (2016)]. In our study, we will be using Hölder-type and the general source conditions with respect to the operator $F'(x_0)$.

1.1.4 Discrepency principle

An a priori choices should be based on some a priori knowledge of the exact solution, namely its smoothness, but unfortunately in practice this information is often not available. This motivates the necessity of looking for a-posteriori parameter choice rules. The most famous a-posteriori choice, the discrepancy principle (introduced for the first time by Morozov (1966)) and some other important improved choices depending both on the noise level and on the noisy data [Vainikko (1982); George (2010a)]. In the context of iterative methods, the discrepancy principle will be the stopping rule.

Stopping Rule: [Hanke (1995)] Assume $\|y - y^\delta\| < \delta$. Fix $\tau > 0$, and terminate the iteration when, for the first time, $\|y^\delta - F(x_k^\delta)\| \leq \tau\delta$. Denote by $k(\delta, y^\delta)$ the resulting stopping index.

1.2 Steepest Descent Method

Steepest descent method is one of the Gradient methods. It was proposed by Cauchy in 1847. To find a local minimum of a function using gradient descent, one takes each step proportional to the negative of the gradient (or of the approximate gradient) of the function at the current point. If instead one takes steps proportional to the positive of the gradient, we approaches a local maximum of that function; the procedure is then known as gradient ascent. This method is based on the observation that if the multi-variable function $f(x)$ is defined and differentiable in a neighborhood of a point a, then f decreases fastest if one goes from a in the direction of the negative gradient of f at a i.e.,$-\nabla f(a)$. It follows that, if $b = a - \gamma\nabla f(a)$, for γ small enough, then $f(a) \geq f(b)$. In other words, the term $\gamma\nabla f(a)$ is subtracted from 'a' because we want to move against the gradient, namely down toward the minimum. So, one starts with a guess x_0 for a local minimum of f, and consider the sequence $x_0, x_1, x_2, \cdots$ such that

$$x_{n+1} = x_n - \gamma_n \nabla f(x_n), \forall n \geq 0.$$

We have $f(x_0) \geq f(x_1) \geq f(x_2) \geq \cdots$, so hopefully the sequence (x_n) converges to the desired local minimum. Note that the value of the step size γ is allowed to change at every iteration. With certain assumptions on the function f (for example, f convex and ∇f Lipschitz) and particular choices of γ (e.g., chosen via a line search that satisfies the Wolfe conditions), convergence to a local minimum can be guaranteed. When the function f is convex, all local minima are also global minima, so in this case gradient descent can converge to the global solution.

1.2.1 Steepest descent method for ill-posed equations

Our study focuses on modified steepest descent method and a modified minimal error method for approximately solving the operator equation (1.0.1), where $F : D(F) \subseteq X \to Y$ is a nonlinear Fréchet differentiable operator between the Hilbert spaces X and Y. Steepest descent method for (1.0.1) is given by

$$x_{k+1} = x_k + \alpha_k s_k, k = 0, 1, 2, \ldots\ldots\ldots \tag{1.2.3}$$

where s_k is the search direction taken as the negative gradient of the minimization functional involved and α_k is the descent.

1.2.2 Previous studies

- Neubauer and Scherzer (1995) considered the steepest descent method (SDM) and the minimal error method (MEM) in the noise free case as:

  steepest descent method

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \;\; (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_k)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|s_k\|^2}{\|F'(x_k)s_k\|^2} \end{aligned} \tag{1.2.4}$$

  minimal error method

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \;\; (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_k)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|F(x_k) - y\|^2}{\|s_k\|^2} \end{aligned} \tag{1.2.5}$$

- Method (1.2.3) was studied by Scherzer (1996) when $s_k = -F'(x_k)^*(F(x_k) - y^\delta)$ and $\alpha_k = \frac{\|s_k\|^2}{\|F'(x_k)^* s_k\|^2}$.

- For linear operator F, Gilyazov (1997) studied ($\alpha-$ process) method (1.2.3) when $s_k = -F'(x_k)^*(F(x_k) - y^\delta)$ and $\alpha_k = \frac{\langle (F^*F)^\alpha s_k, s_k\rangle}{\langle (F^*F)^\alpha s_k, F^*F s_k\rangle}$.

- The TIGRA-method of Ramlau (2003) is of the form (1.2.3) with $s_k = -[F'(x_k)^*(F(x_k) - y^\delta) + \alpha_k(x_0 - x_k)]$ and $\alpha_k = \beta_k$.

- Vasin (2013) considered a regularized version of the steepest descent method in which $s_k = -F'(x_k)^*(F(x_k) - y^\delta) + \alpha(x_k - x_0)$ and $\alpha_k = \frac{\|s_k\|^2}{\|F'(x_k)s_k\|^2 + \alpha\|s_k\|^2}$, x_0 is the initial guess.

Note that, in all these methods, one has to compute Fréchet derivative of F at each iterate x_k in each iteration step which is in general very expensive. To reduce the computational cost, we considered the above iterative methods which involve Fréchet derivative of F at only the initial guess x_0.

1.3 Outline of the thesis

The rest of the thesis is structured as follows.

Chapter 2 deals with the frozen regularized steepest descent method (FRSDM). The regularization parameter is selected through the adaptive parameter choice strategy given in (Pereverzev and Schock, 2005). Finite dimensional realization of this method is also considered in this Chapter. Numerical example shows that the efficiency of our method FRSDM compared to the method considered in (Argyros et al., 2014).

In Chapter 3, we consider modified steepest descent method (MSDM) and modified minimal error method (MMEM) which involves Fréchet derivative of F at only the initial guess x_0. According to our knowledge, no convergence rate results are known for steepest descent method and minimal error method with noisy data. We obtained convergence rate result for both modified methods and existing methods.

Chapter 4 deals with error estimate for steepest descent method and minimal error method under general Hölder-type source condition

$$x_0 - \hat{x} \in R((F'(x_0)^*F'(x_0))^\nu), \text{ for } 0 < \nu < 1. \tag{1.3.6}$$

Using the above source condition we obtain the convergence rate result for MSDM and MMEM for $\nu = \frac{1}{2}$ in Chapter 3. But we could not obtain convergence rate result for MSDM and MMEM with $\nu \neq \frac{1}{2}$ even for SDM and MEM. So we further modified the MSDM and MMEM with exact data y and noisy data y^δ. We obtained convergence rate result for modified form of minimal error method (MFMEM) under the source condition (1.3.6) with $0 < \nu < \frac{1}{2}$ and for modified form of steepest descent method (MFSDM) under the source condition (1.3.6) with $0 < \nu < \frac{1}{4}$.

In Chapter 5, we consider nonlinear Hammerstein type operator equation which is the composition of bounded linear operator and nonlinear operator. The solution of linear operator equation is approximated through Tikhonov regularization and the solution of nonlinear operator equation is approximated using the method considered in Chapter 2.

Chapter 6 gives the conclusion of the thesis and future work.

Chapter 2

NUMERICAL APPROXIMATION OF A TIKHONOV TYPE REGULARIZER

In this Chapter, we present a frozen regularized steepest descent method and its finite dimensional realization for obtaining an approximate solution for the nonlinear ill-posed operator equation $F(x) = y$. The balancing principle considered by Pereverzev and Schock (2005) is used for choosing the regularization parameter. The error estimate is derived under a general source condition and is of 'optimal order'. Numerical example provided proves the efficiency of the proposed method.

2.1 Introduction

Steepest descent method was considered by Scherzer (1996), Neubauer and Scherzer (1995) for approximately solving the operator equation

$$F(x) = y. \tag{2.1.1}$$

In the present study, we consider a modified form of (1.2.3), namely frozen regularized steepest descent method (FRSDM) defined for each $k = 0, 1, 2, \cdots$ by

$$x_{k+1} = x_k - \beta[F'(x_0)^*(F(x_k) - y^\delta) + \alpha(x_k - x_0)], \tag{2.1.2}$$

where x_0 is the initial point, $\beta > 0$ is a fixed parameter and $\alpha > 0$ is the regularization parameter. Further, note that in method (2.1.2), we have frozen the

Fréchet derivative at x_0 throughout the iteration. That is why, we call this method (2.1.2) as frozen regularized steepest descent method. This is one of the advantage of the proposed method. Observe that (2.1.2) is of the form (1.2.3) with $s_k = -[F'(x_0)^*(F(x_k) - y^\delta) + \alpha(x_k - x_0)]$ and $\alpha_k = \beta$, for each $k = 1, 2, \cdots$. Since $\alpha_k = \beta$, one need not have to compute α_k in each step as in the earlier studies such as Scherzer (1996); Neubauer and Scherzer (1995); Vasin (2013). In other-words, the computational work is reduced considerably in the proposed method (2.1.2).

Note that method FRSDM coincide with the method considered in Argyros et al. (2014) when $\beta = 1$, but our convergence analysis is different from that of Argyros et al. (2014) and is based on the property of the norm of a self adjoint operator in Hilbert space (see Section 2.2). Moreover, the condition on the radius of the convergence ball in Argyros et al. (2014) is too restrictive than the condition in this Chapter. The numerical experiments (see comparison Table 2.1) also show that, the method considered in this Chapter provides better error estimate than that of the method considered in Argyros et al. (2014). We also consider the finite dimensional realization of the method FRSDM in Section 2.3. The error analysis and the algorithm for implementing the method FRSDM are given in Section 2.4. Finally the numerical results are given in Section 2.5.

2.2 Convergence analysis of FRSDM

Throughout this Chapter, we assume that the operator F satisfies the following assumptions.

ASSUMPTION 2.2.1.

(a) There exists a constant $k_0 > 0$ such that for every $x \in D(F)$ and $v \in X$, there exists an element $\Phi(x, x_0, v) \in X$ satisfying

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x, x_0, v), \quad \|\Phi(x, x_0, v)\| \leq k_0\|v\|\|x - x_0\|.$$

(b)

$$\|F'(x_0)\| \leq M.$$

Notice that in the literature the stronger than (a) condition

$(a)'$

$$[F'(x) - F'(z)]v = F'(z)\xi(x, z, v), \quad \|\xi(x, z, v)\| \leq K\|v\|\|x - z\|$$

is used for some $\xi(x, z, v) \in X$. However,

$$k_0 \leq K$$

holds in general and $\frac{K}{k_0}$ can be arbitrarily large Argyros (2008). It is also worth noticing that $(a)'$ implies (a) but not necessarily vice versa and element ξ is less accurate and more difficult to find than Φ (see the numerical example in Argyros et al. (2014)).

ASSUMPTION 2.2.2. *There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(x_0)\|^2$ satisfying;*

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0$
- $\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \qquad \forall \alpha \in (0, a].$
- *there exists $v \in X$ with $\|v\| \leq 1$ such that*

$$x_0 - \hat{x} = \varphi(F'(x_0)^* F'(x_0))v.$$

It is known that for $\alpha > 0$,

$$F'(x_0)^*(F(x) - y^\delta) + \alpha(x - x_0) = 0 \tag{2.2.1}$$

has a unique solution x_α^δ in $B(x_0, r)$ provided $0 < r < \frac{1}{k_0}$ (Argyros et al., 2014, Theorem 2.) (see also Jin (2010), Section 4.3). Also it is known (cf. (Argyros et al., 2014, Theorem 4)) that if assumptions 2.2.1 and 2.2.2 are satisfied, then

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{1}{1 - k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right). \tag{2.2.2}$$

Let $\delta_0 > 0$, $a_0 > 0$ be some constants with $\delta_0^2 < a_0$ and $\|x_0 - \hat{x}\| \leq r$. Let $\delta \in (0, \delta_0]$ and $\alpha \in [\delta^2, a_0]$. Further, let β, $q_{\alpha,\beta}$ be the parameters such that

$$\beta \leq \frac{1}{M^2 + a_0} \tag{2.2.3}$$

and

$$q_{\alpha,\beta} = 1 - \alpha\beta + \frac{3\beta M^2 k_0}{2} r. \tag{2.2.4}$$

REMARK 2.2.3.

1. *Suppose* $0 < r < \frac{2\alpha}{3M^2 k_0}$. *Then, we have*

$$\begin{aligned} q_{\alpha,\beta} &= 1 - \alpha\beta + \frac{3\beta M^2 k_0}{2} r \\ &< 1 - \alpha\beta + \frac{3\beta M^2 k_0}{2} \frac{2\alpha}{3M^2 k_0} = 1, \end{aligned}$$

i.e., $q_{\alpha,\beta} < 1$ *for all* $0 < r < \frac{2\alpha}{3M^2 k_0}$.

2. *Notice that, if* $\alpha \longrightarrow 0$, *then* $r \longrightarrow 0$ *and in this case* $x_0 = \hat{x}$. *Further, in practice, we choose* α *from a set* $\{0 < \alpha_0 < \alpha_1, \cdots, \alpha_N < 1\}$ *(see Section 2.4.2) and hence* $r > 0$.

Hereafter, we assume that $0 < r < \min\{\frac{1}{2k_0}, \frac{2\alpha}{3M^2 k_0}\}$.

THEOREM 2.2.4. *Let* (x_n) *be as in (2.1.2) and let* $0 < r < \min\left\{\frac{1}{2k_0}, \frac{2\alpha}{3M^2 k_0}\right\}$. *Then for each* $\delta \in (0, \delta_0]$, $\alpha \in [\delta^2, a_0]$, *the sequence* (x_n) *is in* $B(x_0, 2r)$ *and converges to* x_α^δ *as* $n \to \infty$. *Further,*

$$\|x_{n+1} - x_\alpha^\delta\| \leq q_{\alpha,\beta}^{n+1} \|x_0 - x_\alpha^\delta\|, \tag{2.2.5}$$

where $q_{\alpha,\beta}$ *is as in (2.2.4).*

Proof: Clearly, $x_0 \in \overline{B(x_0, 2r)}$. Let $A_n := \int_0^1 F'(x_\alpha^\delta + t(x_n - x_\alpha^\delta))dt$. Since $x_\alpha^\delta \in B(x_0, r)$, A_0 is well defined. Assume that for some $n > 0, x_n \in B(x_0, 2r)$ and

A_n is well defined. Then, since x_α^δ satisfies the equation (2.2.1), we have,

$$\begin{aligned}
x_{n+1} - x_\alpha^\delta &= x_n - x_\alpha^\delta - \beta\left[F'(x_0)^*(F(x_n) - F(x_\alpha^\delta)) + \alpha(x_n - x_\alpha^\delta)\right] \\
&= x_n - x_\alpha^\delta - \beta\left[F'(x_0)^* A_n + \alpha I\right](x_n - x_\alpha^\delta) \\
&= x_n - x_\alpha^\delta - \beta\left[F'(x_0)^*(A_n - F'(x_0))\right](x_n - x_\alpha^\delta) \\
&\quad -\beta\left[F'(x_0)^* F'(x_0) + \alpha I\right](x_n - x_\alpha^\delta) \\
&= \left[I - \beta(F'(x_0)^* F'(x_0) + \alpha I)\right](x_n - x_\alpha^\delta) \\
&\quad -\beta\left[F'(x_0)^*(A_n - F'(x_0))\right](x_n - x_\alpha^\delta). \qquad (2.2.6)
\end{aligned}$$

Using assumptions 2.2.1, we have

$$\begin{aligned}
x_{n+1} - x_\alpha^\delta &= \left[I - \beta(F'(x_0)^* F'(x_0) + \alpha I)\right](x_n - x_\alpha^\delta) \\
&\quad -\beta F'(x_0)^* F'(x_0)\int_0^1 \Phi(x_\alpha^\delta + t(x_n - x_\alpha^\delta), x_0, x_n - x_\alpha^\delta)dt.
\end{aligned}$$

Now since $I - \beta\left(F'(x_0)^* F'(x_0) + \alpha I\right)$ is a positive self-adjoint operator,

$$\begin{aligned}
&\|I - \beta(F'(x_0)^* F'(x_0) + \alpha I)\| \\
&= \sup_{\|x\|=1} |\langle (I - \beta(F'(x_0)^* F'(x_0) + \alpha I))\, x, x\rangle| \\
&= |\sup_{\|x\|=1} (1 - \beta\alpha)\langle x, x\rangle - \beta\langle F'(x_0)^* F'(x_0)x, x\rangle| \\
&\leq 1 - \alpha\beta. \qquad (2.2.7)
\end{aligned}$$

The last step follows from relation

$$\beta|\langle F'(x_0)^* F'(x_0)x, x\rangle| \leq \beta\|F'(x_0)\|^2 \leq \beta M^2 \leq \frac{1}{M^2+\alpha}M^2 = 1 - \frac{\alpha}{M^2+\alpha} \leq 1 - \beta\alpha.$$

Hence, by assumption 2.2.1, we have

$$\begin{aligned}
\|x_{n+1} - x_\alpha^\delta\| &\leq (1 - \alpha\beta)\|x_n - x_\alpha^\delta\| \\
&\quad +\beta M^2 k_0 \int_0^1 ((1-t)\|x_\alpha^\delta - x_0\| + t\|x_n - x_0\|)dt\|x_n - x_\alpha^\delta\| \\
&\leq \left(1 - \alpha\beta + \frac{3\beta M^2 k_0}{2} r\right)\|x_n - x_\alpha^\delta\| \\
&\leq q_{\alpha,\beta}\|x_n - x_\alpha^\delta\|. \qquad (2.2.8)
\end{aligned}$$

Since $q_{\alpha,\beta} < 1$ (see Remark 2.2.3), we have

$$\|x_{n+1} - x_\alpha^\delta\| < \|x_0 - x_\alpha^\delta\| \leq r$$

and

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - x_\alpha^\delta\| + \|x_0 - x_\alpha^\delta\| \leq 2r$$

i.e., $x_{n+1} \in B(x_0, 2r)$. Also, for $0 \leq t \leq 1$,

$$\|x_\alpha^\delta + t(x_{n+1} - x_\alpha^\delta) - x_0\| = \|(1-t)(x_\alpha^\delta - x_0) + t(x_{n+1} - x_\alpha^\delta)\| < 2r.$$

Hence, $x_\alpha^\delta + t(x_{n+1} - x_\alpha^\delta) \in B(x_0, 2r)$ and A_{n+1} is well defined. Thus, by induction x_n is well defined and remains in $B(x_0, 2r)$ for each $n = 0, 1, 2, \cdots$. By letting $n \to \infty$ in (2.1.2), we obtain the convergence of x_n to x_α^δ. The estimate (2.2.5) now follows from (2.2.8).

□

REMARK 2.2.5.

1. *If assumption 2.2.1 is fulfilled only for all $x \in B(x_0, r) \cap Q \neq \emptyset$, where Q is a convex closed a priori set, for which $\hat{x} \in Q$, then we can modify method (2.1.2) by the following way*

$$x_{n+1,\alpha}^\delta = P_Q(T(x_{n,\alpha}^\delta))$$

to obtain the same estimate in Theorem 2.2.4; here P_Q is the metric projection onto the set Q and T is the step operator in (2.1.2).

2. *Instead of assumption 2.2.1, if we use the following Lipschitz condition:*

$$\|F'(x_1) - F'(x_2)\| \leq L_0\|x_1 - x_2\| \tag{2.2.9}$$

then from (2.2.9) and (2.2.7), one can prove that (2.2.5) holds with $\bar{q}_{\alpha,\beta} := 1 - \alpha\beta + \beta M L_0 r$ instead of $q_{\alpha,\beta}$, provided $0 < r < \frac{\alpha}{ML_0}$.

3. *Also by using (2.2.9) instead of assumption 2.2.1, one can prove that (2.2.1) has a unique solution, if $0 < r < \frac{\sqrt{\alpha}}{L_0}$ and*

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{1}{1 - \frac{L_0 r}{\sqrt{\alpha}}}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right).$$

2.3 Finite dimensional realization of FRSDM

For implementing method FRSDM one needs numerical calculations in finite dimensional spaces. One of the approaches in this regard is through discretization (see Engl et al. (1996), page 63). Here the regularization is achieved by a finite dimensional approximation alone. Regularization of ill-posed problems by projection methods can be found in literature, for e.g in George and Nair (2016); Kaltenbacher et al. (2008); Kirsch (1996); Pereverzev and Prössdorf (2000). This Section is concerned with the finite dimensional realization of the method FRSDM. Precisely, our aim in this Section is to obtain an approximation for x_α^δ, in the finite dimensional space $R(P_h)$ of X. Here $\{P_h\}_{h>0}$ is a family of orthogonal projections of X onto $R(P_h)$, the range of P_h. For the results that follow, we impose the following conditions. Let

$$\epsilon_h := \|F'(x_0)(I - P_h)\|$$

and

$$b_h := \|(I - P_h)\hat{x}\|.$$

We assume that $\lim_{h\longrightarrow 0} \epsilon_h = 0$ and $\lim_{h\to 0} b_h = 0$. The above assumption is satisfied if $P_h \to I$ point-wise and if $F'(.)$ is compact operator. Further, we assume that there exist $\varepsilon_0 > 0$, $b_0 > 0$ and $\delta_0 > 0$ such that $\epsilon_h < \epsilon_0$ and $b_h < b_0$. We have taken the discretized version of (2.1.2) as

$$x_{n+1,\alpha}^{h,\delta} = x_{n,\alpha}^{h,\delta} - \beta P_h \left[F'(x_0)^*(F(x_{n,\alpha}^{h,\delta}) - y^\delta) + \alpha(x_{n,\alpha}^{h,\delta} - x_0^{h,\delta})\right] \tag{2.3.1}$$

where $x_0^{h,\delta} =: P_h x_0$. Let

$$(\delta_0 + \varepsilon_0)^2 < \bar{a}_0.$$

Next we prove that, for $\alpha > 0$

$$P_h F'(x_0)^* \left(F P_h(x) - y^\delta\right) + \alpha P_h(x - x_0) = 0 \tag{2.3.2}$$

has a unique solution $x_\alpha^{h,\delta}$ in $B(x_0, r) \cap R(P_h)$.

THEOREM 2.3.1. *Let $\hat{x}$ be a solution of (2.1.1), assumption 2.2.1 satisfied and let $F : D(F) \subseteq X \to Y$ be Fréchet differentiable in a ball $B(x_0, r) \cap R(P_h) \subseteq D(F)$ with $0 < r < \frac{1}{2k_0}$. Then (2.3.2) possesses a unique solution $x_\alpha^{h,\delta}$ in $B(x_0, r) \cap R(P_h)$.*

Proof: For $x \in B(x_0, r) \cap R(P_h)$, let

$$M_h = \int_0^1 F'(\hat{x} + t(x - \hat{x}))dt.$$

If $P_h F'(x_0)^* M_h P_h + \alpha I$ is invertible, then

$$\begin{aligned}(P_h F'(x_0)^* M_h P_h + \alpha I)(x - P_h \hat{x}) &= \alpha P_h(x_0 - \hat{x}) + P_h F'(x_0)^*(y^\delta - y) \\ &\quad + P_h F'(x_0)^* M_h (I - P_h)\hat{x} \qquad (2.3.3)\end{aligned}$$

has a unique solution $x_\alpha^{h,\delta} \in R(P_h)$. Observe that

$$F(P_h x) - y^\delta = F(P_h x) - F(\hat{x}) + y - y^\delta = M_h(P_h x - \hat{x}) + y - y^\delta$$

and hence

$$\begin{aligned}&P_h F'(x_0)^*(F P_h(x) - y^\delta) + \alpha P_h(x - x_0) \\ = \;& P_h F'(x_0)^* \left(M_h(P_h x - \hat{x}) + y - y^\delta\right) + \alpha P_h(x - x_0) \\ = \;& (P_h F'(x_0)^* M_h P_h + \alpha I) P_h(x - \hat{x}) - \alpha P_h(x_0 - \hat{x}) \\ & - P_h F'(x_0)^* M_h (I - P_h)\hat{x} - P_h F'(x_0)^*(y^\delta - y).\end{aligned}$$

Therefore by (2.3.3) $P_h F'(x_0)^*(F P_h(x) - y^\delta) + \alpha P_h(x - x_0) = 0$ has a unique solution $x_\alpha^{h,\delta}$. Clearly, $x_\alpha^{h,\delta} \in B(x_0, r) \cap R(P_h)$. So, it remains to show that $P_h F'(x_0)^* M_h P_h + \alpha I$ is invertible for $x \in B(x_0, r) \cap R(P_h)$. Note that by assumption 2.2.1, we have

$$\begin{aligned}&\|(P_h F'(x_0)^* F'(x_0) P_h + \alpha I)^{-1} P_h F'(x_0)^*(M_h - F'(x_0)) P_h\| \\ = \;& \sup_{\|v\| \le 1} \|(P_h F'(x_0)^* F'(x_0) P_h + \alpha I)^{-1} P_h F'(x_0)^*(M_h - F'(x_0)) P_h v\|\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{\|v\|\le 1} \| \left(P_h F'(x_0)^* F'(x_0) P_h + \alpha P_h\right)^{-1} P_h F'(x_0)^* \\
&\quad \int_0^1 (F'(\hat{x} + t(x - \hat{x})) - F'(x_0)) dt P_h v\| \\
&\le \sup_{\|v\|\le 1} \| \left(P_h F'(x_0)^* F'(x_0) P_h + \alpha I\right)^{-1} P_h F'(x_0)^* F'(x_0) \\
&\quad \int_0^1 \Phi\left(\hat{x} + t(x - \hat{x}), x_0, P_h v\right) dt\| \\
&\le \sup_{\|v\|\le 1} \| \left(P_h F'(x_0)^* F'(x_0) P_h + \alpha I\right)^{-1} P_h F'(x_0)^* \\
&\quad F'(x_0)[P_h + I - P_h] \int_0^1 \Phi(\hat{x} + t(x - \hat{x}), x_0, P_h v) dt\| \\
&\le \left[k_0 + k_0 \frac{\varepsilon_h}{\sqrt{\alpha}}\right] \int_0^1 \|\hat{x} + t(x - \hat{x}) - x_0\| dt \\
&\le \left[k_0 + k_0 \frac{\varepsilon_h}{\sqrt{\alpha}}\right] \int_0^1 [(1-t)\|\hat{x} - x_0\| + t\|x - x_0\|] dt \\
&\le k_0 \left(1 + \frac{\varepsilon_h}{\sqrt{\alpha}}\right) \frac{r + r}{2} < 2k_0 r < 1.
\end{aligned}
$$

Therefore, $I + \left(P_h F'(x_0)^* F'(x_0) P_h + \alpha I\right)^{-1} P_h F'(x_0)^* (M_h - F'(x_0)) P_h$ is invertible. Now from the relation

$$
\begin{aligned}
&P_h F'(x_0)^* M_h P_h + \alpha I \\
= \; &(P_h F'(x_0)^* F'(x_0) P_h + \alpha I) \\
&\left[I + (P_h F'(x_0)^* F'(x_0) P_h + \alpha I)^{-1} P_h F'(x_0)^* (M_h - F'(x_0)) P_h\right]
\end{aligned}
$$

it follows that $P_h F'(x_0)^* M_h P_h + \alpha I$ is invertible.

□

THEOREM 2.3.2. *Let* $(x_{n,\alpha}^{h,\delta})$ *be as in (2.3.1) and let* $0 < r < \min\left\{\frac{1}{2k_0}, \frac{2\alpha}{3M^2 k_0}\right\}$. *Then for each* $\delta \in (0, \delta_0]$, $\alpha \in ((\delta + \varepsilon_h)^2, \bar{a_0}]$, $\varepsilon_h \le \varepsilon_0$, $(x_{n,\alpha}^{h,\delta})$ *is in* $B(x_0, 2r) \cap R(P_h)$ *and converges to* $x_\alpha^{h,\delta}$ *as* $n \to \infty$. *Further,*

$$
\|x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \le q_{\alpha,\beta}^{n+1} \|P_h x_0 - x_\alpha^{h,\delta}\|, \tag{2.3.4}
$$

where $q_{\alpha,\beta}$ *is as in (2.2.4).*

Proof: Since $x_\alpha^{h,\delta}$ satisfies the equation (2.3.2), we have

$$\begin{aligned}
& x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta} \\
= \; & x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta} - \beta\left[P_h F'(x_0)^*(F(x_{n,\alpha}^{h,\delta}) - F(x_\alpha^{h,\delta})) + \alpha P_h(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta})\right] \\
= \; & x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta} - \beta\left[P_h F'(x_0)^* A_n^h + \alpha P_h\right](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\
= \; & x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta} - \beta\left[P_h F'(x_0)^*(A_n^h - F'(x_0))\right](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\
& -\beta\left[P_h F'(x_0)^* F'(x_0) P_h + \alpha P_h\right](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\
= \; & \left[I - \beta(P_h F'(x_0)^* F'(x_0) P_h + \alpha I)\right](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\
& -\beta\left[P_h F'(x_0)^*(A_n^h - F'(x_0))\right](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}),
\end{aligned}$$

where $A_n^h =: \int_0^1 F'(x_\alpha^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}))dt$. Using assumptions 2.2.1, we have

$$\begin{aligned}
& x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta} \\
= \; & \left[I - \beta(P_h F'(x_0)^* F'(x_0) P_h + \alpha I)\right](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\
& -\beta[P_h F'(x_0)^* F'(x_0) \int_0^1 \Phi(x_\alpha^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}), x_0, (x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}))dt.
\end{aligned}$$

Now since $I - \beta\left(P_h F'(x_0)^* F'(x_0) P_h + \alpha I\right)$ is a positive self-adjoint operator, as in (2.2.7)

$$\|I - \beta\left(P_h F'(x_0)^* F'(x_0) P_h + \alpha I\right)\| \leq 1 - \beta\alpha.$$

Hence,

$$\begin{aligned}
\|x_{n+1}^{h,\delta} - x_\alpha^{h,\delta}\| \; \leq \; & (1 - \alpha\beta)\|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \\
& +\beta M^2 k_0 \int_0^1 \big((1-t)\|x_\alpha^{h,\delta} - x_0\| + t\|x_{n,\alpha}^{h,\delta} - x_0\| dt\big)\, \|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \\
\leq \; & \left(1 - \alpha\beta + \frac{3\beta M^2 k_0}{2} r\right)\|x_{n,\alpha}^h - x_\alpha^{h,\delta}\| \\
\leq \; & \left(1 - \alpha\beta + \frac{3\beta M^2 k_0}{2} r\right)\|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\|.
\end{aligned}$$

The rest of the proof is analogous to the proof of Theorem 2.2.4.

□

REMARK 2.3.3.

Instead of assumption 2.2.1, if we use (2.2.9) then Theorem 2.3.1 holds with $0 < r < \frac{\sqrt{\alpha}}{L_0}$ *and Theorem 2.3.2 holds with* $\bar{q}_{\alpha,\beta} := 1 - \alpha\beta + \beta M L_0 r$ *instead of* $q_{\alpha,\beta}$, *provided* $0 < r < \frac{\alpha}{ML_0}$.

2.4 Error bounds under source conditions

Note that by (2.2.1), we have

$$P_h F'(x_0)^*(F(x_\alpha^\delta) - y^\delta) + \alpha P_h(x_\alpha^\delta - x_0) = 0. \tag{2.4.1}$$

So, by (2.3.2) and (2.4.1), we obtain

$$P_h F'(x_0)^* \left(F(x_\alpha^{h,\delta}) - F(x_\alpha^\delta)\right) + \alpha P_h(x_\alpha^{h,\delta} - x_\alpha^\delta) = 0.$$

That is,

$$\begin{aligned}(P_h F'(x_0)^* F'(x_0) P_h + \alpha I)(x_\alpha^{h,\delta} - P_h x_\alpha^\delta) &= P_h F'(x_0)^*(F'(x_0) - T)(x_\alpha^{h,\delta} - x_\alpha^\delta) \\ &\quad + P_h F'(x_0)^* F'(x_0)(I - P_h) x_\alpha^\delta,\end{aligned}$$

where $T = \int_0^1 F'(x_\alpha^\delta + t(x_\alpha^{h,\delta} - x_\alpha^\delta))dt$. So,

$$\begin{aligned}
&\|x_\alpha^{h,\delta} - P_h x_\alpha^\delta\| \\
= \;& \|(P_h F'(x_0)^* F'(x_0) P_h + \alpha P_h)^{-1}\big[P_h F'(x_0)^*(F'(x_0) - T)(x_\alpha^{h,\delta} - x_\alpha^\delta) \\
& + P_h F'(x_0)^* F'(x_0)(I - P_h) x_\alpha^\delta\big]\| \\
\leq \;& \|(P_h F'(x_0)^* F'(x_0) P_h + \alpha P_h)^{-1} P_h F'(x_0)^* \\
& \times \int_0^1 [F'(x_\alpha^\delta + t(x_\alpha^{h,\delta} - x_\alpha^\delta)) - F'(x_0)]dt(x_\alpha^{h,\delta} - x_\alpha^\delta)\| + \frac{\varepsilon_h}{\sqrt{\alpha}}\|x_\alpha^\delta\| \\
\leq \;& \|(P_h F'(x_0)^* F'(x_0) P_h + \alpha P_h)^{-1} P_h F'(x_0)^* F'(x_0)[P_h + I - P_h] \\
& \times \int_0^1 \varphi(x_\alpha^\delta + t(x_\alpha^{h,\delta} - x_\alpha^\delta)), x_0, x_\alpha^{h,\delta} - x_\alpha^\delta)dt\| + \frac{\varepsilon_h}{\sqrt{\alpha}}\|x_\alpha^\delta\| \\
\leq \;& k_0\left(1 + \frac{\varepsilon_h}{\sqrt{\alpha}}\right)\int_0^1 [(1-t)\|x_\alpha^\delta - x_0\| + t\|x_\alpha^{h,\delta} - x_0\|]dt\|x_\alpha^{h,\delta} - x_\alpha^\delta\|
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_\alpha^\delta - x_0\| + \|x_0\|) \\
\leq\ & 2k_0 r\|x_\alpha^{h,\delta} - x_\alpha^\delta\| + \frac{\varepsilon_h}{\sqrt{\alpha}}(r + \|x_0\|) \\
\leq\ & 2k_0 r\left[\|x_\alpha^{h,\delta} - P_h x_\alpha^\delta\| + \|(I-P_h)x_\alpha^\delta\|\right] + \frac{\varepsilon_h}{\sqrt{\alpha}}(r + \|x_0\|),
\end{aligned}
$$

hence

$$
\|x_\alpha^{h,\delta} - P_h x_\alpha^\delta\| \leq \frac{1}{1-2k_0 r}\left[2k_0 r\|(I-P_h)x_\alpha^\delta\| + \frac{\varepsilon_h}{\sqrt{\alpha}}(r + \|x_0\|)\right]. \tag{2.4.2}
$$

Further, we observe that

$$
\|P_h x_0 - x_\alpha^{h,\delta}\| \leq \|P_h(x_0 - x_\alpha^{h,\delta})\| \leq r. \tag{2.4.3}
$$

Combining the estimates in (2.2.2), (2.4.2), (2.4.3) and Theorem 2.3.2, we obtain the following:

THEOREM 2.4.1. *Let the assumptions in Theorem 2.3.2 hold and let* $x_{n,\alpha}^{h,\delta}$ *be as in (2.3.2). Then*

$$
\begin{aligned}
\|x_{n,\alpha}^{h,\delta} - \hat{x}\| \ \leq\ & q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[b_h + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\| + r)\right] \\
& + \frac{1}{1-2k_0 r}2\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right).
\end{aligned} \tag{2.4.4}
$$

Further if $n_\delta := \min\left\{n : q_{\alpha,\beta}^n < \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}\right\}$ *and* $b_h \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ *then*

$$
\|x_{n_\delta,\alpha}^h - \hat{x}\| \leq C\left(\frac{\delta+\varepsilon_h}{\sqrt{\alpha}} + \varphi(\alpha)\right) \tag{2.4.5}
$$

where $C := r + \frac{1}{1-2k_0 r}\left[1 + \max\{r + \|x_0\|, 2\}\right]$.

Proof: By triangle inequality, we have $\|x_{n,\alpha}^{h,\delta} - \hat{x}\| \leq \|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| + \|x_\alpha^{h,\delta} - x_\alpha^\delta\| + \|x_\alpha^\delta - \hat{x}\|$. Therefore, from (2.2.2),(2.4.2), (2.4.3) and Theorem 2.3.2, we obtain that

$$
\begin{aligned}
\|x_{n,\alpha}^{h,\delta} - \hat{x}\| &\le q_{\alpha,\beta}^n r + \|x_\alpha^{h,\delta} - P_h x_\alpha^\delta\| + \|(I-P_h)x_\alpha^\delta\| + \|x_\alpha^\delta - \hat{x}\| \\
&\le q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[2k_0 r\|(I-P_h)x_\alpha^\delta\| + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right] \\
&\quad + \|(I-P_h)x_\alpha^\delta\| + \frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right) \\
&\le q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[\|(I-P_h)x_\alpha^\delta\| + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right] \\
&\quad + \frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right) \\
&\le q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[\|(I-P_h)(x_\alpha^\delta - \hat{x} + \hat{x})\| + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right] \\
&\quad + \frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right) \\
&\le q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[\|x_\alpha^\delta - \hat{x}\| + \|(I-P_h)\hat{x}\| + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right] \\
&\quad + \frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right) \\
&\le q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[b_h + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right] \\
&\quad + \left(1 + \frac{1}{1-2k_0 r}\right)\frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right) \\
&\le q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[b_h + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right] \\
&\quad + \frac{1}{1-2k_0 r}2\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right).
\end{aligned}
$$

This proves (2.4.4), and (2.4.5) follows from (2.4.4). □

2.4.1 An a priori choice of the parameter

Note that the estimate $\frac{\delta+\varepsilon_h}{\sqrt{\alpha}} + \varphi(\alpha)$ in Theorem 2.4.1 is of optimal order for the choice $\alpha := \alpha_{\delta,h}$, which satisfies $\frac{\delta+\varepsilon_h}{\sqrt{\alpha}} = \varphi(\alpha)$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \le a$. Then $\alpha_{\delta,h} = \varphi^{-1}[\psi^{-1}(\delta+\varepsilon_h)]$ satisfies $\frac{\delta+\varepsilon_h}{\sqrt{\alpha}} = \varphi(\alpha)$.

In view of the above observation, Theorem 2.4.1 leads to the following:

THEOREM 2.4.2. *Let $\psi(\lambda) = \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq a$ and assumptions in Theorem 2.4.1 hold. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}[\psi^{-1}(\delta + \varepsilon_h)]$ and let n_δ be as in Theorem 2.4.1. Then*

$$\|x^{h,\delta}_{n_\delta,\alpha_\delta} - \hat{x}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

2.4.2 Balancing principle

Note that the best function φ measuring the rate of convergence in Theorem 2.4.1 is usually unknown. Therefore, in practical applications, different parameters $\alpha = \alpha_i$ are often selected from some finite set

$$D := \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N < 1\},$$

and corresponding elements $x^{h,\delta}_{n,\alpha_i}, i = 1, 2, \cdots, N$ are studied on line. Let

$$n_i := \min\left\{n : q^n_{\alpha,\beta} \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_i}}\right\}$$

and let $x^h_{\alpha_i} := x^{h,\delta}_{n_i,\alpha_i}$. Then from Theorem 2.4.1, we have

$$\|x^h_{\alpha_i} - \hat{x}\| \leq C\left(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_i}} + \varphi(\alpha_i)\right), i = 1, 2, \cdots N.$$

We choose the regularization parameter α from the set D_N defined by

$$D_N := \{\alpha_i = \mu^{2i}\alpha_0 < 1, i = 1, 2, \cdots N\},$$

where $\alpha_0 = (\delta + \varepsilon_h)^2$ (see Pereverzev and Schock (2005), Semenova (2010)) and $\mu > 1$. Using the ideas in Pereverzev and Schock (2005), we consider all possible functions φ, satisfying assumption 2.2.1 and $\varphi(\alpha_i) \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}$. Any of such functions is called admissible for $\hat{x}$ and it can be used as a measure for the convergence of $x^h_{\alpha_i} \to \hat{x}$ (see Lu et al. (2008)).

The main result of this Section is the following theorem, proof of which is analogous to the proof of Theorem 4.4 in George (2010b).

THEOREM 2.4.3. *Assume that there exists $i \in \{0, 1, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}$. Let assumptions of Theorem 2.4.1 be satisfied and let*

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_i}}\right\} < N,$$

$$k = \max\left\{i : \forall j = 1, 2, \cdots, i; \|x^h_{\alpha_i} - x^h_{\alpha_j}\| \leq 4C\frac{\delta + \varepsilon_h}{\sqrt{\alpha_j}}\right\},$$

where C is as in Theorem 2.4.1. Then $l \leq k$ and

$$\|x^h_{\alpha_k} - \hat{x}\| \leq 6C\mu\psi^{-1}(\delta + \varepsilon_h).$$

REMARK 2.4.4. *The balancing algorithm associated with the choice of the parameter specified in Theorem 2.4.1 involves the following steps:*

- *Choose $\alpha_0 = (\delta + \varepsilon_h)^2$ and $\mu > 1$.*
- *Choose $\alpha_i := \mu^{2i}\alpha_0, i = 0, 1, 2, \cdots, N$.*

1. *Set $i = 0$.*
2. *Choose $n_i := min\left\{n : q^n_{\alpha_i,\beta} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}\right\}$.*
3. *Solve $x_i := x^{h,\delta}_{n_i,\alpha_i}$ by using the iteration (2.3.1).*
4. *If $\|x_i - x_j\| > 4C\frac{1}{\mu^j}, j < i$, then take $k = i - 1$ and return x_k.*
5. *Else, set $i = i + 1$ and go to 2.*

2.5 Numerical example

Let $V_1 \subseteq V_2 \subseteq V_3 \subseteq .....$ be a sequence of finite-dimensional subspaces of X with $\overline{\bigcup_{m\in\mathbb{N}} V_m} = X$ and P_h, $(h = \frac{1}{m})$ is the orthogonal projector of X onto $R(P_h) := V_m \subset D(F)$. Precisely, we choose orthonormal system of box function $\Phi_i(t,\tau) = \Psi_k(t)\Psi_l(\tau), i = (k-1)m+l,\ k = 1, 2, 3, .., m_1,\ l = 1, 2, 3, .., m_1,\quad i = 1, 2, ..., m(= m_1^2)$, where $\Psi_k(t), \Psi_l(\tau)$ are L_2-orthonormalized characteristic functions of the intervals $[k-1,k],[l-1,l]$ (Lu et al. (2008)), respectively, as a basis of V_m in $\Omega = [0,m_1]\times[0,m_1]$. We consider the following integral equation (Inverse gravimetry problem (see V. Vasin and Timerkhanova (1996) and references in it) for the implementation of the method (2.3.1). Let $F : H^1(\Omega)\subset L^2(\Omega)\longrightarrow L^2(\Omega)$ defined by

$$F(u)\equiv -\int\int_\Omega \frac{1}{[(x-x')^2+(y-y')^2+u^2(x',y')]^{1/2}}dx'dy' = f(x,y), \qquad (2.5.1)$$

where $\Omega = [0,m_1]\times[0,m_1]$. The Fréchet-derivative of the operator F at the point $u_0(x,y)$ is expressed by the formula

$$F'(u_0)h = \int\int_\Omega \frac{u_0(x',y')\,h(x',y')}{[(x-x')^2+(y-y')^2+(u_0(x',y'))^2]^{3/2}}dx'dy'. \qquad (2.5.2)$$

Applying to the integral equations (2.5.1) the two-dimensional analogy of rectangle's formula with uniform grid for every variable, we obtain the following system of nonlinear equations:

$$\sum_{i=1}^{m_1}\sum_{j=1}^{m_1}\frac{1}{[(x_k-x'_j)^2+(y_l-y'_i)^2+u^2(x'_j,y'_i)]^{1/2}}\Delta x\Delta y = f(x_k,y_l);$$

$(k=1,2,...,m_1, l=1,2,...,m_1)$. The discrete variant of the derivative $F'(u_0)$ has the form

$$\{F_n^0 h_n\}_{k,l} = \sum_{i=1}^{m_1}\sum_{j=1}^{m_1}\frac{\Delta x\Delta y\, u_0(x'_j,y'_i)h(x'_j,y'_i)}{[(x_k-x'_j)^2+(y_l-y'_i)^2+u_0^2(x'_j,y'_i)]^{3/2}}, \qquad (2.5.3)$$

where $u_0(x,y)\equiv H$ is constant.

We take the exact solution as

$$\hat{u}(x,y) = 5 - 2e^{-[(x/10-3.5)^2(y/10-2.5)^2]} - 3e^{-[(x/10-5.5)^2(y/10-4.5)^2]},$$

and $f^\delta = F(\hat{u})+\delta$. Let $\Delta x = \Delta y = 1,\ m_1 = 35,\ H\equiv 5.$

Note that on the set

$$Q = \{1.0 \le u(x,y) \le 10.0\}$$

$\|F'(u)-F'(u_0)\| \le L_0\|u-u_0\|$ (see V. Vasin and Timerkhanova (1996); Vasin (2014)). The results of numerical experiments are presented in Table 2.1. Here

$\tilde{u}_n$ is the numerical solution obtained by our method; the relative error of solution and residual are

$$\Delta_1 = \frac{\|\hat{u} - \tilde{u}_n\|}{\|\hat{u}_n\|}, \qquad \Delta_2 = \frac{\|F_n(\tilde{u}_n) - f_n\|}{\|f_n\|}$$

respectively, for a noisy right-hand side.

Table 2.1: Comparison table for relative and residual error for method FRSDM and method in Argyros et al. (2014)

	Relative error and residual error for the method FRSDM			Relative error and residual error for the method in Argyros et al. (2014)		
δ	α_k	Δ_1	Δ_2	α_k	Δ_1	Δ_2
0.01	8.3521E-5	1.8576E-4	3.8057E-5	5.0625E-4	8.1214E-4	7.9872E-4
0.001	8.3521E-7	1.8577E-4	3.8123E-5	5.0625E-6	8.1214E-4	7.9871E-4
0.0002	7.3324E-8	1.8578E-4	3.8129E-5	2.0250E-7	8.1214E-4	7.9871E-4
0.0001	8.3521E-9	1.8579E-4	3.8130E-5	5.0625E-8	8.1214E-4	7.9871E-4

Comparison in Table 2.1 shows that the relative and residual error for method FRSDM is smaller than that of the method in Argyros et al. (2014) for a given data error.

Figure 2.1, shows the exact solution, Figure 2.2 shows approximate solution for $\delta = 0.01$ and Figure 2.3 shows the approximate solution for $\delta = 0.0001$.

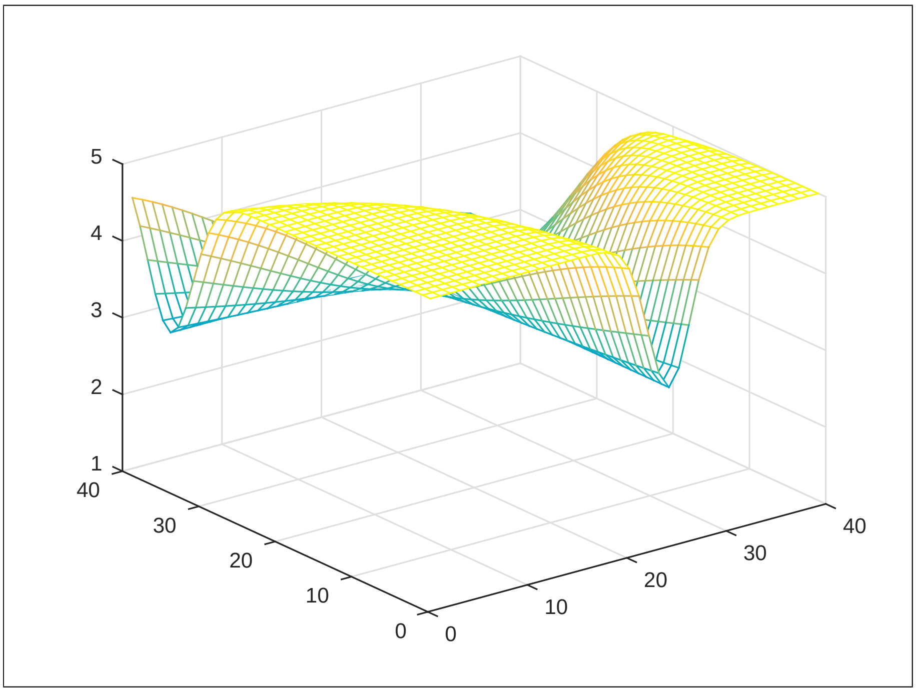

Figure 2.1: Exact solution.

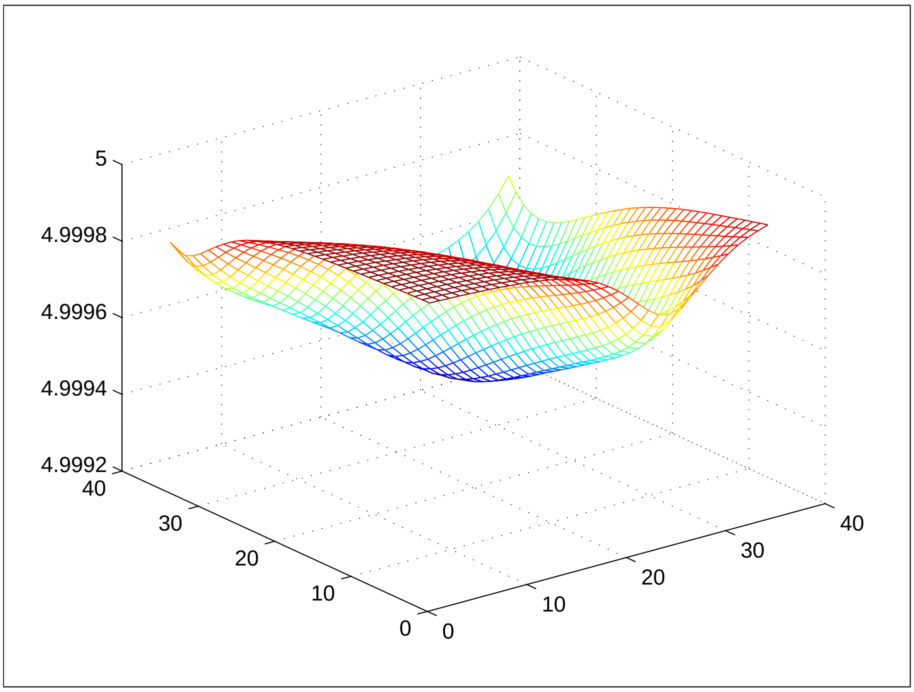

Figure 2.2: Approximate solution for $\delta = 0.01$

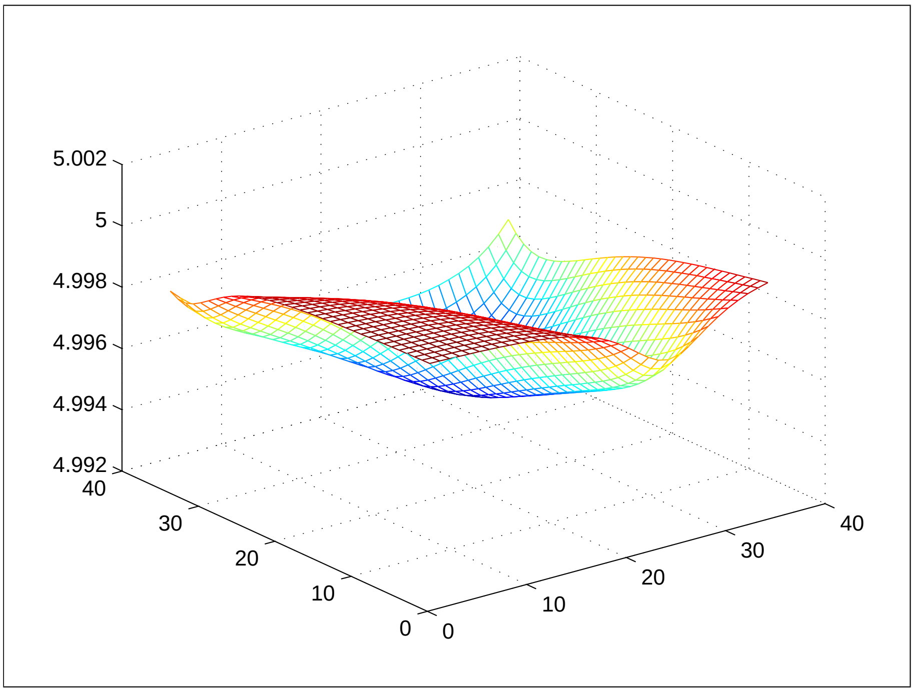

Figure 2.3: Approximate solution for $\delta = 0.0001$.

Chapter 3

MODIFIED STEEPEST DESCENT AND MODIFIED MINIMAL ERROR METHODS

A modified steepest descent method (MSDM) and modified minimal error method (MMEM) for nonlinear ill-posed operator equation have been considered in this Chapter. To our knowledge, convergence rate result for the steepest descent method (SDM) and minimal error method (MEM) with noisy data are not known. We provide convergence rate results for these methods with noisy data. The results in this Chapter are obtained under less computational cost, when compared to the steepest descent method and minimal error method. We provide a numerical example.

3.1 Introduction

Steepest descent method is studied extensively (see Brakhage (1987); Argyros et al. (2016a); Golub and O'Leary (1989); Kammerer and Nashed (1971, 1972); King (1989); Lardy (1990); Louis (1987); Neubauer and Scherzer (1995); Scherzer (1996)). As already stated in Section 1.2.1, the steepest descent method for nonlinear ill-posed operator equation can be written as

$$x_{k+1} = x_k + \alpha_k s_k, \tag{3.1.1}$$

where s_k is the search direction taken as the negative gradient of the minimization functional involved and α_k is the descent. In this Chapter, we provide convergence rate result(with and without noisy data) for the modified steepest descent and modified minimal error method. The same idea can be extended to steepest descent method and minimal error method with noisy data to obtain convergence rate result. The rest of the Chapter is structured as follows. Preliminaries are given in Section 3.2. Convergence analysis of MSDM and MMEM are given in Section 3.3. Convergence analysis of MSDM and MMEM with noisy data are given in Section 3.4 and a numerical example is given in Section 3.5.

3.2 Preliminaries

Neubauer and Scherzer (1995), considered the steepest descent method:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \quad (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_k)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|s_k\|^2}{\|F'(x_k)s_k\|^2} \end{aligned} \tag{3.2.1}$$

and the minimal error method:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \quad (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_k)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|F(x_k) - y\|^2}{\|s_k\|^2} \end{aligned} \tag{3.2.2}$$

in the noise free case and obtained the rate

$$\|x_k - \hat{x}\| = O(k^{-\frac{1}{2}}) \tag{3.2.3}$$

under the assumptions $(\mathcal{A})$:

$(\mathcal{A}_1)$ F has a Lipschitz continuous Fréchet derivative $F'(.)$ in a neighborhood of x_0.

$(\mathcal{A}_2)$ $F'(x) = R_x F'(\hat{x}),\ x \in B(x_0, \rho)$ where $\{R_x : x \in B(x_0, \rho)\}$ is a family of bounded linear operators $R_x : Y \longrightarrow Y$ with

$$\|R_x - I\| \leq C\|x - \hat{x}\|$$

where C is a positive constant and

$(\mathcal{A}_3)$

$$x_0 - \hat{x} = (F'(\hat{x})^* F'(\hat{x}))^{\frac{1}{2}} z$$

for some $z \in X$.

In this Chapter, we consider a modified steepest descent method and a modified minimal error method, in the case of noisy free data, defined by

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \quad (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|s_k\|^2}{\|F'(x_0)s_k\|^2} \end{aligned} \tag{3.2.4}$$

and

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \quad (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|F(x_k) - y\|^2}{\|s_k\|^2}, \end{aligned} \tag{3.2.5}$$

respectively. Let x_0 be the initial guess with $\|x_0 - \hat{x}\| \leq \rho$. Instead of assumptions $(\mathcal{A})$, we use the following assumptions $(\mathcal{C})$:

$(\mathcal{C}_0)$ $\|F'(x)\| \leq m$, for some $m > 0$ and for all $x \in D(F)$.

$(\mathcal{C}_1)$ $F'(\hat{x}) = F'(x_0)G(\hat{x}, x_0)$ where $G(\hat{x}, x_0)$ is a bounded linear operator from $X \longrightarrow X$ with

$$\|G(\hat{x}, x_0) - I\| \leq C_0\rho$$

where C_0 is a positive constant.

($\mathcal{C}_2$) $F'(x) = R(x,y)F'(y)$ $(x, y \in B(x_0, \rho))$ where $\{R(x,y) : x, y \in B(x_0, \rho)\}$ is a family of bounded linear operators $R(x,y) : Y \longrightarrow Y$ with

$$\|R(x,y) - I\| \leq C_1 \|x - y\|$$

for some positive constant C_1.

($\mathcal{C}_3$)

$$x_0 - \hat{x} = (F'(x_0)^* F'(x_0))^{\frac{1}{2}} v$$

for some $v \in X$.

Observe that $x_0 - \hat{x}$ in ($\mathcal{C}_3$) is depending on the known initial guess x_0, where as in ($\mathcal{A}_3$), $x_0 - \hat{x}$ is depending on the unknown $\hat{x}$. Not only this advantage but also in our method one need to compute the Fréchet derivative only at one point x_0 throughout the iteration process. As already mentioned in the introduction, no convergence rate results are known for steepest descent method and minimal error method with noisy data. In other words, it remains an open question whether convergence rate results can be proven for the methods SDM and MEM with noisy data. To answer this question, we considered the methods MSDM and MMEM with noisy data and obtained a convergence rate result. Using the same idea we obtained a convergence rate result for methods SDM and MEM.

3.3 Convergence analysis of MSDM and MMEM

The main purpose of this Section is to obtain an error estimate for $\|x_k - \hat{x}\|$, under the assumptions ($\mathcal{C}$). For this purpose we make use of the following result in Gilyazov (1997)[see (Gilyazov, 1997, Lemma 2)]. Let (v_k) be a sequence in $X, \nu > 0$, be some parameter such that

$$\|A^\nu v_k\|^2 - \|A^\nu v_{k+1}\|^2 \geq \varepsilon_k \langle A^{\nu+1} v_k, A^\nu v_k \rangle$$

for $k = 0, 1, 2, \dots$, where A is a positive self adjoint operator and $\varepsilon_k > 0$. Then

$$\|A^\nu v_k\| \le [2(\nu+1)]^\nu \|v_k\|^{\frac{1}{\nu+1}} \left[\sum_{i=0}^{k-1} \varepsilon_i \|v_i\|^{-\frac{1}{\nu+1}}\right]^{-\nu}. \tag{3.3.1}$$

We shall apply the above result (i.e., (3.3.1)) to $v_k = A^{-\frac{1}{2}}(x_k - \hat{x})$ with $A = F'(x_0)^* F'(x_0)$ and $\nu = \frac{1}{2}$. Therefore, in order to apply (3.3.1), we need to prove;

$$\|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2 \ge \varepsilon_k \langle A(x_k - \hat{x}), x_k - \hat{x}\rangle \tag{3.3.2}$$

for some $\varepsilon_k > 0$ and $\|A^{-\frac{1}{2}}(x_k - \hat{x})\|$ is bounded. Let $\bar{C} = \max\{C_0, C_1\}$.

LEMMA 3.3.1. *Let ($\mathcal{C}$) conditions hold and let $\bar{C}\rho \le \sqrt{5} - 2$. Let (x_k) be as in (3.2.4) or (3.2.5) and $\hat{x} \in B(x_0, \rho)$. Then, $x_k \in B(x_0, 2\rho)$ and*

$$\|x_{k+1} - \hat{x}\|^2 + \alpha_k \left[1 - \bar{C}^2\rho^2 - 4\bar{C}\rho\right] \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \le \|x_k - \hat{x}\|^2 \tag{3.3.3}$$

for all $k = 0, 1, 2, \dots$. Moreover,

$$\sum_{k=0}^{\infty} \alpha_k \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 < \infty.$$

Proof: We shall prove the result using induction. Note that $x_0 \in B(x_0, \rho)$ and suppose $x_k \in B(x_0, \rho)$. Then using (3.2.4) or (3.2.5), we have

$$\begin{aligned}
& \|x_{k+1} - \hat{x}\|^2 - \|x_k - \hat{x}\|^2 \\
= \;& -2\alpha_k \langle x_k - \hat{x}, F'(x_0)^*(F(x_k) - y)\rangle + \alpha_k^2 \|F'(x_0)^*(F(x_k) - y)\|^2 \\
= \;& -2\alpha_k \langle x_k - \hat{x}, F'(x_0)^* \left[F(x_k) - F(\hat{x}) - F'(x_0)(x_k - \hat{x})\right]\rangle \\
& + \alpha_k \Big[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\langle x_k - \hat{x}, F'(x_0)^* F'(x_0)(x_k - \hat{x})\rangle\Big] \\
= \;& -2\alpha_k \langle F'(x_0)(x_k - \hat{x}), \int_0^1 (F'(\hat{x} + \theta(x_k - \hat{x})) - F'(x_0)) d\theta (x_k - \hat{x})\rangle \\
& + \alpha_k \left[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2\right].
\end{aligned} \tag{3.3.4}$$

So by $(\mathcal{C}_2)$, we have

$$\begin{aligned}
& \|x_{k+1}-\hat{x}\|^2 - \|x_k-\hat{x}\|^2 \\
= \;& -2\alpha_k\langle F'(x_0)(x_k-\hat{x}), \int_0^1 \left[R(\hat{x}+\theta(x_k-\hat{x}),x_0)-I\right] d\theta F'(x_0)(x_k-\hat{x})\rangle \\
& +\alpha_k\left[\alpha_k\|F'(x_0)^*(F(x_k)-y)\|^2 - 2\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2\right] \\
\leq \;& 2\alpha_k \int_0^1 \|R(\hat{x}+\theta(x_k-\hat{x}),x_0)-I\|\|F'(x_0)(x_k-\hat{x})\|^2 d\theta \\
& +\alpha_k\left[\alpha_k\|F'(x_0)^*(F(x_k)-y)\|^2 - 2\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2\right] \\
\leq \;& 2\alpha_k\bar{C}\|\hat{x}+\theta(x_k-\hat{x})-x_0\|\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2 \\
& +\alpha_k\left[\alpha_k\|F'(x_0)^*(F(x_k)-y)\|^2 - 2\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2\right]. \qquad (3.3.5)
\end{aligned}$$

Observe that $\alpha_k\|F'(x_0)^*(F(x_k)-y)\|^2 = \|F(x_k)-y\|^2$ in the case of MMEM and in the case of MSDM, we have

$$\alpha_k\|F'(x_0)^*(F(x_k)-y)\|^2 = \frac{\langle F'(x_0)s_k, F(x_k)-y\rangle^2}{\|F'(x_0)s_k\|^2} \leq \|F(x_k)-y\|^2.$$

So for both methods MSDM and MMEM, we have

$$\begin{aligned}
& \alpha_k\|F'(x_0)^*(F(x_k)-y)\|^2 \\
\leq \;& \|F(x_k)-y\|^2 \\
= \;& \|\int_0^1 F'(\hat{x}+\theta(x_k-\hat{x}))d\theta(x_k-\hat{x})\|^2 \\
= \;& \|\int_0^1 \left[R(\hat{x}+\theta(x_k-\hat{x}),x_0)-I+I\right] d\theta F'(x_0)(x_k-\hat{x})\|^2 \\
\leq \;& \left(\bar{C}\|\hat{x}+\theta(x_k-\hat{x})-x_0\|+1\right)^2 \|F'(x_0)(x_k-\hat{x})\|^2 \\
\leq \;& \left(\bar{C}\rho+1\right)^2 \|A^{\frac{1}{2}}(x_k-\hat{x})\|^2. \qquad (3.3.6)
\end{aligned}$$

Therefore, by (3.3.5) and (3.3.6) we have

$$\|x_{k+1}-\hat{x}\|^2 - \|x_k-\hat{x}\|^2 \leq \left[(\bar{C}\rho+1)^2 + 2\bar{C}\rho - 2\right]\alpha_k\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2.$$

This completes the proof.

□

Let

$$p(t) = -2t^3 - \frac{26}{5}t^2 + \frac{56}{5}t - \frac{4}{5}.$$

Note that $p(0) = -\frac{4}{5} < 0$ and $p(1) = \frac{16}{5} > 0$. So $p(t)$ has a zero in $(0,1)$. Let r_0 be the smallest zero of p in $(0,1)$. Next, we shall prove the boundedness of $\|A^{-\frac{1}{2}}(x_k - \hat{x})\|$.

LEMMA 3.3.2. *Let $(\mathcal{C})$ conditions hold and $\bar{C}\rho < \min\left\{\sqrt{5}-2, r_0\right\} = 0.0740$. Let (x_k) be as in (3.2.4) or (3.2.5) and $\hat{x} \in B(x_0, \rho)$. Then $\|A^{-\frac{1}{2}}(x_k - \hat{x})\|$ is bounded.*

Proof: Using $(\mathcal{C}_3)$, one can prove that $x_k - \hat{x} \in R(A^{\frac{1}{2}})$ for all $k = 0, 1, 2, \ldots$. Therefore, we can apply the operator $A^{-\frac{1}{2}}$ to $x_{k+1} - \hat{x}$ and $x_k - \hat{x}$ to obtain

$$\begin{aligned}
& \|A^{-\frac{1}{2}}(x_{k+1} - \hat{x})\|^2 - \|A^{-\frac{1}{2}}(x_k - \hat{x})\|^2 \\
= \; & -2\alpha_k \langle A^{-\frac{1}{2}}(x_k - \hat{x}), A^{-\frac{1}{2}}F'(x_0)^*(F(x_k) - y)\rangle + \alpha_k^2 \|A^{-\frac{1}{2}}F'(x_0)^*(F(x_k) - y)\|^2 \\
= \; & -2\alpha_k \langle A^{-\frac{1}{2}}(x_k - \hat{x}), A^{-\frac{1}{2}}F'(x_0)^*\left(F(x_k) - F(\hat{x}) - F'(\hat{x})(x_k - \hat{x})\right)\rangle \\
& +\alpha_k \Big[\alpha_k \|A^{-\frac{1}{2}}F'(x_0)^*(F(x_k) - y)\|^2 \\
& -2\langle A^{-\frac{1}{2}}(x_k - \hat{x}), A^{-\frac{1}{2}}F'(x_0)^*F'(\hat{x})(x_k - \hat{x})\rangle\Big] \\
= \; & -2\alpha_k \langle A^{-\frac{1}{2}}(x_k - \hat{x}), \int_0^1 \left(F'(\hat{x} + \theta(x_k - \hat{x})) - F'(\hat{x})\right) d\theta (x_k - \hat{x})\rangle \\
& +\alpha_k \left[\alpha_k \|F(x_k) - y\|^2 - 2\langle A^{-\frac{1}{2}}(x_k - \hat{x}), F'(\hat{x})(x_k - \hat{x})\rangle\right]. \qquad (3.3.7)
\end{aligned}$$

So by $(\mathcal{C}_2)$ and (3.3.7), we have

$$\begin{aligned}
& \|A^{-\frac{1}{2}}(x_{k+1} - \hat{x})\|^2 - \|A^{-\frac{1}{2}}(x_k - \hat{x})\|^2 \\
= \; & -2\alpha_k \langle A^{-\frac{1}{2}}(x_k - \hat{x}), \int_0^1 \left(R(\hat{x} + \theta(x_k - \hat{x}), \hat{x}) - I\right) d\theta F'(\hat{x})(x_k - \hat{x})\rangle \\
& +\alpha_k \left[\alpha_k \|F(x_k) - y\|^2 - 2\langle A^{-\frac{1}{2}}(x_k - \hat{x}), F'(\hat{x})(x_k - \hat{x})\rangle\right].
\end{aligned}$$

$$
\begin{aligned}
&= -2\alpha_k\langle A^{-\frac{1}{2}}(x_k-\hat{x}), \int_0^1 [R(\hat{x}+\theta(x_k-\hat{x}),\hat{x})-I]\, d\theta F'(\hat{x})(x_k-\hat{x})\rangle \\
&\quad +\alpha_k\Big[\alpha_k\|F(x_k)-y\|^2 - 2\langle A^{-\frac{1}{2}}(x_k-\hat{x}), F'(x_0)(x_k-\hat{x})\rangle \\
&\quad -2\langle A^{-\frac{1}{2}}(x_k-\hat{x}), [F'(\hat{x})-F'(x_0)](x_k-\hat{x})\rangle\Big] \\
&= -2\alpha_k\langle A^{-\frac{1}{2}}(x_k-\hat{x}), \int_0^1 [R(\hat{x}+\theta(x_k-\hat{x}),\hat{x})-I]\, d\theta F'(\hat{x})(x_k-\hat{x})\rangle \\
&\quad +\alpha_k\Big[\alpha_k\|F(x_k)-y\|^2 - 2\langle A^{-\frac{1}{2}}(x_k-\hat{x}), F'(x_0)(x_k-\hat{x})\rangle \\
&\quad -2\langle A^{-\frac{1}{2}}(x_k-\hat{x}), F'(x_0)[G(\hat{x},x_0)-I](x_k-\hat{x})\rangle\Big].
\end{aligned}
$$

The last step follows from $(\mathcal{C}_1)$. So, we have

$$
\begin{aligned}
&\|A^{-\frac{1}{2}}(x_{k+1}-\hat{x})\|^2 - \|A^{-\frac{1}{2}}(x_k-\hat{x})\|^2 \\
\leq\ & 2\alpha_k\|A^{-\frac{1}{2}}(x_k-\hat{x})\|\bar{C}\int_0^1 \theta\|x_k-\hat{x}\|d\theta\|R(\hat{x},x_0)-I+I\|\|F'(x_0)(x_k-\hat{x})\| \\
&+\alpha_k\left[\alpha_k\|\int_0^1 F'(\hat{x}+\theta(x_k-\hat{x}))d\theta(x_k-\hat{x})\|^2 - 2\|x_k-\hat{x}\|^2 + 2\bar{C}\rho\|x_k-\hat{x}\|^2\right] \\
\leq\ & 2\alpha_k\|A^{-\frac{1}{2}}(x_k-\hat{x})\|\bar{C}\int_0^1 \theta\|x_k-\hat{x}\|d\theta\|R(\hat{x},x_0)-I+I\|\|F'(x_0)(x_k-\hat{x})\| \\
&+\alpha_k\Big[\alpha_k\|\int_0^1 [R(\hat{x}+\theta(x_k-\hat{x}),x_0)-I+I]\, d\theta F'(x_0)(x_k-\hat{x})\|^2 - 2\|x_k-\hat{x}\|^2 \\
&+2\bar{C}\rho\|x_k-\hat{x}\|^2\Big] \\
\leq\ & \alpha_k\|A^{-\frac{1}{2}}(x_k-\hat{x})\|\frac{\bar{C}}{2}\|x_k-\hat{x}\|(1+\bar{C}\rho)\|F'(x_0)(x_k-\hat{x})\| \\
&+\alpha_k\left[\alpha_k(1+\bar{C}\rho)^2\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2 - 2\|x_k-\hat{x}\|^2 + 2\bar{C}\rho\|x_k-\hat{x}\|^2\right].
\end{aligned}
$$

Therefore by Lemma 3.3.1, we have

$$
\begin{aligned}
&\|A^{-\frac{1}{2}}(x_{k+1}-\hat{x})\|^2 - \|A^{-\frac{1}{2}}(x_k-\hat{x})\|^2 \\
\leq\ & \alpha_k\|A^{-\frac{1}{2}}(x_k-\hat{x})\|\bar{C}\|x_k-\hat{x}\|(1+\bar{C}\rho)\|F'(x_0)(x_k-\hat{x})\| - \frac{\alpha_k}{5}\|x_k-\hat{x}\|^2 \\
&+\alpha_k\left[\frac{(1+\bar{C}\rho)^2}{1-4\bar{C}\rho-\bar{C}^2\rho^2} + 2\bar{C}\rho - \frac{9}{5}\right]\|x_k-\hat{x}\|^2 \\
\leq\ & \alpha_k\|A^{-\frac{1}{2}}(x_k-\hat{x})\|\bar{C}\left(1+\bar{C}\rho\right)\|x_k-\hat{x}\|\|F'(x_0)(x_k-\hat{x})\| \\
&-\frac{\alpha_k}{5}\|x_k-\hat{x}\|^2. \qquad (3.3.8)
\end{aligned}
$$

The last step follows from the fact that for $\bar{C}\rho \leq r_0$, we have

$$\frac{(1+\bar{C}\rho)^2}{1-4\bar{C}\rho-\bar{C}^2\rho^2} + 2\bar{C}\rho \leq \frac{9}{5}.$$

Now using the relation $2ab \leq a^2 + b^2$ with

$$a = \sqrt{\frac{1}{5}\alpha_k}\|x_k - \hat{x}\| \text{ and } b = \frac{\sqrt{5\alpha_k}}{2}\bar{C}(1+\bar{C}\rho)\|A^{-\frac{1}{2}}(x_k-\hat{x})\|\|F'(x_0)(x_k-\hat{x})\|$$

in (3.3.8), we have

$$\begin{aligned} & \|A^{-\frac{1}{2}}(x_{k+1}-\hat{x})\|^2 - \|A^{-\frac{1}{2}}(x_k-\hat{x})\|^2 \\ \leq\ & \frac{5}{4}\bar{C}^2(1+\bar{C}\rho)^2\alpha_k\|A^{-\frac{1}{2}}(x_k-\hat{x})\|^2\|F'(x_0)(x_k-\hat{x})\|^2. \end{aligned} \tag{3.3.9}$$

Now, since $\bar{C}\rho \leq \min\{\sqrt{5}-2, r_0\} \leq 0.0740$, we have by (3.3.9)

$$\begin{aligned} & \|A^{-\frac{1}{2}}(x_{k+1}-\hat{x})\|^2 - \|A^{-\frac{1}{2}}(x_k-\hat{x})\|^2 \\ \leq\ & 1.4418\bar{C}^2\alpha_k\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2\|A^{-\frac{1}{2}}(x_k-\hat{x})\|^2. \end{aligned}$$

Set $z_k = \|A^{-\frac{1}{2}}(x_k-\hat{x})\|$. Then

$$z_{k+1}^2 \leq \left(1 + 1.4418\bar{C}^2\alpha_k\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2\right) z_k^2.$$

By induction

$$z_k^2 \leq \prod_{i=0}^{k-1}\left(1 + 1.4418\bar{C}^2\alpha_i\|A^{\frac{1}{2}}(x_i-\hat{x})\|^2\right) z_0^2. \tag{3.3.10}$$

The convergence of $\prod_{i=0}^{\infty}\left(1 + 1.4418\bar{C}^2\alpha_i\|A^{\frac{1}{2}}(x_i-\hat{x})\|^2\right)$ follows from the convergence of $\sum_{i=0}^{\infty}\alpha_i\|A^{\frac{1}{2}}(x_i-\hat{x})\|^2$. By Lemma 3.3.1, $\sum_{i=0}^{\infty}\alpha_i\|A^{\frac{1}{2}}(x_i-\hat{x})\|^2 < \infty$. Therefore, there exists $M > 0$ such that $\sum_{i=0}^{k-1}\bar{C}^2\alpha_i\|A^{\frac{1}{2}}(x_i-\hat{x})\|^2 < M$, which implies that

$$\begin{aligned} \prod_{i=0}^{k-1}(1 + 1.4418\bar{C}^2\alpha_i\|A^{\frac{1}{2}}(x_i-\hat{x})\|^2) &= e^{\sum_{i=0}^{k-1}\ln\left(1+1.4418\bar{C}^2\alpha_i\|A^{\frac{1}{2}}(x_i-\hat{x})\|^2\right)} \\ &\leq e^{1.4418M}. \end{aligned} \tag{3.3.11}$$

From (3.3.10) and (3.3.11), we have $z_k^2 \leq e^{1.4418M} z_0^2$. Since by $(\mathcal{C}_3)$, $z_0 = \|A^{\frac{1}{2}}(x_0 - \hat{x})\| = \|v\|$. So we have

$$z_k^2 \leq e^{1.4418M}\|v\|^2. \tag{3.3.12}$$

This completes the proof.

$\square$

REMARK 3.3.3. *Note that, in (3.3.8), one can split $-2\|x_k - \hat{x}\|^2$ into two parts, say $-c\|x_k - \hat{x}\|^2$ and $(c-2)\|x_k - \hat{x}\|^2$ such that $\frac{(1+\bar{C}\rho)^2}{1-4\bar{C}\rho-\bar{C}^2\rho^2} + 2\bar{C}\rho \leq 2 - c$. In this way one can choose a larger ρ. We choose $c = \frac{1}{5}$ for our convenience.*

THEOREM 3.3.4. *Let $(\mathcal{C})$ conditions hold and $\bar{C}\rho < \min\left\{\sqrt{5} - 2, r_0\right\} = 0.0740$. Let (x_k) be as in (3.2.4) or (3.2.5). Then*

$$\|x_k - \hat{x}\| \leq \tilde{C}k^{-1/2}$$

where $\tilde{C} = \sqrt{3e^{1.4418M}}\epsilon^{-1/2}\|v\|$.

Proof: Observe that $\alpha_k \geq \|F'(x_0)\|^{-2}$. So for $\epsilon_k := \epsilon = 0.6985\|F'(x_0)\|^{-2}$, we have, from Lemma 3.3.1, the conditions $(\mathcal{C})$ and $\bar{C}\rho \leq 0.0740$;

$$\begin{aligned}
\|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2 &\geq \left(1 - 4\bar{C}\rho - \bar{C}^2\rho^2\right)\alpha_k\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\
&\geq 0.6985\|F'(x_0)\|^{-2}\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\
&= \epsilon\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\
&= \epsilon\|F'(x_0)(x_k - \hat{x})\|^2 \\
&= \epsilon\langle A(x_k - \hat{x}), x_k - \hat{x}\rangle.
\end{aligned}$$

An application of (3.3.1), yields

$$\begin{aligned}
\|x_k - \hat{x}\| &\leq \sqrt{3}\|A^{-\frac{1}{2}}(x_k - \hat{x})\|^{2/3}\epsilon^{-1/2}\left[\sum_{i=0}^{k-1}\|A^{-\frac{1}{2}}(x_i - \hat{x})\|^{-2/3}\right]^{-1/2} \\
&= \sqrt{3}z_k^{2/3}\epsilon^{-1/2}\left[\sum_{i=0}^{k-1} z_i^{-2/3}\right]^{-1/2}.
\end{aligned} \tag{3.3.13}$$

So by (3.3.12) and (3.3.13), we have

$$\begin{aligned}\|x_k - \hat{x}\| &\leq \sqrt{3e^{1.4418M}}\epsilon^{-1/2}k^{-1/2}\|v\| \\ &\leq \tilde{C}k^{-1/2}. \end{aligned} \tag{3.3.14}$$

This completes the proof.

□

3.4 Convergence analysis of MSDM and MMEM with noisy data

In this Section, we consider methods MSDM and MMEM with noisy data y^δ instead of y. As already mentioned in the Chapter 1, we assume that

$$\|y - y^\delta\| \leq \delta.$$

Precisely, we define:

$$\begin{aligned} x_{k+1}^\delta &= x_k^\delta + \alpha_k^\delta s_k^\delta \quad (k = 0, 1, 2, \ldots) \\ s_k^\delta &= -F'(x_0)^*(F(x_k^\delta) - y^\delta) \\ \alpha_k^\delta &= \frac{\|s_k^\delta\|^2}{\|F'(x_0)s_k^\delta\|^2} \end{aligned} \tag{3.4.1}$$

and

$$\begin{aligned} x_{k+1}^\delta &= x_k^\delta + \alpha_k^\delta s_k^\delta \quad (k = 0, 1, 2, \ldots) \\ s_k^\delta &= -F'(x_0)^*(F(x_k^\delta) - y^\delta) \\ \alpha_k^\delta &= \frac{\|F(x_k^\delta) - y^\delta\|^2}{\|s_k^\delta\|^2}, \end{aligned} \tag{3.4.2}$$

instead of x_k in (3.2.4) and (3.2.5), respectively. We will use the following assumption together with the assumptions $(\mathcal{C})$:
$(\mathcal{C}_4)$ F satisfies the local property

$$\|F(u) - F(v) - F'(x_0)(u - v)\| \leq \eta\|F(u) - F(v)\| \tag{3.4.3}$$

for all $u, v \in B(x_0, \rho)$ with $\max\{\frac{1}{2} - \frac{\|F'(x_0)\|^2}{2m^2}, 0\} < \eta < \frac{1}{2}$. It was shown in Kammerer and Nashed (1971, 1972); King (1989); Lardy (1990); Louis (1987) (for linear ill-posed problems) that the steepest descent method converges in the case of exact data, but due to the instability of the steepest method it is impossible to use a-priori parameter choice strategies as stopping criteria. Therefore, a-posteriori strategy is used in the literature (Scherzer (1996)) for stopping (3.4.1) and (3.4.2), but no error estimate for $\|x_k^\delta - \hat{x}\|$ was given (as far as the authors are known). In this Section, we propose a discrepancy principle for method (3.4.1) and (3.4.2).

3.4.1 Discrepancy Principle

PROPOSITION 3.4.1. *Let $(\mathcal{C})$ conditions hold. Let (x_k^δ) be as in (3.4.1) or (3.4.2). Then, $x_k^\delta \in B(x_0, 2\rho) \subset D(F)$ for all $k = 0, 1, 2, \ldots$ and if*

$$\|F(x_k^\delta) - y^\delta\| > \tau\delta \tag{3.4.4}$$

where

$$\tau > 2\frac{(1+\eta)}{1-2\eta} > 2, \tag{3.4.5}$$

then, for all $0 \le k < k_*$ with τ as in (3.4.5), we have

$$k_*(\tau\delta)^2 \le \sum_{k=0}^{k_*-1} \|F(x_k^\delta) - y^\delta\|^2 \le \frac{\tau\|F'(x_0)\|^2}{(1-2\eta)\tau - 2(1+\eta)}\|x_0 - \hat{x}\|^2. \tag{3.4.6}$$

Proof: Note that $x_0 \in B(x_0, 2\rho)$. Suppose $x_k^\delta \in B(x_0, 2\rho)$. Using (3.4.1) or (3.4.2), we have

$$\begin{aligned}
& \|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \\
= \;& -2\alpha_k^\delta\langle x_k^\delta - \hat{x}, F'(x_0)^*(F(x_k^\delta) - y^\delta)\rangle + {\alpha_k^\delta}^2\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 \\
= \;& 2\alpha_k^\delta\langle F(x_k^\delta) - y^\delta - F'(x_0)(x_k^\delta - \hat{x}), F(x_k^\delta) - y^\delta\rangle \\
& +\alpha_k^\delta\left[\alpha_k^\delta\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2\right] \\
\le \;& 2\alpha_k^\delta\|F(x_k^\delta) - F(\hat{x}) + y - y^\delta - F'(x_0)(x_k^\delta - \hat{x})\|\|F(x_k^\delta) - y^\delta\| \\
& +\alpha_k^\delta\left[\alpha_k^\delta\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2\right].
\end{aligned} \tag{3.4.7}$$

So by $(\mathcal{C}_4)$, we have by (3.4.7)

$$\begin{aligned}
& \|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \\
\leq\ & 2\alpha_k^\delta \left(\eta\|F(x_k^\delta) - F(\hat{x})\| + \delta\right) \|F(x_k^\delta) - y^\delta\| \\
& + \alpha_k^\delta \left[\alpha_k^\delta \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2\right] \\
\leq\ & 2\alpha_k^\delta \left[\eta\|F(x_k^\delta) - y^\delta\| + (1+\eta)\delta\right] \|F(x_k^\delta) - y^\delta\| \\
& + \alpha_k^\delta \left[\alpha_k^\delta \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2\right] \\
=\ & \alpha_k^\delta (2\eta - 1)\|F(x_k^\delta) - y^\delta\|^2 + \alpha_k^\delta 2(1+\eta)\delta\|F(x_k^\delta) - y^\delta\| \\
& + \alpha_k^\delta \left[\alpha_k^\delta \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - \|F(x_k^\delta) - y^\delta\|^2\right].
\end{aligned}$$

In both methods, i.e., (3.4.1) and (3.4.2),

$$\alpha_k^\delta \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 \leq \|F(x_k^\delta) - y^\delta\|^2.$$

Therefore, we have

$$\begin{aligned}
& \|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \\
\leq\ & \alpha_k^\delta \left[(2\eta - 1)\|F(x_k^\delta) - y^\delta\|^2 + 2(1+\eta)\delta\|F(x_k^\delta) - y^\delta\|\right],
\end{aligned} \tag{3.4.8}$$

so, by (3.4.4), we have

$$\begin{aligned}
& \|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \\
\leq\ & \alpha_k^\delta \left((2\eta - 1) + 2\frac{(1+\eta)}{\tau}\right) \|F(x_k^\delta) - y^\delta\|^2 < 0.
\end{aligned} \tag{3.4.9}$$

This implies $\|x_{k+1}^\delta - \hat{x}\| < \|x_k^\delta - \hat{x}\| < \|x_0 - \hat{x}\| < \rho$. Thus $\|x_{k+1}^\delta - x_0\| \leq \|x_{k+1}^\delta - \hat{x}\| + \|x_0 - \hat{x}\| < 2\rho$ i.e., $x_{k+1}^\delta \in B(x_0, 2\rho) \subset D(F)$ for all $k = 0, 1, 2, \ldots$.

Now since $\alpha_k^\delta \geq \|F'(x_0)\|^{-2}$, we have by (3.4.9)

$$\begin{aligned}
& \|F'(x_0)\|^{-2} \left((1 - 2\eta) - 2\frac{(1+\eta)}{\tau}\right) \|F(x_k^\delta) - y^\delta\|^2 \\
\leq\ & \|x_k^\delta - \hat{x}\|^2 - \|x_{k+1}^\delta - \hat{x}\|^2.
\end{aligned} \tag{3.4.10}$$

Adding the inequality (3.4.10) for k from 0 through $k_* - 1$, we obtain

$$\|F'(x_0)\|^{-2} \left((1 - 2\eta) - 2\frac{(1+\eta)}{\tau}\right) \sum_{k=0}^{k_*-1} \|F(x_k^\delta) - y^\delta\|^2 \leq \|x_0 - \hat{x}\|^2 - \|x_{k_*}^\delta - \hat{x}\|^2. \tag{3.4.11}$$

This completes the proof.

□

REMARK 3.4.2. *Note that (3.4.11) implies, for $y^\delta \neq y$, there must be a unique index k_* such that (3.4.4) holds for all $k < k_*$ but is violated at $k = k_*$ [see also (Engl et al., 1996, page 282)].*

Let $\Omega := \|F'(x_0)\|^{-2}\left((1-2\eta) - 2\frac{(1+\eta)}{\tau}\right)$ Let $\Omega := \|F'(x_0)\|^{-2}\left((1-2\eta) - 2\frac{(1+\eta)}{\tau}\right)$
and

$$q = 1 - \Omega\left(\frac{m}{\tau-1}\right)^2. \tag{3.4.12}$$

Now, we shall prove that $q < 1$ for $\tau > 2$. Note that, to prove $q < 1$, it is enough to prove that

$$\Omega\left(\frac{m}{\tau-1}\right)^2 = \|F'(x_0)\|^{-2}\left((1-2\eta) - 2\frac{(1+\eta)}{\tau}\right)\left(\frac{m}{\tau-1}\right)^2 < 1,$$

for $\tau > 2$. That is to prove that

$$p(\tau) := \tau^3 - 2\tau^2 + (1 - \|F'(x_0)\|^{-2}(1-2\eta)m^2)\tau + 2(1+\eta)m^2\|F'(x_0)\|^{-2} > 0,$$

for $\tau > 2$. This follows from the condition $\eta > \frac{1}{2} - \frac{\|F'(x_0)\|^2}{2m^2}$.

THEOREM 3.4.3. *Let ($\mathcal{C}$) conditions hold and $\rho < \min\left\{\frac{(\tau-1)^2\delta}{m}, \frac{2}{m\sqrt{\Omega}}\right\}$. Let x_{k+1}^δ be as in (3.4.1) or in (3.4.2). Then for $0 \leq k < k_*$,*

$$\|x_{k+1}^\delta - \hat{x}\| = \begin{cases} O(q^{\frac{k+1}{2}}) & if \quad \delta < q^{k+1} \\ O(\delta^{\frac{1}{2}}) & if \quad q^{k+1} \leq \delta \end{cases} \tag{3.4.13}$$

where $q := 1 - \frac{\Omega m^2}{(\tau-1)^2}$.

Proof: By the definition of k_*, we have for $k \leq k_*$;

$$\begin{aligned} \tau\delta &< \|F(x_k^\delta) - y^\delta\| \\ &\leq \|F(x_k^\delta) - F(\hat{x})\| + \|y - y^\delta\|. \end{aligned} \tag{3.4.14}$$

So, we have

$$\|F(x_k^\delta) - F(\hat{x})\| > (\tau-1)\delta. \tag{3.4.15}$$

Again by (3.4.14), we have

$$\begin{aligned}\tau\delta &< \|F(x_k^\delta) - y^\delta\| \\ &< \|\int_0^1 F'(\hat{x} + \theta(x_k^\delta - \hat{x}))d\theta(x_k^\delta - \hat{x})\| + \delta \\ &\leq m\|x_k^\delta - \hat{x}\| + \delta,\end{aligned}$$

i.e.,

$$\delta < \frac{m\|x_k^\delta - \hat{x}\|}{\tau - 1}. \tag{3.4.16}$$

Thus, by (3.4.15) and (3.4.16), we have

$$\begin{aligned}\|F(x_k^\delta) - y^\delta\| &\geq \|F(x_k^\delta) - F(\hat{x})\| - \delta \\ &\geq (\tau - 1)\delta - \frac{m\|x_k^\delta - \hat{x}\|}{\tau - 1} \\ &\geq (\tau - 1)\delta - \frac{m\rho}{\tau - 1} > 0.\end{aligned} \tag{3.4.17}$$

Thus by (3.4.17), we have

$$\begin{aligned}\|F(x_k^\delta) - y^\delta\|^2 \;\geq\; & (\tau - 1)^2\delta^2 + \left(\frac{m\|x_k^\delta - \hat{x}\|}{\tau - 1}\right)^2 \\ & -2\delta m\|x_k^\delta - \hat{x}\|.\end{aligned} \tag{3.4.18}$$

So by (3.4.18) and (3.4.10), we have

$$\begin{aligned}\|x_{k+1}^\delta - \hat{x}\|^2 \;\leq\; & \left(1 - \Omega\left(\frac{m}{\tau - 1}\right)^2\right)\|x_k^\delta - \hat{x}\|^2 \\ & -\Omega(\tau - 1)^2\delta^2 + 2\Omega\delta m\|x_k^\delta - \hat{x} \\ \leq\; & \left(1 - \Omega\left(\frac{m}{\tau - 1}\right)^2\right)\|x_k^\delta - \hat{x}\|^2 \\ & -\Omega(\tau - 1)^2\delta^2 + 2\Omega\delta m\rho \\ \leq\; & \left(1 - \Omega\left(\frac{m}{\tau - 1}\right)^2\right)\|x_k^\delta - \hat{x}\|^2 \\ & +2\Omega\delta m\rho.\end{aligned} \tag{3.4.19}$$

Therefore we have,

$$\|x_{k+1}^\delta - \hat{x}\|^2 \leq q\|x_k^\delta - \hat{x}\|^2 + L\delta$$

where $L = 2\Omega m\rho$. Then,

$$\begin{aligned} \|x_{k+1}^\delta - \hat{x}\|^2 &\leq q^{k+1}\|x_0^\delta - \hat{x}\|^2 + q^k L\delta + ... + qL\delta + L\delta \\ &\leq q^{k+1}\rho^2 + \frac{L\delta}{1-q}. \end{aligned} \tag{3.4.20}$$

This completes the proof.

□

3.4.2 Convergence rate result for steepest descent method and minimal error method with noisy data

In this Section, we considered the steepest descent method and minimal error method with noisy data and obtained a convergence rate result which is not available in the literature. The steepest descent method and minimal error method with noisy data are defined by

$$\begin{aligned} x_{k+1}^\delta &= x_k^\delta + \alpha_k^\delta s_k^\delta \;\; (k = 0, 1, 2, \ldots) \\ s_k^\delta &= -F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta) \\ \alpha_k^\delta &= \frac{\|s_k^\delta\|^2}{\|F'(x_k^\delta)s_k^\delta\|^2} \end{aligned} \tag{3.4.21}$$

and

$$\begin{aligned} x_{k+1}^\delta &= x_k^\delta + \alpha_k^\delta s_k^\delta \;\; (k = 0, 1, 2, \ldots) \\ s_k^\delta &= -F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta) \\ \alpha_k^\delta &= \frac{\|F(x_k^\delta) - y^\delta\|^2}{\|s_k^\delta\|^2}, \end{aligned} \tag{3.4.22}$$

respectively. We have the following convergence rate result.

THEOREM 3.4.4. *Let ($\mathcal{C}$) conditions hold and* $\rho < \min\left\{\frac{(\tau-1)^2\delta}{m}, \frac{2}{m\sqrt{\Omega}}\right\}$. *Let* x_{k+1}^δ *be as in (3.4.21) or (3.4.22). Then for* $0 \leq k < k_*$,

$$\|x_{k+1}^\delta - \hat{x}\| = \begin{cases} O(q^{\frac{k+1}{2}}) & if \quad \delta < q^{k+1} \\ O(\delta^{\frac{1}{2}}) & if \quad q^{k+1} \leq \delta \end{cases} \tag{3.4.23}$$

where

$$q := 1 - \frac{\omega m^2}{(\tau - 1)^2}.$$

with

$$\omega := \|F'(x_k^\delta)\|^{-2}\left((1-2\eta) - 2\frac{(1+\eta)}{\tau}\right).$$

Proof: Simply follow the proof of Theorem 3.4.3.

□

3.5 Example

In this Section, we implement MSDM and MMEM for both noisy data and noisy free data through below example.

EXAMPLE 3.5.1. *(cf. Hoang and Ramm (2010)) Consider a nonlinear operator equation* $F : L^2[0,1] \to L^2[0,1]$ *defined by*

$$F(x) := (\arctan(x))^2. \tag{3.5.24}$$

The Fréchet derivative of F is

$$F'(x)w = \frac{2arctan(x)}{1+x^2}w.$$

If $x(t)$ vanishes on a set of positive Lebesgue measure, then $F'(x)$ is not boundedly invertible. If $x \in C[0,1]$ vanishes even at one point t_0, then $F'(x)$ is not boundedly invertible in $L^2[0,1]$.

Note that

$$F'(\hat{x})w = F'(x_0)G(\hat{x}, x_0)w,$$

and

$$F'(x)w = R(x, x_0)F'(x_0)w$$

with

$$G(\hat{x}, x_0) = \frac{1 + x_0^2}{1 + \hat{x}^2} \frac{arctan(\hat{x})}{arctan(x_0)}$$

and

$$R(x, x_0) = \frac{1 + x_0^2}{1 + x^2} \frac{arctan(x)}{arctan(x_0)},$$

respectively. Further, for $x_0 \neq 0$,

$$\|G(\hat{x}, x_0) - I\| \leq \left[\frac{1}{\|arctan(x_0)\|} + 2\max\{\|\hat{x}\|, \|x_0\|\} \right] \|\hat{x} - x_0\|$$

and

$$\|R(x, x_0) - I\| \leq \left[\frac{1}{\|arctan(x_0)\|} + 2\max\{\|x\|, \|x_0\|\} \right] \|x - x_0\|.$$

That is, assumptions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ are satisfied. Let $\hat{x}(t) = t, t \in [0, 1]$ and $y(t) = arctan(t)^2$. Initial guess $x_0(t) = t/2$ and $\tau = 2.1$. For nosiy free data error estimates are given in Table 3.1 and approximate of solutions are given in Figure 3.1 and Figure 3.6. For noisy data, error estimates are given in 3.2 with different values of δ. Approximate solutions are given in Figures 3.2-3.5 for MSDM and in Figure 3.7-3.10 for MMEM.

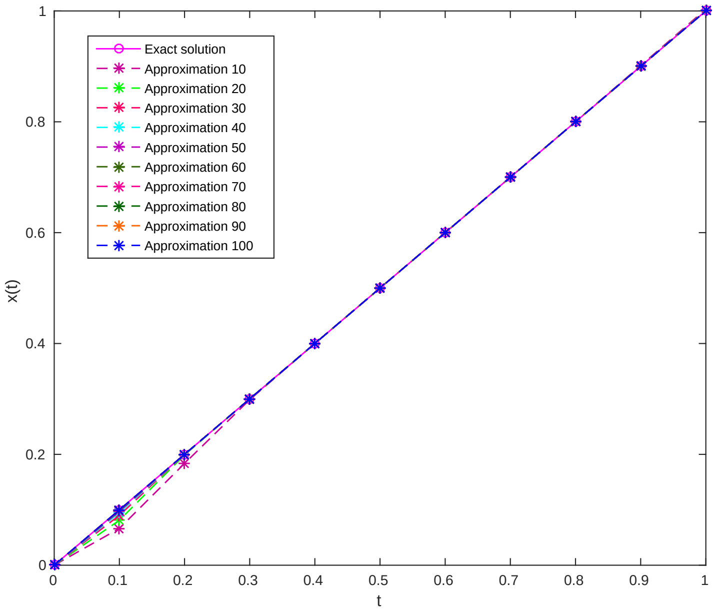

Figure 3.1: Approximate solutions of MSDM with exact data

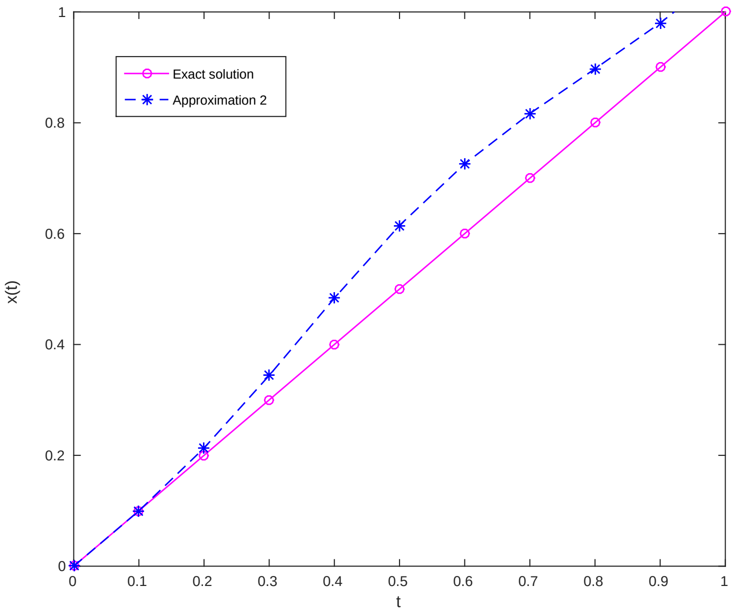

Figure 3.2: Approximate solution of MSDM with $\delta = 0.1$

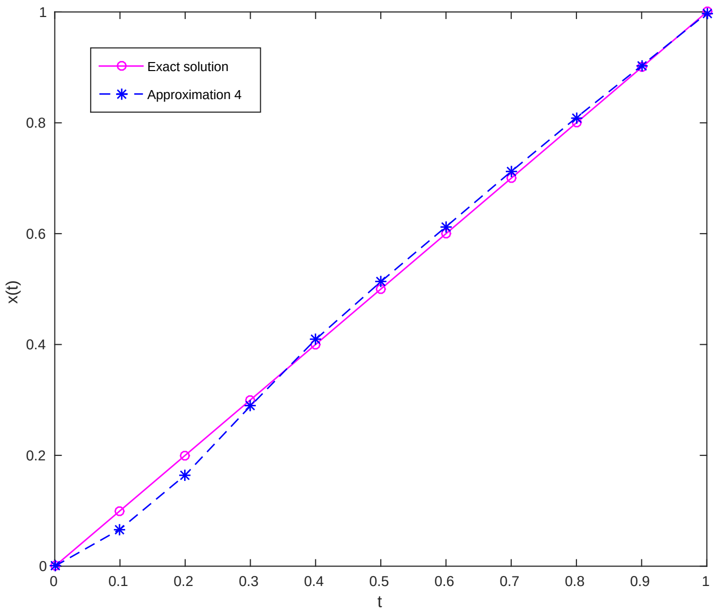

Figure 3.3: Approximate solution of MSDM with $\delta = 0.01$

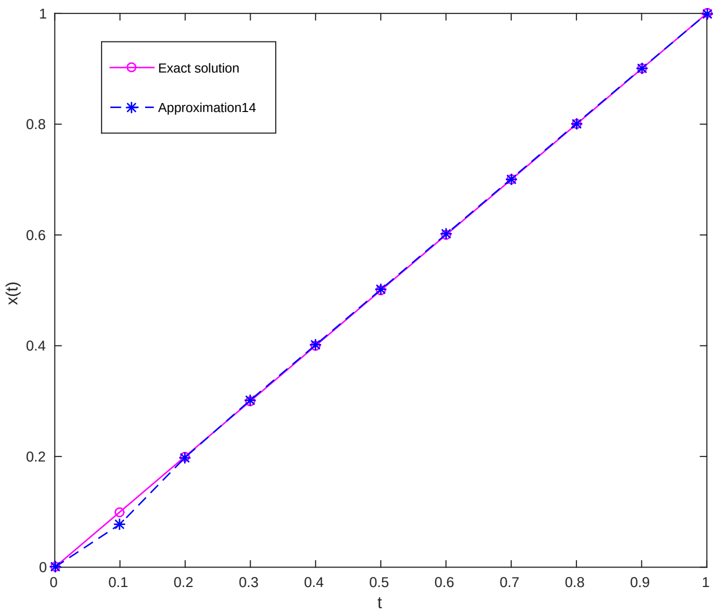

Figure 3.4: Approximate solution of MSDM with $\delta = 0.001$

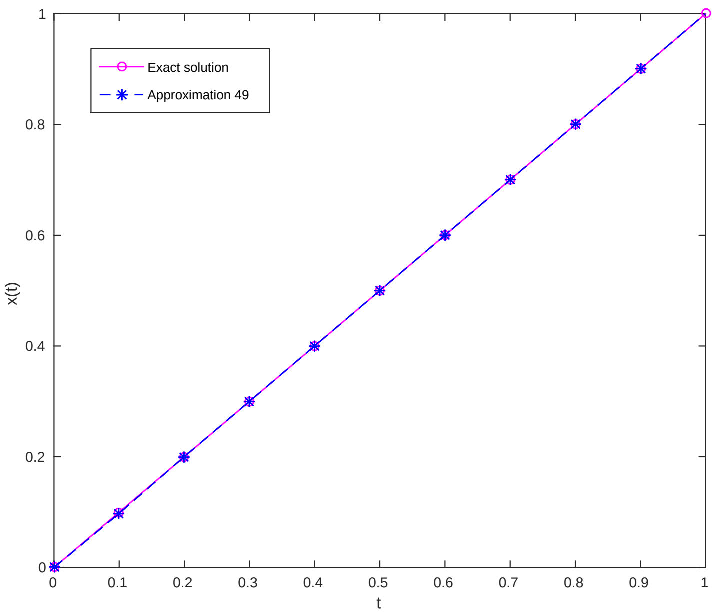

Figure 3.5: Approximate solution of MSDM with $\delta = 0.0001$

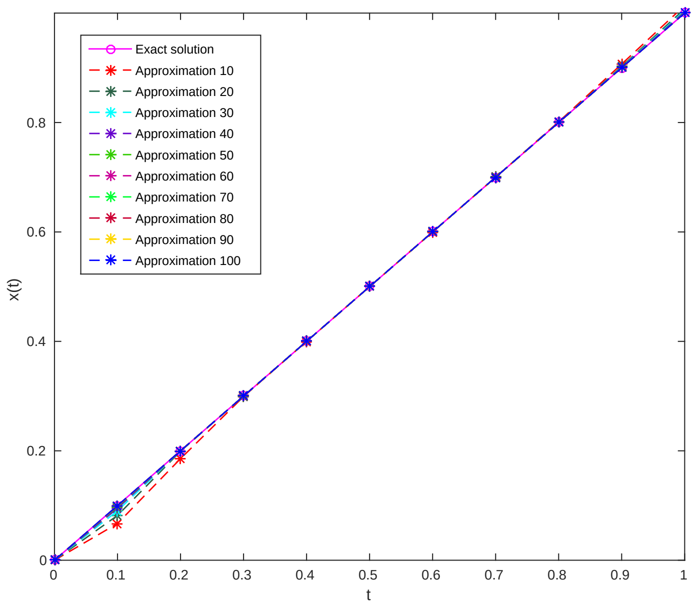

Figure 3.6: Approximate solutions of MMEM for exact data

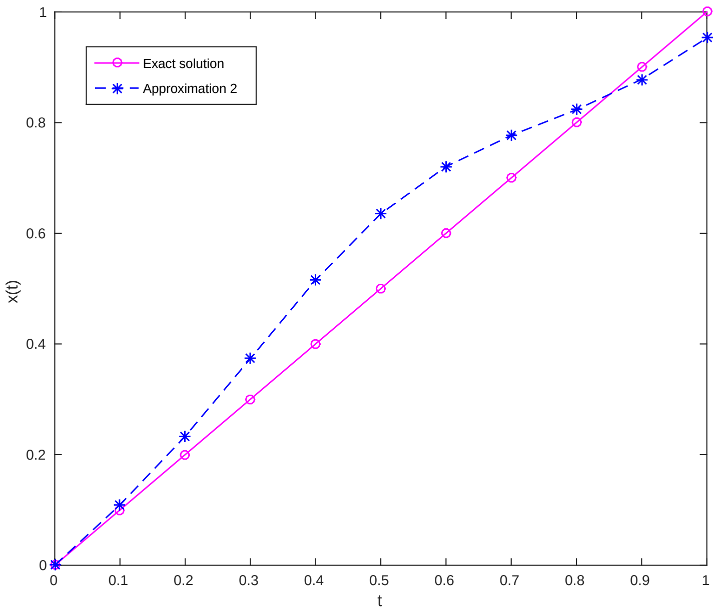

Figure 3.7: Approximate solution of MMEM with $\delta = 0.1$

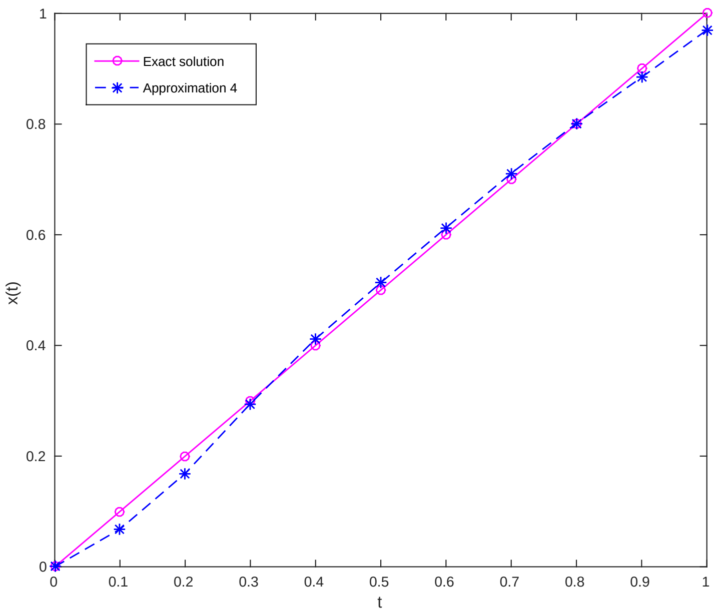

Figure 3.8: Approximate solution of MMEM with $\delta = 0.01$

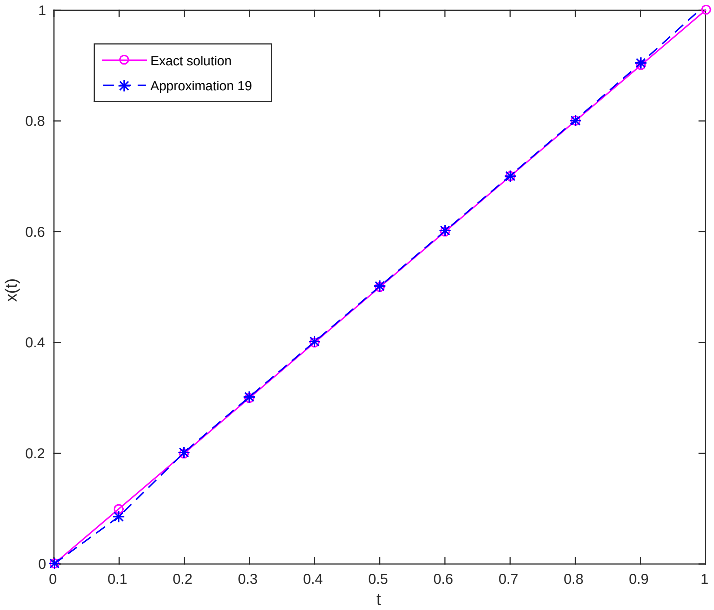

Figure 3.9: Approximate solution of MMEM with $\delta = 0.001$

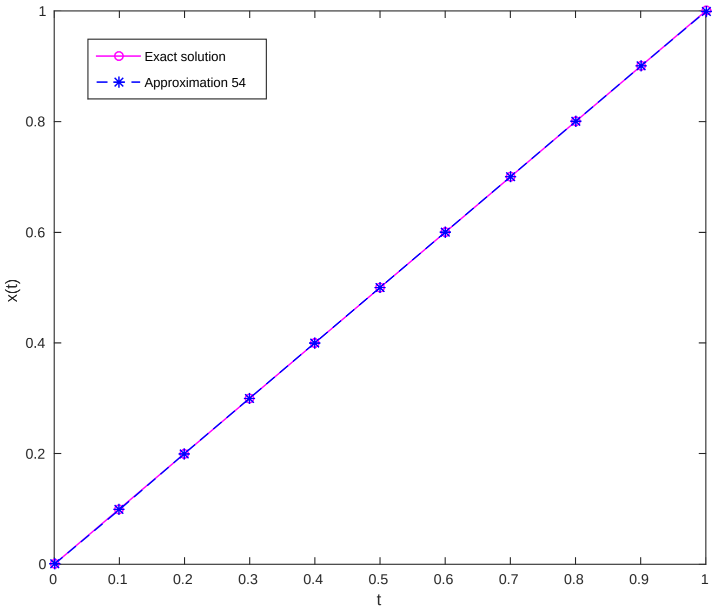

Figure 3.10: Approximate solution of MMEM with $\delta = 0.0001$

Table 3.1: Error estimate for MSDM and MMEM with exact data

k	MSDM		MMEM	
	$\|x_k^\delta - \hat{x}\|$	$\frac{\|x_k^\delta - \hat{x}\|}{\sqrt{k}}$	$\|x_k^\delta - \hat{x}\|$	$\frac{\|x_k^\delta - \hat{x}\|}{\sqrt{k}}$
10	1.3031E-02	4.1208E-03	1.3044E-02	4.1248E-03
20	6.9204E-03	1.5474E-03	6.8454E-03	1.5307E-03
30	3.6819E-03	6.7222E-04	3.6393E-03	6.6445E-04
40	1.9186E-03	3.0336E-04	1.8962E-03	2.9981E-04
50	9.8853E-04	1.3980E-04	9.7693E-04	1.3816E-04
60	5.0629E-04	6.5362E-05	5.0036E-04	6.4596E-05
70	2.5850E-04	3.0897E-05	2.5548E-04	3.0535E-05
80	1.3177E-04	1.4733E-05	1.3023E-04	1.4561E-05
90	6.7118E-05	7.0748E-06	6.6334E-05	6.9923E-06
100	3.4172E-05	3.4172E-06	3.3773E-05	3.3773E-06

Table 3.2: Error estimate for MSDM and MMEM with noisy data

δ	MSDM			MMEM		
	k	$\|x_k^\delta - \hat{x}\|$	$\frac{\|x_k^\delta - \hat{x}\|}{\sqrt{\delta}}$	k	$\|x_k^\delta - \hat{x}\|$	$\frac{\|x_k^\delta - \hat{x}\|}{\sqrt{\delta}}$
0.1	2	1.5130E-01	4.7845E-01	2	2.0772E-01	1.4688E-01
0.01	4	2.7174E-02	2.7174E-01	4	3.4426E-02	1.7213E-02
0.001	14	8.9047E-03	2.8159E-01	19	5.9390E-03	1.3625E-03
0.0001	49	8.7804E-04	8.7804E-02	54	5.9459E-04	8.0914E-05

Chapter 4

ERROR ESTIMATES FOR MSDM AND MMEM UNDER A GENERAL HÖLDER-TYPE SOURCE CONDITION

An error estimate for steepest descent and minimal error method for nonlinear ill-posed problems, under a general Hölder-type source condition is not known. We consider modified form of a steepest descent method (MFSDM) and modified form of minimal error method (MFMEM) for nonlinear ill-posed problems. Using a general Hölder-type source condition, we obtained an error estimate. We also consider the methods MFSDM and MFMEM with noisy data and provide an error estimate. A numerical example is also provided.

4.1 Introduction

In Chapter 3, we have studied modified steepest descent method:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \ \ (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|s_k\|^2}{\|F'(x_0)s_k\|^2} \end{aligned} \tag{4.1.1}$$

and modified minimal error method:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \ \ (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|F(x_k) - y\|^2}{\|s_k\|^2}. \end{aligned} \tag{4.1.2}$$

As far as the authors are known, for a steepest descent method and a minimal error method, no error estimate is known under a general Hölder-type source condition

$$x_0 - \hat{x} = (F'(\hat{x})^* F'(\hat{x}))^\nu v \tag{4.1.3}$$

or

$$x_0 - \hat{x} = (F'(x_0)^* F'(x_0))^\nu v \tag{4.1.4}$$

for $\nu \neq \frac{1}{2}$. Let x_0 is the initial guess such that $\|x_0 - \hat{x}\| < \rho$. In order to obtain error estimate under the general source condition (4.1.4), we consider the following modified form of steepest descent method:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|s_k\|^2}{\|A^q s_k\|^2} \end{aligned} \tag{4.1.5}$$

and the following modified form of minimal error method:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \ \ (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|F(x_k) - y\|^2}{\|A^q (F(x_k) - y)\|^2} \end{aligned} \tag{4.1.6}$$

where $A = F'(x_0)^* F'(x_0)$ and $0 < q < \frac{1}{2}$.

REMARK 4.1.1.

(a) For $q = \frac{1}{2}$, the methods (4.1.5) and (4.1.6) are reduced to modified steepest descent method and modified minimal error method considered in Chapter 3, but proof in this paper cannot be applied for the method considered in Chapter 3.

(b) Note that, for q close to zero, in MFMEM, ν is close to $\frac{1}{2}$, i.e., we obtained the error estimate $O(k^{-\nu})$, for $0 < \nu < \frac{1}{2}$ (see Theorem 4.2.4) and in MFSDM, ν is close to $\frac{1}{4}$, i.e., we obtained the error estimate $O(k^{-\nu})$ for $0 < \nu < \frac{1}{4}$ (see Theorem 4.2.6).

The rest of the Chapter is organized as follows. Convergence analysis of methods MFSDM and MFMEM is given in Section 4.2 and convergence rate result of methods MFSDM and MFMEM with noisy data is given in Section 4.3. Numerical example is given in Section 4.4.

4.2 Convergence analysis of MFSDM and MFMEM

To obtain an error estimate for $\|x_k - \hat{x}\|$ under the assumption (4.1.4), we use (3.3.1). To apply (3.3.1) with $v_k = A^{-\nu}(x_k - \hat{x})$, one has to prove that

$$\|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2 \geq \varepsilon_k \langle A(x_k - \hat{x}), x_k - \hat{x}\rangle \tag{4.2.1}$$

for some $\varepsilon_k > 0$ and $\|A^{-\nu}(x_k - \hat{x})\|$ is bounded.

Let $B = \|A^{\frac{1}{2}-q}\| < \sqrt{2}$ and $D = \frac{\sqrt{1+4B^2}-(B^2+1)}{B^2}$.

LEMMA 4.2.1. *Let (x_k) be as in (4.1.5) or in (4.1.6). If assumption $(\mathcal{C}_2)$ holds and $0 < C_1\rho < D$, then $x_k \in B(x_0, 2\rho)$ and*

$$\|x_{k+1} - \hat{x}\|^2 + \alpha_k \Gamma \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \leq \|x_k - \hat{x}\|^2 \tag{4.2.2}$$

with

$$\Gamma = 2 - (B^2C_1^2\rho^2 + 2(B^2+1)C_1\rho + B^2), \tag{4.2.3}$$

for all $k = 0, 1, 2, \ldots$. Moreover,

$$\sum_{k=0}^{\infty} \alpha_k \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 < \infty.$$

Proof: We shall prove the result using induction. Note that $x_0 \in B(x_0, 2\rho)$ and suppose $x_k \in B(x_0, 2\rho)$. Then using (4.1.5) or (4.1.6), we have

$$\begin{aligned}
& \|x_{k+1} - \hat{x}\|^2 - \|x_k - \hat{x}\|^2 \\
= \; & -2\alpha_k \langle x_k - \hat{x}, F'(x_0)^*(F(x_k) - y)\rangle + \alpha_k^2 \|F'(x_0)^*(F(x_k) - y)\|^2 \\
= \; & -2\alpha_k \langle x_k - \hat{x}, F'(x_0)^*[F(x_k) - F(\hat{x}) - F'(x_0)(x_k - \hat{x})]\rangle \\
& + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\langle x_k - \hat{x}, F'(x_0)^* F'(x_0)(x_k - \hat{x})\rangle] \\
= \; & -2\alpha_k \langle F'(x_0)(x_k - \hat{x}), \int_0^1 (F'(\hat{x} + t(x_k - \hat{x})) - F'(x_0)) dt (x_k - \hat{x})\rangle \\
& + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2]. \qquad (4.2.4)
\end{aligned}$$

So by $(\mathcal{C}_2)$, we have

$$\begin{aligned}
& \|x_{k+1} - \hat{x}\|^2 - \|x_k - \hat{x}\|^2 \\
= \; & -2\alpha_k \langle F'(x_0)(x_k - \hat{x}), \int_0^1 [R(\hat{x} + t(x_k - \hat{x}), x_0) - I] dt F'(x_0)(x_k - \hat{x})\rangle \\
& + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2] \\
\leq \; & 2\alpha_k \int_0^1 \|R(\hat{x} + t(x_k - \hat{x}), x_0) - I\| \|F'(x_0)(x_k - \hat{x})\|^2 dt \\
& + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2] \\
\leq \; & 2\alpha_k C_1 \|\hat{x} + t(x_k - \hat{x}) - x_0\| \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\
& + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2]. \qquad (4.2.5)
\end{aligned}$$

By the definition of α_k, we have by MFSDM

$$\begin{aligned}
\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 &= \frac{\langle A^q s_k, A^{-q} s_k\rangle^2}{\|A^q s_k\|^2} \\
&\leq \frac{\|A^q s_k\|^2 \|A^{-q} s_k\|^2}{\|A^q s_k\|^2} \\
&\leq \|A^{\frac{1}{2}-q}\|^2 \|F(x_k) - y\|^2
\end{aligned}$$

and by MFMEM

$$\begin{aligned}
\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 &= \alpha_k \|A^{\frac{1}{2}-q} A^q (F(x_k) - y)\|^2 \\
&\leq \|A^{\frac{1}{2}-q}\|^2 \|F(x_k) - y\|^2.
\end{aligned}$$

So in both methods MFSDM and MFMEM

$$\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 \leq \|A^{\frac{1}{2}-q}\|^2 \|F(x_k) - y\|^2.$$

Therefore,

$$\begin{aligned} & \alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 \\ = \; & B^2 \| \int_0^1 F'(\hat{x} + t(x_k - \hat{x}))dt(x_k - \hat{x})\|^2 \\ = \; & B^2 \| \int_0^1 [R(\hat{x} + t(x_k - \hat{x}), x_0) - I + I] dt F'(x_0)(x_k - \hat{x})\|^2 \\ \leq \; & B^2 \int_0^1 (C_1 \|\hat{x} + t(x_k - \hat{x}) - x_0\| + 1)^2 dt \|F'(x_0)(x_k - \hat{x})\|^2 \\ \leq \; & B^2 (C_1 \rho + 1)^2 \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2. \end{aligned} \tag{4.2.6}$$

Hence by (4.2.5) and (4.2.6) we have

$$\|x_{k+1} - \hat{x}\|^2 - \|x_k - \hat{x}\|^2 \leq -\Gamma \alpha_k \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2.$$

This completes the proof.

□

Next, we will prove the boundedness of $\|A^{-\nu}(x_k - \hat{x})\|$. Let $B_1 = \|A^{\frac{1}{2}-\nu-q}\|$ with $0 < \nu < \frac{1}{2} - q$ and $0 < q < \frac{1}{2}$. We have the following Lemma for MFMEM.

LEMMA 4.2.2. *Let (x_k) be as in (4.1.6). Assume that the assumptions $(\mathcal{C}_0)$ and $(\mathcal{C}_2)$ hold and $0 < C_1\rho < D$. If the source condition (4.1.4) holds with $0 < \nu < \frac{1}{2} - q, 0 < q < \frac{1}{2}$, then $\|A^{-\nu}(x_k - \hat{x})\|$ is bounded.*

Proof: By using (4.1.4), one can prove that, $x_k - \hat{x} \in R(A^\nu)$ for all $k = 0, 1, 2, \ldots$. So, we can apply $A^{-\nu}$ to $x_{k+1} - \hat{x}$ and $x_k - \hat{x}$, as in Lemma 3.3.2. Then, we have

$$\begin{aligned} & \|A^{-\nu}(x_{k+1} - \hat{x})\|^2 - \|A^{-\nu}(x_k - \hat{x})\|^2 \\ = \; & -2\alpha_k \langle A^{-\nu}(x_k - \hat{x}), A^{-\nu}F'(x_0)^*(F(x_k) - y)\rangle \\ & + \alpha_k^2 \|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|^2 \\ \leq \; & 2\alpha_k \|A^{-\nu}(x_k - \hat{x})\| \|A^{-\nu}F'(x_0)^*(F(x_k) - y)\| \\ & + \alpha_k^2 \|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|^2. \end{aligned}$$

i.e.,

$$\|A^{-\nu}(x_{k+1} - \hat{x})\| \le \|A^{-\nu}(x_k - \hat{x})\| + \alpha_k \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\|. \tag{4.2.7}$$

By the definition of α_k, we have

$$\begin{aligned}
& \alpha_k \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\|^2 \\
= \; & \alpha_k \|A^{\frac{1}{2}-\nu-q} A^q (F(x_k) - y)\|^2 \\
\le \; & \|A^{\frac{1}{2}-\nu-q}\|^2 \|F(x_k) - y\|^2 \\
= \; & \|A^{\frac{1}{2}-\nu-q}\|^2 \|\int_0^1 F'(\hat{x} + t(x_k - \hat{x})) dt (x_k - \hat{x})\|^2.
\end{aligned} \tag{4.2.8}$$

Using assumption $(\mathcal{C}_2)$ in (4.2.8), we get

$$\begin{aligned}
& \alpha_k \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\|^2 \\
= \; & \|A^{\frac{1}{2}-\nu-q}\|^2 \|\int_0^1 [R(\hat{x} + t(x_k - \hat{x}), x_0) - I + I] dt F'(x_0)(x_k - \hat{x})\|^2 \\
\le \; & \|A^{\frac{1}{2}-\nu-q}\|^2 \int_0^1 (C_1 \|\hat{x} + t(x_k - \hat{x}) - x_0\| + 1)^2 dt \|F'(x_0)(x_k - \hat{x})\|^2 \\
\le \; & B_1^2 (C_1 \rho + 1)^2 \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2
\end{aligned} \tag{4.2.9}$$

so,

$$\sqrt{\alpha_k} \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\| \le B_1 (C_1 \rho + 1) \|A^{\frac{1}{2}}(x_k - \hat{x})\|. \tag{4.2.10}$$

Therefore by (4.2.10) and (4.2.7), we have

$$\|A^{-\nu}(x_{k+1} - \hat{x})\| \le \|A^{-\nu}(x_k - \hat{x})\| + \sqrt{\alpha_k} B_1 (C_1 \rho + 1) \|A^{\frac{1}{2}}(x_k - \hat{x})\|. \tag{4.2.11}$$

Let $z_k = \|A^{-\nu}(x_k - \hat{x})\|$. Then by (4.2.11), we have

$$z_{k+1} \le z_k + B_1 (C_1 \rho + 1) \sqrt{\alpha_k} \|A^{\frac{1}{2}}(x_k - \hat{x})\|,$$

i.e., we have

$$z_k \le z_0 + B_1 (C_1 \rho + 1) \sum_{i=0}^{k-1} \sqrt{\alpha_i} \|A^{\frac{1}{2}}(x_i - \hat{x})\|. \tag{4.2.12}$$

By Lemma 4.2.1, we have

$$z_k \leq z_0 + B_1(C_1\rho + 1)M,$$

where M is such that

$$\sum_{k=0}^{\infty} \alpha_k \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \leq M^2.$$

Now since $z_0 = \|A^{-\nu}(x_0 - \hat{x})\| = \|A^{-\nu}A^{\nu}v\| = \|v\|$, we have

$$z_k \leq \|v\| + B_1(C_1\rho + 1)M. \tag{4.2.13}$$

This completes the proof.

□

Let $B_0 = \|A^{\nu}\|\|A^{\frac{1}{2}-2\nu-q}\|$ with $0 < 2\nu < \frac{1}{2} - q$ and $0 < q < \frac{1}{2}$. Then we have the following Lemma for MFSDM .

LEMMA 4.2.3. *Let (x_k) be as in (4.1.5). Assume that the assumptions $(\mathcal{C}_0)$ and $(\mathcal{C}_2)$ hold and $0 < C_1\rho < D$. If the source condition (4.1.4) holds with $0 < 2\nu < \frac{1}{2} - q, 0 < q < \frac{1}{2}$, then $\|A^{-\nu}(x_k - \hat{x})\|$ is bounded.*

Proof: Analogous to the proof of (4.2.7), one can prove

$$\|A^{-\nu}(x_{k+1} - \hat{x})\| \leq \|A^{-\nu}(x_k - \hat{x})\| + \alpha_k\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|. \tag{4.2.14}$$

By definition of α_k,

$$\begin{aligned}
\alpha_k\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|^2 &\leq \frac{\|A^{\nu}\|^2\|A^{-\nu}s_k\|^2}{\|A^q s_k\|^2}\|A^{-\nu}s_k\|^2 \\
&= \frac{\|A^{\nu}\|^2}{\|A^q s_k\|^2}\langle A^q s_k, A^{-2\nu-q}s_k\rangle^2 \\
&\leq \|A^{\nu}\|^2\|A^{\frac{1}{2}-2\nu-q}\|^2\|F(x_k) - y\|^2 \\
&\leq B_0^2\|F(x_k) - y\|^2. \qquad (4.2.15)
\end{aligned}$$

Using assumption $(\mathcal{C}_2)$ in (4.2.15), we get

$$\begin{aligned} & \alpha_k \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\|^2 \\ = \; & B_0^2 \| \int_0^1 [R(\hat{x} + t(x_k - \hat{x}), x_0) - I + I] \, dt F'(x_0)(x_k - \hat{x})\|^2 \\ \leq \; & B_0^2 \int_0^1 (C_1 \|\hat{x} + t(x_k - \hat{x}) - x_0\| + 1)^2 \, dt \|F'(x_0)(x_k - \hat{x})\|^2 \\ \leq \; & B_0^2 (C_1 \rho + 1)^2 \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \end{aligned}$$

so,

$$\sqrt{\alpha_k} \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\| \leq B_0 (C_1 \rho + 1) \|A^{\frac{1}{2}}(x_k - \hat{x})\|. \qquad (4.2.16)$$

Therefore by (4.2.16) and (4.2.14), we have

$$\|A^{-\nu}(x_{k+1} - \hat{x})\| \leq \|A^{-\nu}(x_k - \hat{x})\| + \sqrt{\alpha_k} B_0 (C_1 \rho + 1) \|A^{\frac{1}{2}}(x_k - \hat{x})\|. \qquad (4.2.17)$$

Let $z_k = \|A^{-\nu}(x_k - \hat{x})\|$. Then as in (4.2.13), one can prove that

$$z_k \leq \|v\| + B_0 (C_1 \rho + 1) M. \qquad (4.2.18)$$

This completes the proof.

□

By Lemma 4.2.2 and Lemma 4.2.3, $\|A^{-\nu}(x_k - \hat{x})\|$ is bounded if x_k is in MFMEM or MFSDM.

THEOREM 4.2.4. *Let (x_k) be as in (4.1.6). Assume that the assumptions $(\mathcal{C}_0)$ and $(\mathcal{C}_2)$ hold and $0 < C_1 \rho < D$. If the source condition (4.1.4) holds with $0 < \nu < \frac{1}{2} - q, 0 < q < \frac{1}{2}$, then*

$$\|x_k - \hat{x}\| \leq \tilde{C} k^{-\nu}$$

where $\tilde{C} = [2(\nu + 1)]^\nu \epsilon^{-\nu} (\|v\| + B_1 (C_1 \rho + 1) M)$.

Proof: Note that $\alpha_k \geq \|A^q\|^{-2}$. Since $(\mathcal{C}_2)$ and (4.1.4) for $0 < \nu < \frac{1}{2} - q$ hold and $C_1 \rho < D$, set $\epsilon_k := \epsilon = \Gamma \|A^q\|^{-2}$ where Γ is as in (4.2.3). Now Lemma 4.2.2

implies that,

$$\begin{aligned}
\|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2 &\geq \Gamma\alpha_k\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\
&\geq \Gamma\|A^q\|^{-2}\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\
&= \epsilon\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\
&= \epsilon\langle F'(x_0)^*F'(x_0)(x_k - \hat{x}), x_k - \hat{x}\rangle \\
&= \epsilon\langle A(x_k - \hat{x}), x_k - \hat{x}\rangle.
\end{aligned}$$

Therefore by (3.3.1), we have

$$\begin{aligned}
\|x_k - \hat{x}\| &\leq [2(\nu+1)]^\nu \|A^{-\nu}(x_k - \hat{x})\|^{\frac{1}{\nu+1}} \left[\sum_{i=0}^{k-1} \epsilon_i \|A^{-\nu}(x_i - \hat{x})\|^{\frac{-1}{\nu+1}}\right]^{-\nu} \\
&\leq [2(\nu+1)]^\nu z_k^{\frac{1}{\nu+1}} \epsilon^{-\nu} \left[\sum_{i=0}^{k-1} z_i^{-\frac{1}{\nu+1}}\right]^{-\nu}. \qquad (4.2.19)
\end{aligned}$$

So by (4.2.13) and (4.2.19), we have

$$\begin{aligned}
\|x_k - \hat{x}\| &\leq [2(\nu+1)]^\nu \epsilon^{-\nu} \left(\|v\| + B_1(C_1\rho + 1)M\right) k^{-\nu} \\
&\leq \tilde{C}k^{-\nu}.
\end{aligned}$$

□

REMARK 4.2.5. *Note that as $q \to 0$, $\nu \to \frac{1}{2}$. So, we obtain the error estimate $\|x_k - \hat{x}\| = O(k^{-\nu})$, for $0 < \nu < \frac{1}{2}$ under a Hölder-type source condition (4.1.4) for the method MFMEM.*

THEOREM 4.2.6. *Let (x_k) be as in (4.1.5). Assume that the assumptions $(\mathcal{C}_0)$ and $(\mathcal{C}_2)$ hold and $0 < C_1\rho < D$. If the source condition (4.1.4) holds with $0 < 2\nu < \frac{1}{2} - q, 0 < q < \frac{1}{2}$ then*

$$\|x_k - \hat{x}\| \leq \tilde{C}k^{-\nu}$$

where $\tilde{C} = [2(\nu+1)]^\nu \epsilon^{-\nu}(\|v\| + B_1(C_1\rho + 1)M)$.

Proof: Proof is analogous to the proof of Theorem 4.2.4.

□

REMARK 4.2.7. *Note that, as* $q \to 0$, $\nu \to \frac{1}{4}$. *So we obtain the error estimate* $\|x_k - \hat{x}\| = O(k^{-\nu})$, *for* $0 < \nu < \frac{1}{4}$ *under a Hölder-type source condition (4.1.4) for the method MFSDM.*

4.3 MFSDM and MFMEM with noisy data

In this Section, we study MFSDM and MFMEM with noisy data y^δ, instead of exact data y. We assume that $\|y - y^\delta\| \le \delta$, as stated in the Chapter 1. MFSDM and MFMEM with noisy data are defined by

$$\begin{aligned} x_{k+1}^\delta &= x_k^\delta + \alpha_k^\delta s_k^\delta \ \ (k = 0, 1, 2, \ldots) \\ s_k^\delta &= -F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta) \\ \alpha_k^\delta &= \frac{\|s_k^\delta\|^2}{\|A^q s_k^\delta\|^2} \end{aligned} \tag{4.3.1}$$

and

$$\begin{aligned} x_{k+1}^\delta &= x_k^\delta + \alpha_k^\delta s_k^\delta \ \ (k = 0, 1, 2, \ldots) \\ s_k^\delta &= -F'(x_0)^*(F(x_k^\delta) - y^\delta) \\ \alpha_k^\delta &= \frac{\|F(x_k^\delta) - y^\delta\|^2}{\|A^q(F(x_k^\delta) - y^\delta)\|^2}. \end{aligned} \tag{4.3.2}$$

respectively. As in Chapter 3, we assume:
$(\mathcal{C}_5)$ F satisfies the local property

$$\|F(u) - F(v) - F'(x_0)(u - v)\| \le \eta \|F(u) - F(v)\|, \tag{4.3.3}$$

for all $u, v \in B(x_0, \rho)$ with $\max\{\frac{1-B^2}{3}, 1 - \frac{B^2}{2} - \frac{\|A^q\|^2}{2m^2}, 0\} < \eta < 1 - \frac{B^2}{2}$.

4.3.1 Discrepancy Principle

PROPOSITION 4.3.1. *Let the assumption* $(\mathcal{C}_5)$ *holds. Let* x_k^δ *be as in (4.3.1) or in (4.3.2). Then,* $x_k^\delta \in B(x_0, 2\rho) \subset D(F)$, *for all* $k = 0, 1, 2, \ldots$ *and if*

$$\|F(x_k^\delta) - y^\delta\| > \tau\delta \tag{4.3.4}$$

where

$$\tau > 2\frac{(1+\eta)}{2-2\eta-B^2} > 2, \tag{4.3.5}$$

then, for all $0 \le k < k_*$ *with* τ *as in (4.3.5), we have*

$$k_*(\tau\delta)^2 \le \sum_{k=0}^{k_*-1} \|F(x_k^\delta) - y^\delta\|^2 \le \frac{\tau\|F'(x_0)\|^2}{(2-2\eta-B^2)\tau - 2(1+\eta)}\|x_0 - \hat{x}\|^2. \tag{4.3.6}$$

Proof: Note that $x_0 \in B(x_0, 2\rho)$. Suppose $x_k^\delta \in B(x_0, 2\rho)$. Using (4.3.1) or (4.3.2), we have

$$\begin{aligned}
& \|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \\
= \; & -2\alpha_k^\delta \langle x_k^\delta - \hat{x}, F'(x_0)^*(F(x_k^\delta) - y^\delta)\rangle + {\alpha_k^\delta}^2 \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 \\
= \; & 2\alpha_k^\delta \langle F(x_k^\delta) - y^\delta - F'(x_0)(x_k^\delta - \hat{x}), F(x_k^\delta) - y^\delta \rangle \\
& + \alpha_k^\delta \left[\alpha_k^\delta \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2\right] \\
\le \; & 2\alpha_k^\delta \|F(x_k^\delta) - F(\hat{x}) + y - y^\delta - F'(x_0)(x_k^\delta - \hat{x})\| \|F(x_k^\delta) - y^\delta\| \\
& + \alpha_k^\delta \left[\alpha_k^\delta \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2\right].
\end{aligned} \tag{4.3.7}$$

So by $(\mathcal{C}_5)$ and (4.3.7), we have

$$\begin{aligned}
& \|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \\
\le \; & 2\alpha_k^\delta \left(\eta\|F(x_k^\delta) - F(\hat{x})\| + \delta\right) \|F(x_k^\delta) - y^\delta\| \\
& + \alpha_k^\delta \left[\alpha_k^\delta \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2\right] \\
\le \; & 2\alpha_k^\delta \left[\eta\|F(x_k^\delta) - y^\delta\| + (1+\eta)\delta\right] \|F(x_k^\delta) - y^\delta\| \\
& + \alpha_k^\delta \left[\alpha_k^\delta \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2\right] \\
= \; & \alpha_k^\delta (2\eta - 2)\|F(x_k^\delta) - y^\delta\|^2 + \alpha_k^\delta 2(1+\eta)\delta \|F(x_k^\delta) - y^\delta\| \\
& + (\alpha_k^\delta)^2 \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2.
\end{aligned}$$

Note that

$$\begin{aligned}
\alpha_k^\delta \|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 &= \alpha_k^\delta \|A^{\frac{1}{2}}(F(x_k^\delta) - y^\delta)\|^2 \\
&\le \alpha_k^\delta \|A^{\frac{1}{2}-q}\|^2 \|A^q(F(x_k^\delta) - y^\delta)\|^2 \\
&\le B^2 \|F(x_k^\delta) - y^\delta\|^2.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \\
\leq\ & \alpha_k^\delta \left[(2\eta + B^2 - 2)\|F(x_k^\delta) - y^\delta\|^2 + 2(1+\eta)\delta\|F(x_k^\delta) - y^\delta\|\right], \quad (4.3.8)
\end{aligned}$$

so by (4.3.8),

$$\begin{aligned}
&\|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \\
\leq\ & \alpha_k^\delta \left((2\eta + B^2 - 2) + 2\frac{(1+\eta)}{\tau}\right)\|F(x_k^\delta) - y^\delta\|^2 < 0. \quad (4.3.9)
\end{aligned}$$

This implies $\|x_{k+1}^\delta - \hat{x}\| < \|x_k^\delta - \hat{x}\| < \|x_0 - \hat{x}\| < \rho$. Thus, $\|x_{k+1}^\delta - x_0\| \leq \|x_{k+1}^\delta - \hat{x}\| + \|x_0 - \hat{x}\| < 2\rho$ i.e., $x_{k+1}^\delta \in B(x_0, 2\rho) \subset D(F)$ for all $k = 0, 1, 2, \ldots$. Now since $\alpha_k^\delta \geq \|A^q\|^{-2}$, we have by (4.3.9)

$$\begin{aligned}
&\|A^q\|^{-2}\left((2 - 2\eta - B^2) - 2\frac{(1+\eta)}{\tau}\right)\|F(x_k^\delta) - y^\delta\|^2 \\
\leq\ & \|x_k^\delta - \hat{x}\|^2 - \|x_{k+1}^\delta - \hat{x}\|^2. \quad (4.3.10)
\end{aligned}$$

Adding the inequality (4.3.10) for k from 0 through $k_* - 1$, we obtain

$$\|A^q\|^{-2}\left((2 - 2\eta - B^2) - 2\frac{(1+\eta)}{\tau}\right)\sum_{k=0}^{k_*-1}\|F(x_k^\delta) - y^\delta\|^2 \leq \|x_0 - \hat{x}\|^2 - \|x_{k_*}^\delta - \hat{x}\|^2. \quad (4.3.11)$$

This completes the proof.

□

REMARK 4.3.2. *Note that (4.3.11) implies that, for $y^\delta \neq y$, there must be a unique index k_* such that, (4.3.4) holds for all $k < k_*$ but is violated at $k = k_*$ (see also (Engl et al., 1996, page 282)).*

Let $\Omega := \|A^q\|^{-2}\left((2 - 2\eta - B^2) - 2\frac{(1+\eta)}{\tau}\right)$.

THEOREM 4.3.3. *Let the assumptions ($\mathcal{C}_2$) and ($\mathcal{C}_5$) hold and $\rho < \min\left\{\frac{(\tau-1)^2\delta}{m}, \frac{2}{m\sqrt{\Omega}}\right\}$. Let x_{k+1}^δ be as in (4.3.1) or in (4.3.2). Then for $0 \leq k < k_*$,*

$$\|x_{k+1}^\delta - \hat{x}\| = \begin{cases} O(q_0^{\frac{k+1}{2}}) & if \quad \delta < q_0^{k+1} \\ O(\delta^{\frac{1}{2}}) & if \quad q_0^{k+1} \leq \delta \end{cases} \quad (4.3.12)$$

where $q_0 := 1 - \frac{\Omega m^2}{(\tau-1)^2}$.

Proof: Analogous to the proof of Theorem 3.4.4 in Chapter 3. □

4.4 Example

In this Section, we consider the Example 3.5.1 to implement the method (4.1.5) and (4.1.6).

EXAMPLE 4.4.1. *Returning back to the example 3.5.1, We take* $\hat{x}(t) = t, t \in [0,1]$ *and* $y(t) = arctan(t)^2$*. We have taken initial guess* $x_0(t) = t/2$ *and* $q = \frac{1}{4}$*. Then* $\nu < \frac{1}{8}$ *for MFSDM and* $\nu < \frac{1}{4}$ *for MFMEM. Error estimates for exact data are given in Table 4.1 and for noisy data, we have taken* $\tau = 2.1$ *and the error estimates are given in Table 4.2 with different values of* δ*. For MFSDM approximate solutions are given in Figures 4.1-4.5 and for MFMEM approximate solutions are given in Figures 4.6-4.10.*

Table 4.1: Error estimate for MFSDM and MFMEM with exact data

k	MFSDM		MFMEM	
	$\|x_k - \hat{x}\|$	$\frac{\|x_k-\hat{x}\|}{k^{\frac{1}{8}}}$	$\|x_k - \hat{x}\|$	$\frac{\|x_k-\hat{x}\|}{k^{\frac{1}{4}}}$
10	1.2173E-02	9.1287E-03	1.3044E-02	7.3350E-03
20	6.3958E-03	4.3981E-03	6.8454E-03	3.2370E-03
30	3.3920E-03	2.2172E-03	3.6393E-03	1.5550E-03
40	1.7654E-03	1.1132E-03	1.8962E-03	7.5398E-04
50	9.0892E-04	5.5739E-04	9.7693E-04	3.6738E-04
60	4.6533E-04	2.7893E-04	5.0036E-04	1.7978E-04
70	2.3753E-04	1.3966E-04	2.5548E-04	8.8323E-05
80	1.2107E-04	7.0008E-05	1.3023E-04	4.3546E-05
90	6.1662E-05	3.5135E-05	6.6334E-05	2.1537E-05
100	3.1393E-05	1.7654E-05	3.3773E-05	1.0680E-05

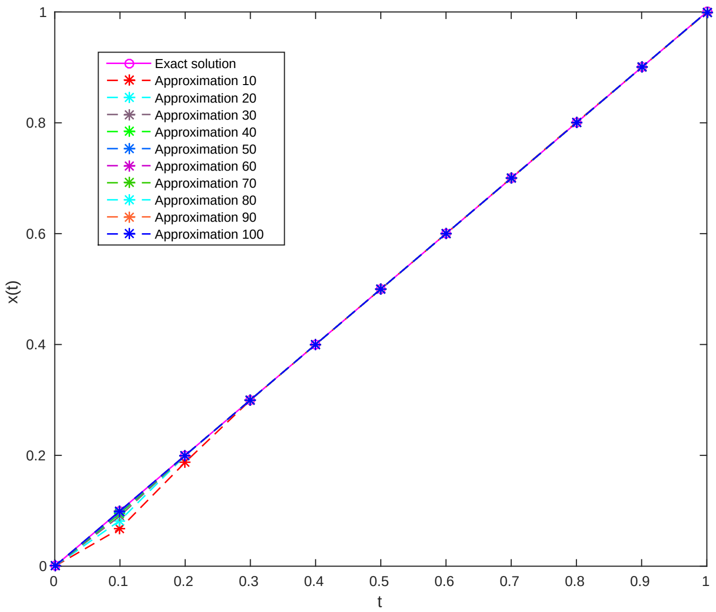

Figure 4.1: Approximate solutions of MFSDM for exact data

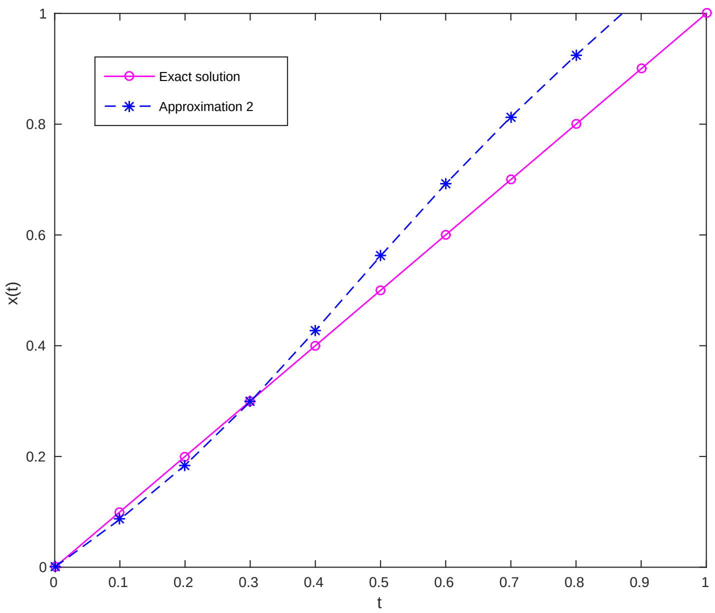

Figure 4.2: Approximate solution of MFSDM with $\delta = 0.1$

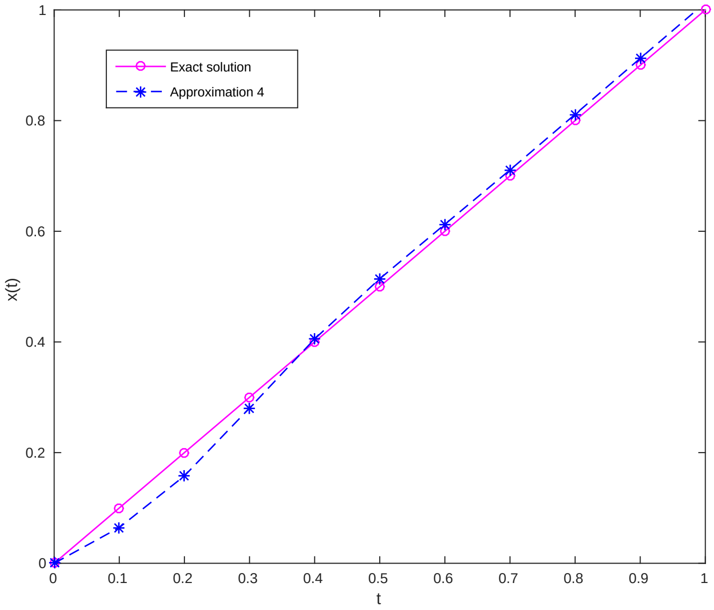

Figure 4.3: Approximate solution of MFSDM with $\delta = 0.01$

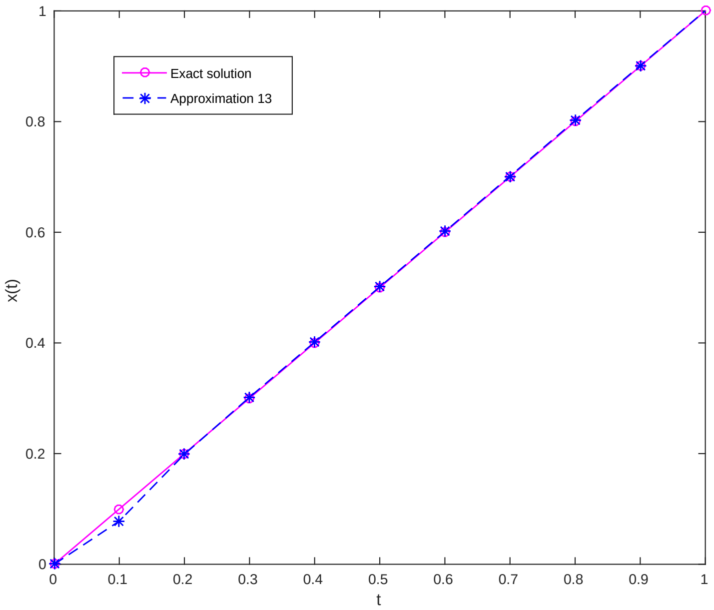

Figure 4.4: Approximate solution of MFSDM with $\delta = 0.001$

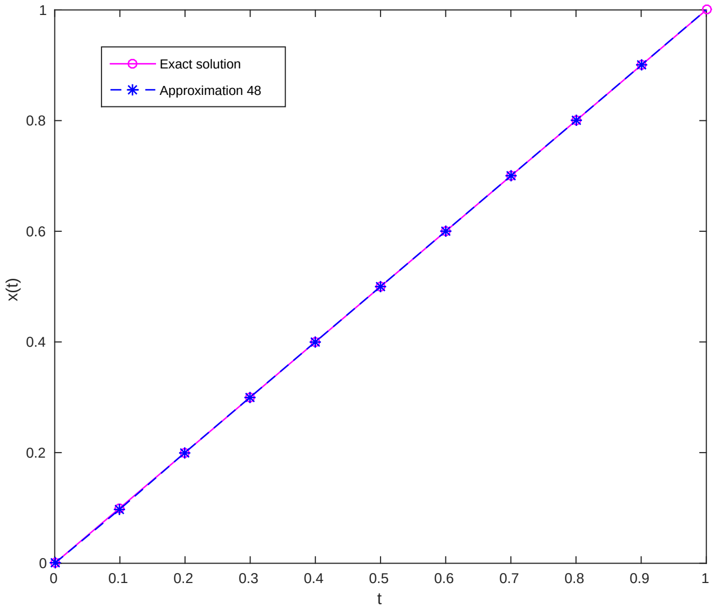

Figure 4.5: Approximate solution of MFSDM with $\delta = 0.0001$

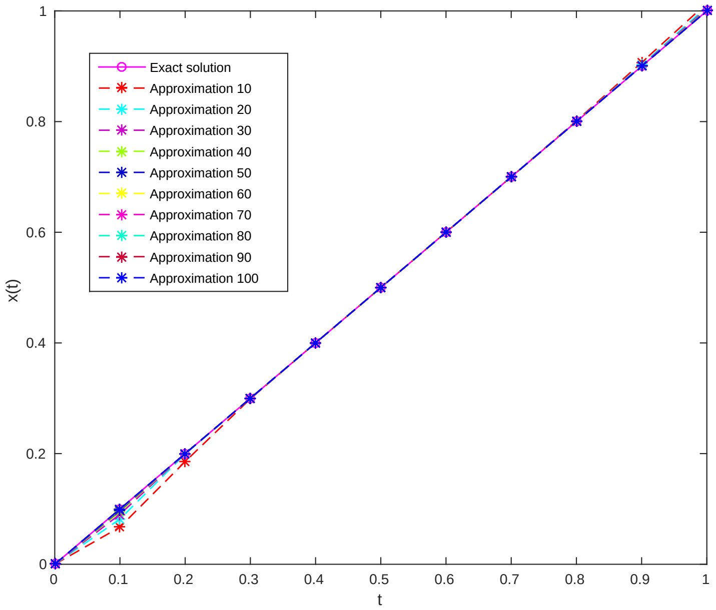

Figure 4.6: Approximate solutions of MFMEM with exact data

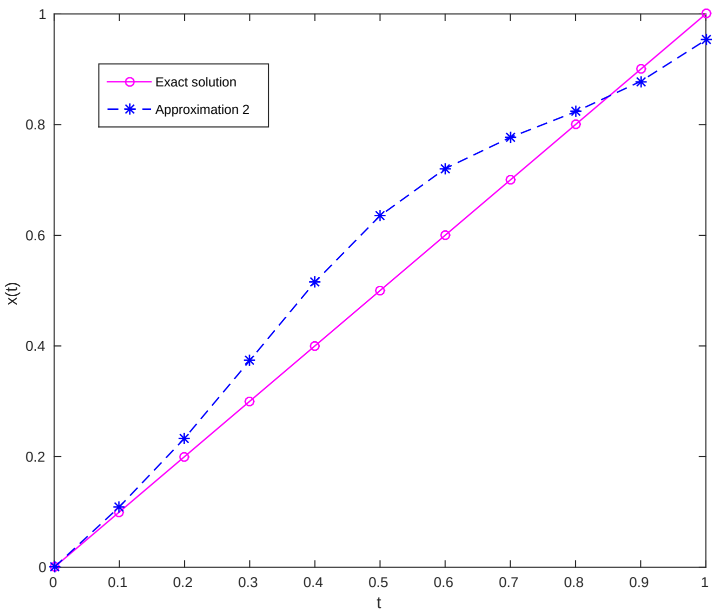

Figure 4.7: Approximate solution of MFMEM with $\delta = 0.1$

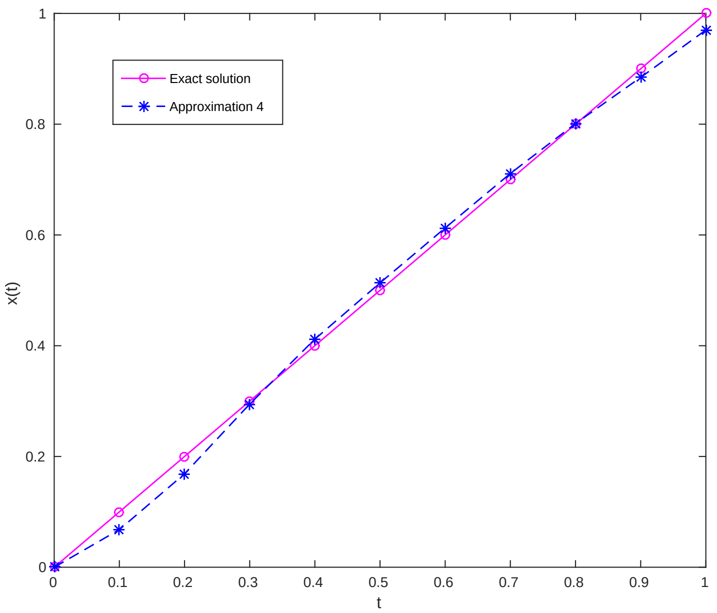

Figure 4.8: Approximate solution of MFMEM with $\delta = 0.01$

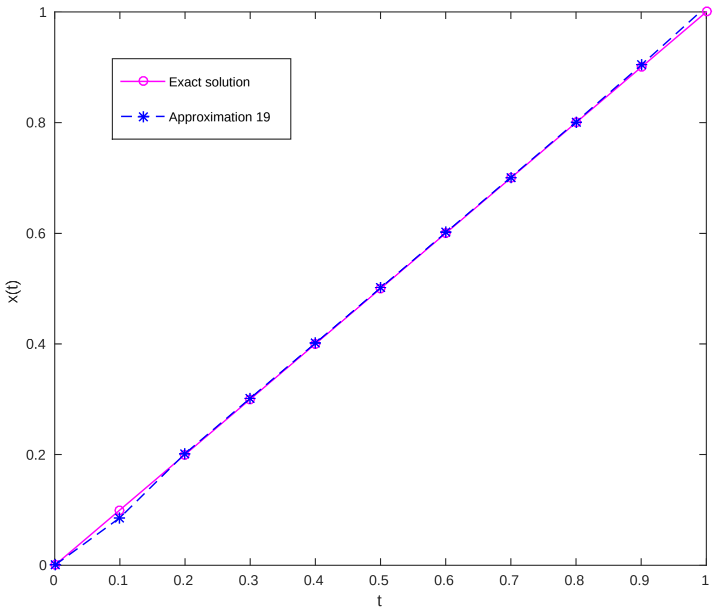

Figure 4.9: Approximate solution of MFMEM with $\delta = 0.001$

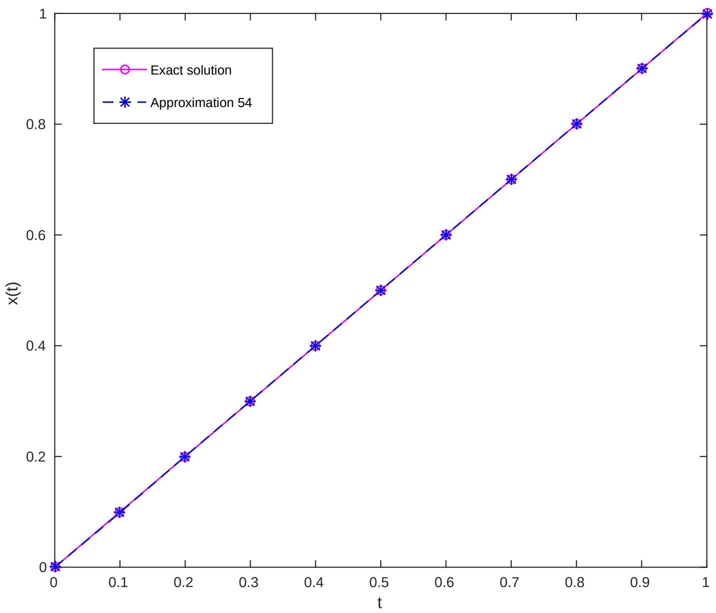

Figure 4.10: Approximate solution of MFMEM with $\delta = 0.0001$

Table 4.2: Error estimate for MFSDM and MFMEM with noisy data

δ	MFSDM			MFMEM		
	k	$\|x_k^\delta - \hat{x}\|$	$\frac{\|x_k^\delta - \hat{x}\|}{\sqrt{\delta}}$	k	$\|x_k^\delta - \hat{x}\|$	$\frac{\|x_k^\delta - \hat{x}\|}{\sqrt{\delta}}$
0.1	2	5.3985E-02	1.7072E-01	2	2.0772E-01	1.4688E-01
0.01	4	3.0498E-02	3.0498E-01	4	3.4426E-02	1.7213E-02
0.001	13	8.7840E-03	2.7778E-01	19	5.9390E-03	1.3625E-03
0.0001	48	8.6070E-04	8.6070E-02	54	5.9459E-04	8.0914E-05

Chapter 5

FROZEN STEEPEST DESCENT METHOD FOR NONLINEAR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS

In this study, we consider an inverse free iterative method for approximating a solution of the nonlinear ill-posed Hammerstein type equation $KF(x) = y$. Our approach is to solve $Kz = y$ and then $F(x) = z$. We use Tikhonov regularization method for approximating the solution of $Kz = y$ and Frozen steepest descent method for approximating the solution of $F(x) = z$. The adaptive parameter choice strategy of Pereverzev and Schock (2005) is used for choosing the regularization parameter.

5.1 Introduction

In this Chapter, we considered the problem of approximating the solution $\hat{x}$ of the nonlinear ill-posed Hammerstein type equation

$$KF(x) = y \tag{5.1.1}$$

where $F : D(F) \subseteq X \to Z$ is a Fréchet differentiable nonlinear operator, $K : Z \to Y$ is a bounded linear operator and X, Y, Z are Hilbert spaces. A typical example of the Hammerstein type equation (5.1.1) is

$$KF(x)(t) := \int_o^1 k(s,t)x^3(s)ds$$

where $K : L^2[0,1] \to L^2[0,1]$ is a bounded linear operator defined by

$$Kz(t) = \int_0^1 k(s,t)z(s)ds,$$

with kernel $k(s,t) \in L^2([0,1] \times [0,1])$ and $F : D(F) \subseteq L^2[0,1] \to L^2[0,1]$ is the nonlinear operator defined by

$$Fx(s) = x^3(s).$$

In general (5.1.1) is ill-posed in the sense that the solution need not depend continuously on the right-hand side data y. George (2006) studied an iterative Newton-Tikhonov regularization (NTR) method for approximating a x_0 -minimum norm solution $\hat{x}$ of (5.1.1). Further in practice, only an approximation of y, say y^δ with $\|y - y^\delta\| \leq \delta$ are available. So one has to consider

$$KF(x) = y^\delta \tag{5.1.2}$$

instead of (5.1.1). As in Argyros et al. (2016b); George (2006); George and Nair (2008); George and Kunhanandan (2009); George and Shobha (2012, 2014); Shobha et al. (2014), we approach the problem (5.1.2) by solving the equation

$$Kz = y^\delta \tag{5.1.3}$$

first and then

$$F(x) = z. \tag{5.1.4}$$

For approximating $\hat{x}$, iterative regularization methods are studied by Argyros et al. (2016b,c), George (2006), George and Nair (2008), George and Kunhanandan (2009), George and Shobha (2014) and Shobha et al. (2014). Note that, in all these methods, one has to compute the inverse involving Fréchet derivative of F at each iterate x_k or at initial guess x_0.

In the present study, we apply Tikhonov regularization to solve the linear operator equation (5.1.3) and then we consider the inverse free iterative method to solve the non-linear operator equation (5.1.4). The method involves, Fréchet derivative of F only at x_0 (see (5.3.2)).

The rest of the Chapter is organized as follows: Section 5.2 contains preliminaries, Section 5.3 contains convergence analysis of inverse free iterative method, Section 5.4 contains error bounds and source conditions and Section 5.5 contains finite dimensional realization of inverse free iterative method. Finally the Chapter ends with an academic example in Section 5.6.

5.2 Preliminaries

The following assumption is used for obtaining the error estimate.

ASSUMPTION 5.2.1. *There exists a continuous, strictly monotonically increasing function* $\varphi : (0, a] \to (0, \infty)$ *with* $a \geq \|K\|^2$ *satisfying;*

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0$
- $\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq \varphi(\alpha), \qquad \forall \alpha \in (0, a].$
- *there exists* $v \in X$ *with* $\|v\| \leq 1$ *such that*

$$F(\hat{x}) - F(x_0) = \varphi(K^*K)v.$$

Let

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*(y^\delta - KF(x_0)) + F(x_0). \tag{5.2.1}$$

It is known that (see (4.3) in George and Kunhanandan (2009)) under the assumption 5.2.1

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}. \tag{5.2.2}$$

5.3 Convergence analysis

Let $\delta_0 > 0$, $a_0 > 0$ be some constants with $\delta_0^2 < a_0$ and $\|x_0 - \hat{x}\| \leq r$. Let $\delta \in (0, \delta_0]$ and $\alpha \in [\delta_0^2, a_0]$. Then as in Argyros et al. (2014), for $\alpha > 0$, one can prove that

$$F'(x_0)^*(F(x) - z_\alpha^\delta) + \frac{\alpha}{c}(x - x_0) = 0 \tag{5.3.1}$$

has a unique solution x_α^δ in $B_r(x_0)$ provided $0 < r < \frac{1}{2k_0}$. To obtain an approximation for x_α^δ, we consider the iteration defined for $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - \beta[F'(x_0)^*(F(x_n) - z_\alpha^\delta) + \frac{\alpha}{c}(x_n - x_0)]. \tag{5.3.2}$$

We need the following assumption for the convergence analysis of (5.3.2).

ASSUMPTION 5.3.1.

(a) There exists a constant $k_0 > 0$ such that for every $x \in D(F)$ and $v \in X$, there exists an element $\Phi(x, x_0, v) \in X$ satisfying

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x, x_0, v), \;\; \|\Phi(x, x_0, v)\| \leq k_0\|v\|\|x - x_0\|.$$

(b)

$$\forall x \in B(\hat{x}, r), \|F'(x)\| \leq m.$$

Further, let β, $q_{\alpha,\beta}$ be parameters such that

$$\beta \leq \frac{1}{m^2 + \frac{a_0}{c}} \tag{5.3.3}$$

and

$$q_{\alpha,\beta} = 1 - \frac{\alpha\beta}{c} + \frac{3\beta m^2 k_0}{2} r. \tag{5.3.4}$$

The main result of this Chapter is the following theorem.

THEOREM 5.3.2. *Let assumption 5.3.1 holds and let (x_n) be as in (5.3.2) and $0 < r < \min\{\frac{1}{2k_0}, \frac{2\alpha}{3m^2k_0}\}$. Then for each $\delta \in (0, \delta_0]$ and $c \leq \alpha$. Then (x_n) is in $B(x_0, 2r)$ and converges to x_α^δ as $n \to \infty$. Further,*

$$\|x_{n+1} - x_\alpha^\delta\| \leq q_{\alpha,\beta}^{n+1}\|x_0 - x_\alpha^\delta\|, \tag{5.3.5}$$

where $q_{\alpha,\beta}$ is as in (5.3.4).

Proof: Clearly, $x_0 \in \overline{B(x_0, 2r)}$. Let $M_n := \int_0^1 F'(x_\alpha^\delta + t(x_n - x_\alpha^\delta))dt$. Since $x_\alpha^\delta \in B_r(x_0)$, M_0 is well defined. Assume that for some $n > 0, x_n \in B(x_0, 2r)$ and M_n is well defined. Then, since x_α^δ satisfies the equation (5.3.1), we have

$$\begin{aligned} x_{n+1} - x_\alpha^\delta &= x_n - x_\alpha^\delta - \beta\left[F'(x_0)^*(F(x_n) - F(x_\alpha^\delta)) + \frac{\alpha}{c}(x_n - x_\alpha^\delta)\right] \\ &= x_n - x_\alpha^\delta - \beta\left[F'(x_0)^* M_n + \frac{\alpha}{c} I\right](x_n - x_\alpha^\delta) \\ &= x_n - x_\alpha^\delta - \beta\left[F'(x_0)^*(M_n - F'(x_0))\right](x_n - x_\alpha^\delta) \\ &\quad -\beta\left[F'(x_0)^* F'(x_0) + \frac{\alpha}{c} I\right](x_n - x_\alpha^\delta) \\ &= \left[I - \beta\left(F'(x_0)^* F'(x_0) + \frac{\alpha}{c} I\right)\right](x_n - x_\alpha^\delta) \\ &\quad -\beta[F'(x_0)^*(M_n - F'(x_0))](x_n - x_\alpha^\delta). \end{aligned} \tag{5.3.6}$$

Using assumptions 5.3.1, we have

$$\begin{aligned} x_{n+1} - x_\alpha^\delta &= \left[I - \beta\left(F'(x_0)^* F'(x_0) + \frac{\alpha}{c} I\right)\right](x_n - x_\alpha^\delta) \\ &\quad -\beta F'(x_0)^* F'(x_0) \int_0^1 \Phi(x_\alpha^\delta + t(x_n - x_\alpha^\delta), x_0, x_n - x_\alpha^\delta)dt. \end{aligned}$$

Now since $I - \beta\left(F'(x_0)^* F'(x_0) + \frac{\alpha}{c} I\right)$ is a positive self-adjoint operator,

$$\begin{aligned} &\|I - \beta\left(F'(x_0)^* F'(x_0) + \frac{\alpha}{c} I\right)\| \\ =\ & \sup_{\|x\|=1} |\langle\left(I - \beta\left(F'(x_0)^* F'(x_0) + \frac{\alpha}{c} I\right)\right)x, x\rangle| \\ =\ & |\sup_{\|x\|=1}\left(1 - \beta\frac{\alpha}{c}\right)\langle x, x\rangle - \beta\langle F'(x_0)^* F'(x_0)x, x\rangle| \\ \le\ & 1 - \frac{\alpha\beta}{c}. \end{aligned} \tag{5.3.7}$$

The last step follows from the relation

$$\begin{aligned} \beta|\langle F'(x_0)^* F'(x_0)x, x\rangle| &\le \beta\|F'(x_0)\|^2 \\ &\le \beta m^2 \\ &\le \frac{1}{m^2 + \frac{a_0}{c}} m^2 \\ &\le \frac{1}{m^2 + \frac{\alpha}{c}} m^2 = 1 - \frac{\alpha/c}{m^2 + \alpha/c} \le 1 - \frac{\alpha\beta}{c}. \end{aligned}$$

Hence, by assumption 5.3.1, we have

$$\begin{aligned}\|x_{n+1}-x_\alpha^\delta\| &\leq \left(1-\frac{\alpha\beta}{c}\right)\|x_n-x_\alpha^\delta\| \\ &\quad +\beta m^2k_0\int_0^1((1-t)\|x_\alpha^\delta-x_0\|+t\|x_n-x_0\|)dt\|x_n-x_\alpha^\delta\| \\ &\leq \left(1-\frac{\alpha\beta}{c}+\beta\frac{3k_0m^2r}{2}\right)\|x_n-x_\alpha^\delta\| \\ &\leq q_{\alpha,\beta}\|x_n-x_\alpha^\delta\|. \qquad (5.3.8)\end{aligned}$$

Since $q_{\alpha,\beta}<1$, we have

$$\|x_{n+1}-x_\alpha^\delta\| < \|x_0-x_\alpha^\delta\| \leq r$$

and

$$\|x_{n+1}-x_0\| \leq \|x_{n+1}-x_\alpha^\delta\| + \|x_0-x_\alpha^\delta\| \leq 2r$$

i.e., $x_{n+1}\in B(x_0,2r)$. Also, for $0\leq t\leq 1$,

$$\|x_\alpha^\delta+t(x_{n+1}-x_\alpha^\delta)-x_0\| = \|(1-t)(x_\alpha^\delta-x_0)+t(x_{n+1}-x_\alpha^\delta)\| < 2r.$$

Hence, $x_\alpha^\delta+t(x_{n+1}-x_\alpha^\delta)\in B(x_0,2r)$ and M_{n+1} is well defined. Thus, by induction x_n is well defined and remains in $B(x_0,2r)$ for each $n=0,1,2,\ldots$. By letting $n\to\infty$ in (5.3.2), we obtain the convergence of x_n to x_α^δ. The estimate (5.3.5) now follows from (5.3.8).

□

5.4 Error bounds under source conditions

In this Section, we need the following assumptions in addition to the earlier assumptions to obtain the error bound.

ASSUMPTION 5.4.1. *There exists a continuous, strictly monotonically increasing function $\varphi_1:(0,b]\to(0,\infty)$ with $b\geq\|F'(x_0)\|^2$ satisfying;*

- $\lim_{\lambda\to 0}\varphi_1(\lambda)=0$

- $\sup_{\lambda \geq 0} \frac{\alpha\varphi_1(\lambda)}{\lambda+\alpha} \leq \varphi_1(\alpha), \qquad \forall \alpha \in (0, b].$
- *there exists* $v \in X$ *with* $\|v\| \leq 1$ *such that*

$$x_0 - \hat{x} = \varphi_1(F'(x_0)^* F'(x_0))v.$$

ASSUMPTION 5.4.2. *For each* $x \in B(x_0, r)$*, there exists a bounded linear operator* $G(x, x_0)$ *such that*

$$F'(x) = F'(x_0)G(x, x_0)$$

with $\|G(x, x_0)\| \leq k_1$

Let $k_1 < \frac{1-k_0 r}{1-c}$ and assume that $\varphi_1(\alpha) \leq \varphi(\alpha)$. Proof of the following Theorems 5.4.3, 5.4.4 and 5.4.5 are analogous to the proof of Theorems 3.14, 3.15 and 3.16 in Argyros et al. (2016b).

THEOREM 5.4.3. *(cf. Argyros et al. (2016b), Theorem 3.14) Let* x_α^δ *be the solution of (5.3.1) and assumption 5.4.1 and assumption 5.4.2 hold. Let* $0 < r < \min\left\{\frac{1}{2k_0}, \frac{2\alpha}{3cm^2 k_0}\right\}$ *and* $k_1 < \frac{1-k_0 r}{1-c}$*. Then*

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{\varphi_1(\alpha) + \|F(\hat{x}) - z_\alpha^\delta\|}{1 - k_0 r - (1-c)k_1}.$$

THEOREM 5.4.4. *(cf. Argyros et al. (2016b), Theorem 3.15) Let* (x_n) *be as in (5.3.2).If assumption 2.1 and assumptions in Theorem 5.4.3 and Theorem 5.3.2 hold and* $\varphi_1(\alpha) \leq \varphi(\alpha)$*, then*

$$\|x_n - \hat{x}\| \leq q_{\alpha,\beta}^n r + K\left(2\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}\right)$$

where $K = \frac{1}{1-K_0 r-(1-c)k_1}$.

THEOREM 5.4.5. *(cf. Argyros et al. (2016b), Theorem 3.16) Let* (x_n) *be as in (5.3.2) and assumptions in Theorem 5.4.4 hold. Let*

$$n_k = \min\left\{n : q_{\alpha,\beta}^n \leq \frac{\delta}{\sqrt{\alpha}}\right\}.$$

Then

$$\|x_{n_k} - \hat{x}\| = \bar{K}\left(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}\right).$$

where $\bar{K} = \max\{2K, r + K\}$.

5.5 Finite dimensional realization of FRSDM

Let $V_1 \subseteq V_2 \subseteq V_3 \subseteq$ ..... be a sequence of finite-dimensional subspaces of X with $\overline{U_{n\in N} V_n} = X$ and P_h is the orthogonal projection of X onto V_n. Let

$$\varepsilon_h := \|K(I - P_h)\|,$$

$$\tau_h := \|F'(x)(I - P_h)\|, \quad \forall x \in D(F).$$

Let $\{b_h : h > 0\}$ is such that $\lim\limits_{h\to 0} \frac{\|(I-P_h)x_0\|}{b_h} = 0$, $\lim\limits_{h\to 0} \frac{\|(I-P_h)F(x_0)\|}{b_h} = 0$ and $\lim\limits_{h\to 0} b_h = 0$. We assume that $\varepsilon_h \to 0$ and $\tau_h \to 0$ as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ point wise and if K and $F'(x)$ are compact operators. Further we assume that $\varepsilon_h < \varepsilon_0$, $\tau_h \leq \tau_0$, $b_h \leq b_0$.

In the discretized Tikhonov regularization method for solving equation (5.2.1), the solution of $z_\alpha^{h,\delta}$ of the equation

$$\left(P_h K^* K P_h + \frac{\alpha}{c} P_h\right)(z_\alpha^{h,\delta} - P_h F(x_0)) = P_h K^*[y^\delta - KF(x_0)] \tag{5.5.1}$$

is taken as an approximation for $F(\hat{x})$.

THEOREM 5.5.1. *(See George and Shobha (2012), Theorem 2.4) Let $z_\alpha^{h,\delta}$ be as in (5.5.1). Further, if $b_h \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ and assumption 5.2.1 holds, then*

$$\|F(\hat{x}) - z_{\alpha,h}^\delta\| \leq C\left(\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\sqrt{\alpha}}\right). \tag{5.5.2}$$

where $C = \max\{mr, 1\} + 1$.

5.5.1 An a priori choice of the parameter

Note that the estimate $\varphi(\frac{\alpha}{c}) + \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ in (5.5.2) is of optimal order for the choice $\alpha := \alpha(\delta, h)$ which satisfies $\varphi(\alpha(\delta, h)) = \frac{\delta+\varepsilon_h}{\sqrt{\alpha(\delta,h)}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq a$. Then we have $\delta + \varepsilon_h = \sqrt{\alpha(\delta, h)}\varphi(\alpha(\delta, h)) = \psi(\varphi(\alpha(\delta, h)))$ and

$$\alpha(\delta, h) = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h)).$$

So from (5.5.2) we have $\|F(\hat{x}) - z_\alpha^{h,\delta}\| \leq 2C\psi^{-1}(\delta + \varepsilon_h)$.

5.5.2 An adaptive choice of the parameter

Let

$$D_N = \{\alpha_i = \mu^i \alpha_0 : i = 1, 2, \ldots N, \mu > 1, \alpha_0 > 0\}$$

be the set of possible values of the parameter α.

Let

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_i}}\right\} < N, \tag{5.5.3}$$

$$k = \max\left\{i : \alpha_i \in D_N^+\right\} \tag{5.5.4}$$

where $D_N^+ = \left\{\alpha_i \in D_N : \|z_{\alpha_i}^{\delta} - z_{\alpha_j}^{\delta}\| \leq \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}, j = 0, 1, 2, ..., i-1\right\}$.

THEOREM 5.5.2. *(cf. George and Shobha (2012), Theorem 2.5) Let l be as in (5.5.3), k be as in (5.5.4) and $z_{\alpha_k}^{h,\delta}$ be as in (5.5.1) with $\alpha = \alpha_k$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\| \leq C\left(2 + \frac{4\mu}{\mu - 1}\right)\mu\psi^{-1}(\delta + \varepsilon_h).$$

Proof: Analogous to the proof of Theorem 2.5 in George and Shobha (2012).

□

The discretized version of (5.3.2) is defined as:

$$x_{n+1,\alpha_k}^{h,\delta} = x_{n,\alpha_k}^{h,\delta} - \beta P_h\left[F'(x_0)^*(F(x_{n,\alpha_k}^{h,\delta}) - z_\alpha^{h,\delta}) + \frac{\alpha_k}{c}(x_{n,\alpha_k}^{h,\delta} - x_0^{h,\delta})\right] \tag{5.5.5}$$

where $x_0^{h,\delta} =: P_h x_0$ and $c \leq \alpha_k$. Let

$$(\delta_0 + \varepsilon_0)^2 < \bar{a}_0.$$

It is known that (George and Shobha, 2012, Theorem 3.7.) under the assumption 5.2.1

$$P_h F'(x_0)^*(F P_h(x) - z_\alpha^{h,\delta}) + \frac{\alpha_k}{c} P_h(x - x_0) = 0 \tag{5.5.6}$$

has a unique solution $x_{\alpha_k}^{h,\delta}$ in $B(x_0, r) \cap R(P_h)$ and the following Theorems hold.

THEOREM 5.5.3. *(cf. George and Shobha (2012), Theorem 3.8) Suppose $x_{\alpha_k}^{h,\delta}$ is the solution of 5.5.6 and assumption 5.2.1 and Theorem 5.4.3 hold. In addition if $\tau_0 < 1$, then*

$$\|x_{\alpha_k}^{h,\delta} - x_{\alpha_k}^{\delta}\| \leq \frac{1}{1-\tau_0}\left(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}\right).$$

Proof: Proof is analogous to the proof of Theorem 3.8 in George and Shobha (2012).

□

The proof of the following Theorem 5.5.4 is analogous to the proof of Theorem 5.3.2 in Section 5.3.

THEOREM 5.5.4. *Let* $x^{h,\delta}_{n,\alpha_k}$ *be as in (5.5.5) and let* $0 < r < \min\left\{\frac{2\alpha}{3M^2ck_0}, \frac{1}{2k_0}\right\}$. *Then for each* $\delta \in (0, \delta_0]$, $\alpha_k \in ((\delta+\varepsilon_h)^2, \bar{a_0}], \varepsilon_h \le \varepsilon_0$ *the sequence* $\{x^{h,\delta}_{n,\alpha_k}\}$ *is in* $B(x_0, 2r) \cap R(P_h)$ *and converges to* $x^{h,\delta}_{\alpha_k}$ *as* $n \to \infty$. *Further,*

$$\|x^{h,\delta}_{n+1,\alpha_k} - x^{h,\delta}_{\alpha_k}\| \le q^{n+1}_{\alpha_k,\beta}\|P_h x_0 - x^{h,\delta}_{\alpha_k}\|, \qquad (5.5.7)$$

where $q_{\alpha_k,\beta}$ *is as in (5.3.4) with* $\alpha = \alpha_k$.

THEOREM 5.5.5. *Let* $x^{h,\delta}_{\alpha_k}$ *be the solution of (5.5.6) and assumptions in Theorem 5.4.3, Theorem 5.5.3 and Theorem 5.5.4 hold. If* $\varphi_1(\alpha) \le \varphi(\alpha)$, *then*

$$\|x^{h,\delta}_{n,\alpha_k} - \hat{x}\| \le q^n_{\alpha_k,\beta} r + \left(\left(K + \frac{1}{1-\tau_0}\right) + KC\left(2 + \frac{4\mu}{\mu-1}\right)\right)\mu\psi^{-1}(\delta+\varepsilon_h).$$

where $q_{\alpha_k,\beta}$ *is as in (5.3.4) with* $\alpha = \alpha_k$.

By combing the results in Theorem 5.5.4 and Theorem 5.5.5, we obtain the following Theorem.

THEOREM 5.5.6. *Let* $x^{h,\delta}_{n,\alpha_k}$ *be as in (5.5.5) and assumptions in Theorem 5.5.5 holds. Let*

$$n_k = \min\left\{n : q^n_{\alpha_k,\beta} \le \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\right\}.$$

Then

$$\|x^{h,\delta}_{n_k,\alpha_k} - \hat{x}\| = O(\psi^{-1}(\delta+\varepsilon_h)).$$

5.5.3 Algorithm

The balancing algorithm associated with the choice of the parameter specified in this Section involves the following steps:

- For $i, j \in \{0, 1, 2, \ldots, N\}$

$$z_{\alpha_i}^{\delta} - z_{\alpha_j}^{\delta} = (\alpha_j - \alpha_i)(K^*K + \alpha_i I)^{-1}[K^*(y^{\delta} - KF(x_0))];$$

- Choose $\alpha_0 = (\delta + \varepsilon_h)^2$ and $\mu > 1$;
- Choose $\alpha_i := \mu^{2i}\alpha_0, i = 0, 1, 2, \cdots, N$;
- Solve for $w_i : (K^*K + \alpha_i I)w_i = K * (y^{\delta} - KF(x_0);$
- Solve for $j < i, z_{ij} : (K^*K + \alpha_i I)z_{ij} = (\alpha_j - \alpha_i)w_i;$
- $\|z_{ij}\| > 4C\frac{(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$, then take k=i-1;
- Otherwise repeat with i+1 in place of i;
- Choose $n_k := min\left\{n : q_{\alpha_k,\beta}^n \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\right\}$;
- Solve $x_k := x_{n_k,\alpha_k}^{h,\delta}$ by using the iteration (5.5.5).

5.6 Numerical Example

We consider the space $X = Y = L^2(0, 1)$ and the operator $KF : X \to Y$, where $F : D(F) \subseteq X \to Y$ is a nonlinear operator defined by

$$F(u) = \int_0^1 k(t, s)u^3(s)ds$$

and $K : X \to Y$ is a bounded linear operator defined by

$$K(x)(t) = \int_0^1 k(t, s)x(s)ds.$$

Here

$$k(t, s) = \begin{cases} (1 - t)s, & 0 \leq s \leq t \leq 1 \\ (1 - s)t & 0 \leq s \leq t \leq 1. \end{cases}$$

The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)u^2(s)w(s)ds.$$

We have taken the exact solution $\hat{x}(t) = 0.5 + t^3$ and initial guess $x_0(t) = 0.5 + t^3 - \frac{3}{56}(t - t^8)$. Let $\beta = 1/1000$. Then the error estimates are given in Table 5.1 and approximate and exact solutions for various values of δ are given in Figures 5.1-5.6 .

Table 5.1: Error estimate

N	k	α_k	$\|x^{h,\delta}_{n_k,\alpha_k} - \hat{x}\|$	$\frac{\|x^{h,\delta}_{n_k,\alpha_k} - \hat{x}\|}{\sqrt{\delta+\varepsilon_h}}$
32	4	1.06E-01	2E-02	7.47E-02
62	4	1.06E-01	1.67E-02	5.28E-02
124	4	1.06E-01	1.18E-02	3.74E-02
256	4	1.06E-01	8.35E-03	2.64E-02
512	4	1.06E-01	5.91E-03	1.87E-02
1024	4	1.06E-01	4.18E-03	1.32E-02

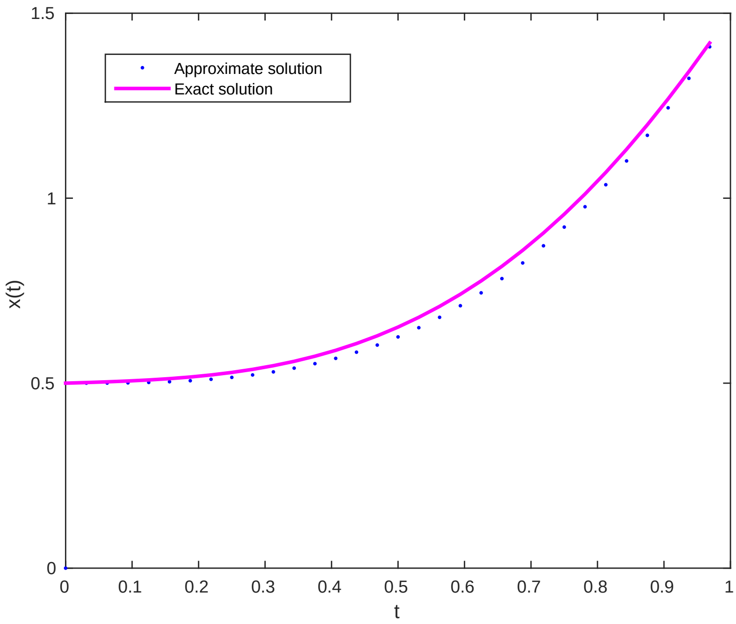

Figure 5.1: N=32

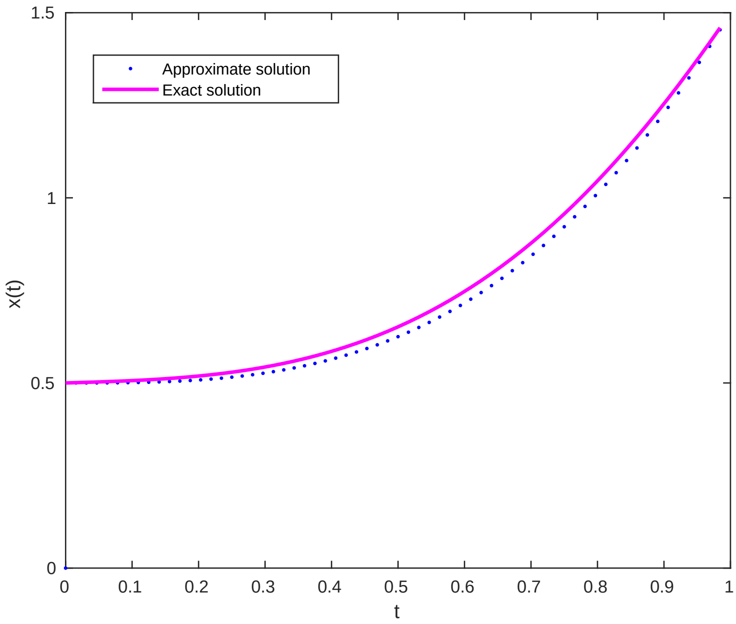

Figure 5.2: N=64

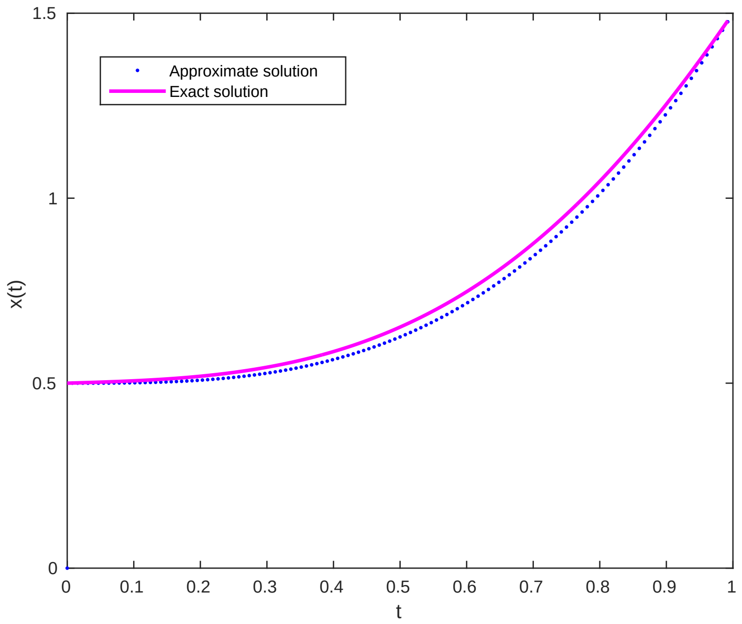

Figure 5.3: N=128

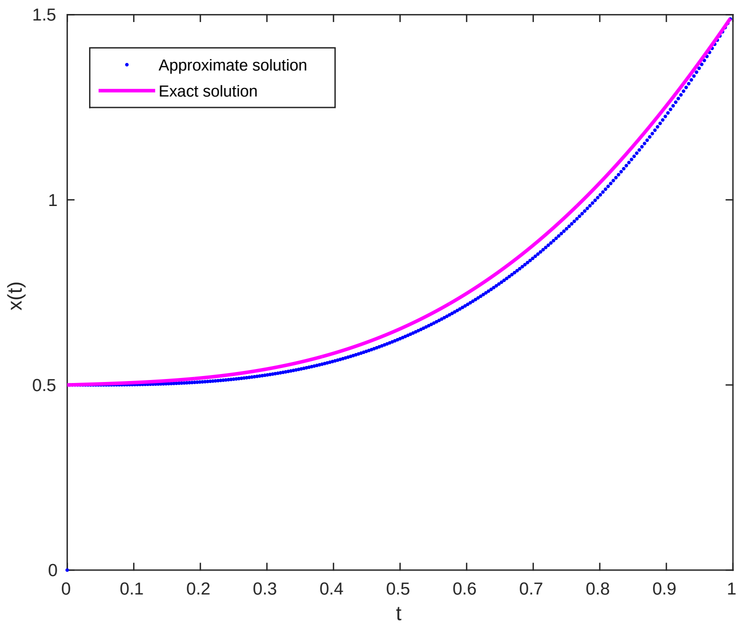

Figure 5.4: N=256

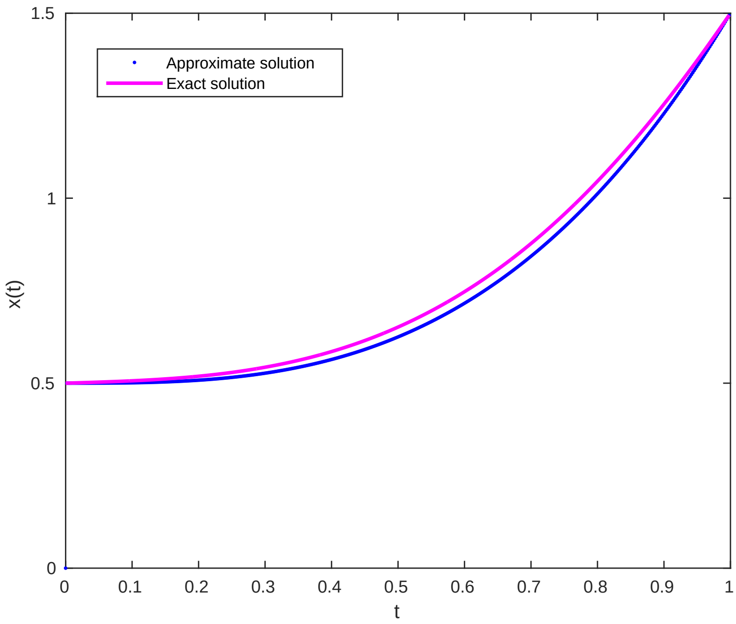

Figure 5.5: N=512

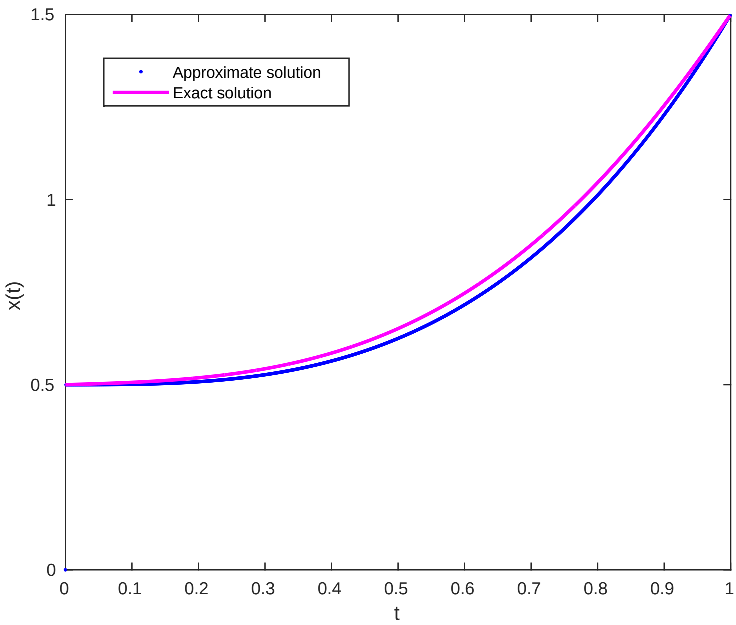

Figure 5.6: N=1024

Chapter 6

CONCLUSION

Several iterative methods have been studied for solving nonlinear ill-posed operator equations. Our Study in this thesis is focused on steepest descent method and minimal error method.

In Chapter 2, we have studied the frozen regularized steepest descent method. Our convergence analysis in this method is based on the property of norm of self adjoint operator. The balancing principle considered by Pereverzev and Schock (2005) was used for choosing the regularization parameter. Finite dimensional realization of this method is also studied. We applied the method considered in this Chapter to nonlinear ill-posed Hammerstein type operator equation in Chapter 5.

In Chapter 3 and Chapter 4, we have studied modified steepest descent method and modified minimal error method.

Some of the problems that were thought about and where further research may be possible, are discussed below.

In order to improve the order of convergence, many authors [George and Nair (1997); Egger and Neubauer (2005); George et al. (2013); Goldenshluger and Pereverzev (2000); Kreĭ n and Petunin (1966); Lu et al. (2010); Natterer (1984); Tautenhahn (1996); Neubauer (2000); George and Nair (2003)] studied ill-posed problems in the setting of Hilbert scales. Even though other iterative method are studied in the Hilbert scale setting, steepest descent method and minimal error method are not studied in the setting of Hilbert scale.

It is proposed to study, steepest descent-type method in the setting of Hilbert scale.

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2. Santhosh, George & Sabari, M (2017). Numerical approximation of a Tikhonov type regularizer by a discretized frozen steepest descent method, *Journal of Computational and Applied Mathematics* , 330, 488-498.

3. Sabari, M & Santhosh, George (2017). Modified minimal error method for nonlinear ill-posed problems, *Computational Methods in Applied Mathematics*, DOI:10.1515/cmam-2017-0024.

4. Santhosh, George & Sabari,M (2017). Error Estimate for Modified Steepest Descent Method for Nonlinear Ill-Posed Problems Under Holder-Type Source Condition, *Mathematical Inverse Problems*, 4, 1-11.

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