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Symmetric multistep methods with zero phase-lag for periodic initial value problems of second order differential equations

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Abstract

We present in this paper two-step and four-step symmetric multistep methods involving a parameter p to solve a special class of initial value problems associated with second order ordinary differential equations in which the first derivative does not appear explicitly. It is shown that the methods have zero phase-lag when p is chosen as 2π times the frequency of the given initial value problem. The periodicity intervals are given in terms of expressions involving the parameter p. As p increases, the periodicity intervals increase and for large p, the methods are almost P-stable.

Keywords: Symmetric multistep methods; Periodicity interval; Phase-lag; P-stable; Second order initial value problems

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1. Introduction

In this paper, we discuss the numerical integration of a special class of initial value problems associated to second order ordinary differential equations

$$y'' = f(x, y), \quad y(0) = y_0, \quad y'(0) = y'_0, \tag{1}$$

in which the first derivative does not appear explicitly. The numerical integration methods for (1) can be divided into two distinct classes: (a) problems for which the solution period is known (even approximately) in advance; (b) problems for which the period is not known. There is a vast literature available for the numerical solution of this problem. Computational methods involving a parameter proposed by Gautschi [7], Jain et al. [8], Stiefel and Bettis [11] yield the numerical solution to the problem of the first class. Numerical treatment to the problem of the second class have been presented by Chawla and Rao [1,2], Chawla and Zanaidi [3], Dahlquist [5], Franco [6], Lambert and Watson [9], Tsitouras and Simos [13].

Lambert and Watson [9] have developed linear, symmetric multistep methods of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+1-j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+1-j}, \quad k \geqslant 2, \tag{2}$$

where h (>0) is the step length of integration and $\alpha_j = \alpha_{k-j}$, $\beta_j = \beta_{k-j}$, $j = 0(1)k$, on the discrete point set $\{x_n : x_n = nh, n = 0, 1, \ldots\}$, for finding the numerical solution of the special initial value problem (1). They derive methods for $k = 2$, 4 and 6. Further by applying the methods to the test equation

$$y'' = -\lambda^2 y, \lambda \in \mathbf{R}, \tag{3}$$

with nontrivial initial conditions on y and y', they obtained intervals of periodicity from the characteristic polynomial

$$\Omega(z; H^2) = \rho(z) + H^2 \zeta(z), \quad H = \lambda h, \tag{4}$$

where

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^{k-j}, \quad \zeta(z) = \sum_{j=0}^{k} \beta_j z^{k-j},$$

based on the following definition.

Definition 1. A symmetric multistep method (2) with characteristic polynomial $\Omega(z, H^2)$ is said to have an *interval of periodicity* $(0, H_0^2)$ if, for all $H^2 \in (0, H_0^2)$, the roots z_j, $j = 1(1)k$, satisfy

$$z_1 = e^{i\theta(H)}, \quad z_2 = e^{-i\theta(H)}, \quad \text{and} \quad |z_j| \leqslant 1, \;\; j = 3(1)k,$$

where $\theta(H)$ is a real function of H.

Definition 2. The method (2) is said to be *P-stable* if its interval of periodicity is $(0, \infty)$.

Further, the phase-error or phase-lag analysis of symmetric multistep methods is based on the following definition.

Definition 3. For any symmetric multistep method (2) with characteristic polynomial $\Omega(z; H^2)$ given by (4) the phase-lag is

$$t(H) = H - \theta(H) = cH^{q+1} + \mathrm{O}(H^{q+2}),$$

where c is the *phase-lag constant* and q is the *phase-lag order*.

Lambert and Watson [9] have proved that the method described by (2) has a nonvanishing interval of periodicity only if it is symmetric and for P-stability the order cannot exceed 2. Further, the method is implicit. Later Chawla and Rao [1] noted that Numerov method has phase-lag error of $H^6/480$ and derived a Numerov type method of algebraic order four with minimal phase-lag $H^6/12{,}096$ and having an interval of periodicity $(0, 2.71)$. This method is implicit and its implementation involves the computations of Jacobians and solution of nonlinear systems of equations. So subsequently many authors proposed explicit modifications of Numerov method.

In the present paper we derive two-step and four-step methods involving a parameter p. In Section 2, the derivation is given and also the local truncation error. In Section 3, the intervals of periodicity is determined in terms of p. We observe that as p increases, the length of the intervals of periodicity $(0, H_0^2)$ increase. For large p, the methods are almost P-stable having a zero phase-lag. These methods have zero phase-lag when the parameter p is chosen as 2π times the frequency of the initial value problem, as shown in Section 4. In the subsequent Section 5 the phase-lag analysis is presented for an arbitrary p. Numerical illustrations are appended in Section 6.

2. Derivation of the methods

For the numerical integration of (1) we consider two-step symmetric methods of the form (2),

$$\alpha_0 y_{n+1} + \alpha_1 y_n + \alpha_0 y_{n-1} = h^2(\beta_0 f_{n+1} + \beta_1 f_n + \beta_0 f_{n-1}). \tag{5}$$

On applying the necessary and sufficient condition for consistency, [9], viz., $\rho(1) = \rho'(1) = 0$ and $\rho''(1) = 2\zeta(1)$, the coefficients α_j, β_j, $j = 0, 1$ are chosen as follows:

$$\alpha_0 = 1, \quad \alpha_1 = -2, \quad 2\beta_0 + \beta_1 = 1. \tag{6}$$

Then a family of two-step symmetric methods would follow from (5) if we agree to take one of the coefficients β_0 or β_1 as a free parameter. For instance, taking β_0 as a free parameter, with $\beta_0 = 1/4$, $\beta_0 = 1/12$ we obtain Dahlquist, Numerov methods of algebraic order two, four, respectively. Presently, we shall determine the coefficients β_0, β_1 in (5) by substituting $y(x) = \cos(rpx)$, $r = 1, 2$, with p as a parameter and take $x_n = 0$. This leads to a linear system of two equations for β_0, β_1. Denoting $\sigma = ph/2$, the values of β_0, β_1 are found to be,

$$\beta_0 = \frac{1}{12}\left(\frac{\sin\sigma}{\sigma}\right)^3 \frac{3\sigma}{\sin(3\sigma)}, \quad \beta_1 = \left(\frac{\sin\sigma}{\sigma}\right)^2 - 2\beta_0\cos(2\sigma). \tag{7}$$

The above expressions for β_0, β_1 satisfy the linear relation in (6), prescribed by the consistency requirement, in the limiting case as $\sigma \to 0$. The symmetric two-step method (5) with coefficients given by (6), (7) has local truncation error,

$$\begin{aligned}\mathrm{LTE} = &\left[1 - \left(\frac{\sin\sigma}{\sigma}\right)^2 - \frac{1}{3}\left(\frac{\sin\sigma}{\sigma}\right)^3 \frac{3\sigma}{\sin(3\sigma)}\sin^2\sigma\right] h^2 y''(x) \\ &+ \left[1 - \left(\frac{\sin\sigma}{\sigma}\right)^3 \frac{3\sigma}{\sin(3\sigma)}\right] \frac{h^4}{12} y^{(4)}(x) \\ &+ \left[1 - \frac{5}{2}\left(\frac{\sin\sigma}{\sigma}\right)^3 \frac{3\sigma}{\sin(3\sigma)}\right] \frac{h^6}{360} y^{(6)}(x) + \mathrm{O}(h^8).\end{aligned} \tag{8}$$

We note that as $\sigma \to 0$ the method (5) reduces to the familiar Numerov method,

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12}(f_{n+1} + 10f_n + f_{n-1}), \tag{9}$$

with algebraic order 4, since the local truncation error (8) now reduces to

$$\mathrm{LTE} = -\frac{h^6}{240} y^{(6)}(x) + \mathrm{O}(h^8).$$

Consider the four-step symmetric methods of the form (2),

$$\begin{aligned}&\alpha_0 y_{n+2} + \alpha_1 y_{n+1} + \alpha_2 y_n + \alpha_1 y_{n-1} + \alpha_0 y_{n-2} \\ &\quad = h^2(\beta_0 f_{n+2} + \beta_1 f_{n+1} + \beta_2 f_n + \beta_1 f_{n-1} + \beta_0 f_{n-2}).\end{aligned} \tag{10}$$

Using the consistency conditions, as stated earlier, the coefficients α_j, β_j, $j = 0, 1, 2$, subjected to the restrictions $\alpha_1 \in (-4, 0)$ and $2\alpha_1 + \alpha_2 = -2$, could be chosen as follows:

$$\alpha_0 = 1, \quad \alpha_1 = -2, \quad \alpha_2 = 2, \quad 2\beta_0 + 2\beta_1 + \beta_2 = 2. \tag{11}$$

To determine β_0, β_1, β_2 in (10) we substitute $y(x) = \cos(rpx)$, $r = 1, 2, 3$, with p as a parameter and take $x_n = 0$. This leads to a linear system of three equations for the undetermined coefficients. The values of β_0, β_1, β_2 can be easily determined to be

$$\begin{aligned}
\beta_0 &= \left(\frac{3}{40} - \frac{229}{540}\sin^2\sigma + \frac{13}{15}\sin^4\sigma - \frac{34}{45}\sin^6\sigma + \frac{32}{135}\sin^8\sigma\right) \\
&\quad \times \left(\frac{\sin\sigma}{\sigma}\right)^6 \prod_{k=2}^{5} \frac{k\sigma}{\sin(k\sigma)}, \\
\beta_1 &= \frac{1}{6}\left(7 - 16\sin^2\sigma + 8\sin^4\sigma\right)\left(\frac{\sin\sigma}{\sigma}\right)^3 \frac{3\sigma}{\sin 3\sigma} - 4\beta_0\cos\sigma\cos 3\sigma, \\
\beta_2 &= 2\left(\frac{\sin\sigma}{\sigma}\right)^2 \cos 2\sigma - 2\beta_0\cos 4\sigma - 2\beta_1\cos 2\sigma.
\end{aligned} \tag{12}$$

The expressions for β_0, β_1, β_2 satisfy the linear relation in (11), prescribed by the consistency condition, in the limiting case as $\sigma \to 0$.

The symmetric four-step method (10) with coefficients given by (11) and (12) has local truncation error,

$$\begin{aligned}
\mathrm{LTE} &= (2 - 2\beta_0 - 2\beta_1 - \beta_2)h^2 y''(x) + \left(\tfrac{7}{6} - 4\beta_0 - \beta_1\right)h^4 y^{(4)}(x) \\
&\quad + \left(\tfrac{31}{180} - \tfrac{4}{3}\beta_0 - \tfrac{1}{12}\beta_1\right)h^6 y^{(6)}(x) \\
&\quad + \left(\tfrac{127}{10{,}080} - \tfrac{8}{45}\beta_0 - \tfrac{1}{360}\beta_1\right)h^8 y^{(8)}(x) + \mathrm{O}(h^{10}).
\end{aligned} \tag{13}$$

We note that as $\sigma \to 0$ the coefficients β_0, β_1, β_2 tend to 3/40, 13/15, 7/60, respectively, and the method (10) reduces to the familiar Lambert–Watson method, [9], viz.,

$$\begin{aligned}
&y_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} + y_{n-2} \\
&\quad = \frac{h^2}{120}(9f_{n+2} + 104f_{n+1} + 14f_n + 104f_{n-1} + 9f_{n-2}).
\end{aligned} \tag{14}$$

In this case the local truncation error (13) simplifies to

$$\mathrm{LTE} = -\frac{19}{6048}h^8 y^{(8)}(x) + \mathrm{O}(h^{10}),$$

which justifies that the method (14) has an algebraic order 6.

3. Intervals of periodicity

Applying the method (5) with coefficients given by (6) and (7) to the test equation (3), we obtain the characteristic polynomial

$$\Omega(z; H^2) = (1 + \beta_0 H^2)z^2 - (2 - \beta_1 H^2)z + (1 + \beta_0 H^2), \quad H = \lambda h. \tag{15}$$

The roots of this polynomial will be a complex conjugate pair lying on the unit circle if

$$\left|\frac{2-\beta_1 H^2}{2+2\beta_0 H^2}\right| < 1.$$

The above condition gives the interval of periodicity $(0, H_0^2)$, where

$$H_0^2 = \frac{4}{\beta_1 - 2\beta_0}.$$

To find the interval of periodicity $(0, H_0^2)$ for the four-step method (10) with coefficients (11) and (12), we consider the associated characteristic polynomial given by

$$\Omega(z; H^2) \equiv A(H)z^4 - B(H)z^3 + C(H)z^2 - B(H)z + A(H), \quad H = \lambda h, \quad (16)$$

where

$$A(H) = (1+\beta_0 H^2), \quad B(H) = (2-\beta_1 H^2), \quad C(H) = (2+\beta_2 H^2).$$

The roots of the characteristic polynomial (16) should be complex conjugate pairs lying on the unit circle $|z| = 1$. For this requirement we shall determine the condition on H^2. So we use the transformation $z = (1+\xi)/(1-\xi)$, which maps the circle $|z| = 1$ into the line $\operatorname{Re}\xi = 0$, and the region $|z| \leqslant 1$ into $\operatorname{Re}\xi \leqslant 0$, which transforms (16) to

$$\begin{aligned}\Omega(\xi; H^2) \equiv{}& (2A(H) + 2B(H) + C(H))\xi^4 + 2(6A(H) - C(H))\xi^2 \\ &+ (2A(H) - 2B(H) + C(H)).\end{aligned}$$

The transformed polynomial must have purely imaginary roots, and this is possible under the requirement

$$2A(H) + 2B(H) + C(H) > 0,$$

which gives the interval of periodicity $(0, H_0^2)$, where

$$H_0^2 = \frac{8}{2\beta_1 - 2\beta_0 - \beta_2}.$$

The length of the interval of periodicity $(0, H_0^2)$ for the methods (5) and (10) increases as σ increases. The values of H_0^2 corresponding to various values of σ are listed in Table 1.

4. Phase-lag error

Consider the characteristic polynomial (15) obtained on applying the two-step method (5), with coefficients given by (6) and (7), to the test equation (3). The roots of the characteristic polynomial are

Table 1
Intervals of periodicity $(0, H_0^2)$ for the methods (5) and (10) for various σ

σ	H_0^2 for method (5)	H_0^2 for method (10)
5	97	99
15	3807	3512
25	214,719	200,113
50	220,682	221,269
100	250,253	360,657

$$z_{1,2} = \mathrm{e}^{\pm \mathrm{i}\theta(H)},$$

where $\theta(H)$ is given by

$$\theta(H) = \cos^{-1}\left(\frac{2 - \beta_1 H^2}{2 + 2\beta_0 H^2}\right). \tag{17}$$

Taking $p = \lambda$, we have $\sigma = \lambda h/2 = H/2$. The coefficients β_0, β_1 could be expressed in terms of H and a simple manipulation leads to

$$\frac{2 - \beta_1 H^2}{2 + 2\beta_0 H^2} = \cos H.$$

Thus $\theta(H) = H$, proving that the phase-lag is zero. It is well known that the phase-lag is half the truncation error (see [12]). Applying the method (5) to the test equation (3), we obtain

$$y_{n+1} - 2y_n + y_{n-1} = -H^2(\beta_0 y_{n+1} + \beta_1 y_n + \beta_0 y_{n-1}).$$

With the above linear multistep method, we associate the linear difference operator $\mathscr{L}$ defined by

$$\mathscr{L}[y(x); h] = (1 + \beta_0 H^2)y(x + h) - (2 - \beta_1 H^2)y(x) + (1 + \beta_0 H^2)y(x - h),$$

where $y(x)$ is an adequately smooth arbitrary test function. Expanding the test function in a Taylor series about x and collecting the like derivatives,

$$\begin{aligned}\mathscr{L}[y(x); h] &= 2(1 + \beta_0 H^2)\left(y(x) + \frac{h^2}{2!}y''(x) + \frac{h^4}{4!}y^{(4)}(x) + \cdots\right) - (2 - \beta_1 H^2)y(x)\\ &= \left[2(1 + \beta_0 H^2)\cos H - (2 - \beta_1 H^2)\right]y(x) = 0.\end{aligned}$$

Thus the method (5) with coefficients given by (6) and (7) has zero truncation error and so the algorithm generates exact solution at the grid points.

On applying the four-step method (10) with coefficients (11) and (12), to the test equation (3), we obtain the characteristic polynomial (16). Its complex roots of unit modules are given by

$$z_{1,2} = \mathrm{e}^{\pm \mathrm{i}\theta(H)},$$

where $\theta(H)$ is given by

$$\cos(\theta(H)) = \frac{B(H) + \sqrt{B^2(H) - 4A(H)C(H) + 8A^2(H)}}{4A(H)}. \tag{18}$$

Taking $p = \lambda$, we have $\sigma = \lambda h/2 = H/2$. Expressing $C(H)$ in terms of $A(H)$, $B(H)$ we obtain,

$$C(H) = 2(B(H)\cos H - A(H)\cos(2H)).$$

The expression for $\cos(\theta(H))$ simplifies to $\cos H$, whence $\theta(H) = H$, proving that the phase-lag is zero. Thus, if $p = \lambda$, both the methods (5) and (10) have zero phase-lag.

The truncation error of the linear multistep method (10) is determined by considering the associated linear difference operator $\mathscr{L}$ defined by

$$\begin{aligned}\mathscr{L}[y(x); h] = A(H)y(x+2h) - B(H)y(x+h) + C(H)y(x)\\ - B(H)y(x-h) + A(H)y(x-2h),\end{aligned}$$

where $y(x)$ is an adequately smooth arbitrary test function. Expanding the test function in a Taylor series about x and collecting the like derivatives,

$$\begin{aligned}\mathscr{L}[y(x); h] &= 2A(H)\left[y(x) + \frac{(2h)^2}{2!}y''(x) + \frac{(2h)^4}{4!}y^{(4)}(x) + \cdots\right]\\ &\quad - 2B(H)\left[y(x) + \frac{h^2}{2!}y''(x) + \frac{h^4}{4!}y^{(4)}(x) + \cdots\right] + C(H)y(x)\\ &= (2A(H)\cos(2H) - 2B(H)\cos H + C(H))y(x) = 0.\end{aligned}$$

Thus the method (10) with coefficients given by (11) and (12) has zero truncation error and so the algorithm generates exact solution at the grid points.

5. Phase-lag errors for an arbitrary *p*

To find the phase-lag error of the method (5) with coefficients given by (6) and (7), we use the expression for $\theta(H)$ given in (17), viz.,

$$\cos(\theta(H)) = \frac{2 - \beta_1 H^2}{2 + 2\beta_0 H^2},$$

and expanding $\cos H - \cos(\theta(H))$, [4], we arrive at

$$\cos H - \cos(\theta(H)) = \sum_{j=1}^{\infty}(-1)^j\left[\frac{1}{(2j)!} - \beta_0^{j-1}\left(\beta_0 + \frac{\beta_1}{2}\right)\right]H^{2j}.$$

As observed in Section 2, the method (5) reduces to Numerov method (9) as $\sigma \to 0$ and in this case the phase-lag is given by

$$\cos H - \cos(\theta(H)) = \tfrac{1}{480}H^6 + \mathrm{O}(H^8).$$

So when $p \neq \lambda$, the phase-lag of the method (5) tends to the phase-lag of the Numerov method as observed in [1]. The local truncation error of the method (5) tends to

$$\tfrac{1}{240}H^6 y(x) + \mathrm{O}(H^8),$$

as $\sigma \to 0$, justifying the fact that phase-lag constant is half of the truncation error constant.

To determine the phase-lag error of the method (10) with coefficients (11) and (12), we use the expression for $\theta(H)$ given by (18), viz.,

$$\cos(\theta(H)) = \frac{B(H) + \sqrt{B^2(H) - 4A(H)C(H) + 8A^2(H)}}{4A(H)},$$

where $A(H)$, $B(H)$ and $C(H)$ are as stated earlier. On substituting the expressions for $A(H)$, $B(H)$ and $C(H)$, the expansion for $\cos H - \cos(\theta(H))$ simplifies to

$$\cos H - \cos(\theta(H)) = s_1 H^2 + s_2 H^4 + s_3 H^6 + s_4 H^8 + \mathrm{O}(H^{10}),$$

where

$$\begin{aligned}
s_1 &= -\tfrac{1}{2!} + \tfrac{1}{4}(2\beta_0 + 2\beta_1 + \beta_2),\\
s_2 &= \tfrac{1}{4!} - \tfrac{1}{16}(6\beta_0 - \beta_2)(2\beta_0 + 2\beta_1 + \beta_2),\\
s_3 &= -\tfrac{1}{6!} + \tfrac{1}{32}[8\beta_0^2 + (4\beta_0 - \beta_1 - \beta_2)(2\beta_0 - \beta_2)](2\beta_0 + 2\beta_1 + \beta_2),\\
s_4 &= \tfrac{1}{8!} - \tfrac{1}{256}[64\beta_0^3 + (2\beta_0 - \beta_2)\{4\beta_1^2 + 8\beta_0(4\beta_0 - \beta_1 - \beta_2)\}\\
&\quad + (2\beta_0 - \beta_2)^2(6\beta_0 - 10\beta_1 - 5\beta_2)](2\beta_0 + 2\beta_1 + \beta_2).
\end{aligned}$$

As $\sigma \to 0$, we note that $s_1, s_2, s_3 \to 0$ and in this case the method (10) reduces to Lambert–Watson method (14) with the phase-lag,

$$\cos H - \cos(\theta(H)) = -7.853835979 \times 10^{-4} H^8 + \mathrm{O}(H^{10}).$$

So when $p \neq \lambda$, the phase-lag of the method (10) agrees with the phase-lag of Lambert–Watson method. The local truncation error of the method (10) tends to

$$-3.141534392 \times 10^{-3} H^8 y(x) + \mathrm{O}(H^{10}),$$

as $\sigma \to 0$, justifying that the phase-lag constant is a fourth of the truncation error constant.

6. Numerical illustrations

To illustrate that the new methods derived here have zero phase-lag and are almost P-stable we consider an inhomogeneous IVP that is well known in literature [13].

$$\textbf{IVP 1}: \quad y'' = -v^2 y + (v^2 - 1)\sin x, \\ y(0) = 1, \quad y'(0) = v + 1, \quad x \in [0, v\pi].$$

The analytical solution is: $y(x) = \cos(vx) + \sin(vx) + \sin x$; $v \gg 1$. (Here we take $v = 10$.) This solution consists of rapidly and slowly oscillating functions, the slowly oscillating function is due to the inhomogeneous term. We integrated the problem in the interval $x \in [0, 10\pi]$ for various values of the step size. The value $p = 10$ was chosen in each of the new methods. In the tabulations given below, the two-step and four-step methods derived above with their respective coefficients are identified as methods I and II, respectively. The results are compared with Numerov method (9) and Lambert–Watson method (14) which are identified as methods III and IV, respectively. The absolute errors at $x = 10\pi$ are tabulated in Table 2.

To illustrate that the new methods derived here produce accurate results comparable with Numerov and Lambert–Watson methods, we consider an inhomogeneous, nonlinear IVP well known in literature, viz., the Duffing equation forced by a harmonic function,

$$\textbf{IVP 2}: \quad y'' = -y - y^3 + \frac{1}{500}\cos(1.01x), \\ y(0) = 0.200426728067, \quad y'(0) = 0, \quad x \in \left[0, \tfrac{40.5}{1.01}\pi\right].$$

A very accurate approximation of the theoretical solution, [14], is given by

$$y(x) = 0.200179477536\cos(1.01x) + 0.000246946143\cos(3.03x) \\ + 0.304014 \times 10^{-6}\cos(5.05x) + 0.374 \times 10^{-9}\cos(7.07x),$$

on neglecting those coefficients smaller than 10^{-12}. The value $p = 1$ was chosen in each of the new methods. The results are compared with Numerov method

Table 2
Absolute errors at $x = 10\pi$ for the IVP 1

h	Method I	Method II	Method III	Method IV
$\pi/50$	0.6661 (−13)	0.1316 (−06)	0.9818 (−01)	0.1844 (−01)
$\pi/100$	0.6339 (−13)	0.5913 (−09)	0.6380 (−02)	0.2480 (−03)
$\pi/200$	0.9279 (−12)	0.1060 (−11)	0.3988 (−03)	0.3747 (−04)
$\pi/300$	0.2667 (−11)	0.6253 (−12)	0.7874 (−04)	0.3270 (−06)
$\pi/400$	0.2260 (−11)	0.2615 (−11)	0.2491 (−04)	0.5807 (−07)

Table 3
Absolute errors at $x = \frac{40.5}{1.01}\pi$ for the IVP 2

$\frac{40.5\pi}{1.01h}$	Method I	Method II	Method III	Method IV
500	0.8190 (−05)	0.1561 (−05)	0.1346 (−03)	0.3220 (−05)
1000	0.5061 (−06)	0.2340 (−07)	0.8399 (−05)	0.4940 (−07)
2000	0.3155 (−07)	0.3691 (−09)	0.5246 (−06)	0.7783 (−09)
3000	0.6236 (−08)	0.3870 (−10)	0.1036 (−06)	0.7468 (−10)
4000	0.1977 (−08)	0.1261 (−10)	0.3278 (−07)	0.1900 (−10)
5000	0.8141 (−09)	0.8414 (−11)	0.1342 (−07)	0.1011 (−10)

Table 4
Absolute errors at $x = 4.5$ for the IVP 3

$\frac{4.5}{h}$	Method I	Method II	Method III	Method IV
250	0.5449 (−04)	0.8452 (−06)	0.5139 (−04)	0.7743 (−06)
500	0.3642 (−05)	0.1679 (−07)	0.3438 (−05)	0.1545 (−07)
1000	0.2353 (−06)	0.3001 (−09)	0.2223 (−06)	0.2716 (−09)
2000	0.1499 (−07)	0.1007 (−10)	0.1415 (−07)	0.5101 (−11)

and Lambert–Watson method. The absolute errors at $x = \frac{40.5}{1.01}\pi$ are tabulated in Table 3.

In Table 4 the absolute errors at $x = 4.5$ are tabulated for the homogeneous, nonlinear IVP taken from [10],

$$\textbf{IVP 3}: \quad y'' = \frac{8y^2}{1+2x}, \quad y(0) = 1, \quad y'(0) = -2, \quad x \in [0, 4.5],$$

having the analytical solution, $y(x) = 1/(1+2x)$. The value $p = 1$ was chosen.

7. Concluding remarks

- In this paper, two-step and four-step symmetric multistep methods involving a parameter are derived for the numerical integration of the special IVP(1), including those having oscillatory solutions. The methods were designed to fit cosines of given frequencies.
- The methods have zero phase-lag when the parameter is suitably chosen and hence fit equations with oscillatory nature and are useful for long interval integration. The methods perform well when applied to nonlinear IVP as well.
- The cost of solving a nonlinear IVP, in each step, required an inner iteration process. We used the second order Newton–Raphson scheme. The choice of the initial approximation could be easily decided from the given differential equation.

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