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Reconstruction of Signals by Standard Tikhonov Method

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Abstract

In this work we propose a standard Tikhonov regularization approach for obtaining the signal f from the observed signal y^e. The observed signal is distorted by an additive noise or error e. Deviating from the usual assumption on the bound on $\|e\|$, we assume that the available noise is e^δ with $\|e - e^\delta\| \leq \delta$ and prove that the error $\|x_\alpha^\delta - \hat{f}\|$ between the regularized approximation x_α^δ and the solution $\hat{f}$ of the noise free equation $Kf = y$ is of optimal order. The regularization parameter α is chosen using a balancing principle considered in [10]. The computational results provided endorses the reliability and effectiveness of our method.

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1 Introduction

Consider the problem of restoration of the signal from the noisy data. If we model the signal by a function $f(t)$ and the observed signal by another function

$y(t)$, where t is the time parameter, we can set the problem of inferring $f(t)$ from $y(t)$ as a linear inverse problem, in which one has to solve the equation

$$Kf = y. \tag{1.1}$$

Note that here onwards we drop the parameter t for a simplified notation. Observe further that the equation (1.1) and the problem of solving it make sense only when placed in an appropriate framework. So we shall assume that both f and y are in some Hilbert space. Let $f \in X$ and $y \in Y$ where X and Y are Hilbert spaces with corresponding inner product $\langle ., . \rangle$ and norm $\|.\|$ respectively and $K : X \longrightarrow Y$ is a bounded linear operator with non-closed range $R(K)$. This non-closedness is reflected in the discontinuity of the inverse operator K^{-1}, if it exists. In general, the generalized solution $\hat{f} = K^{\dagger}y$, where $K^{\dagger}$ is the Moore-Penrose inverse of K [3, 6], does not depend continuously on the right-hand side y.

At the same time in many applications the observed data y^e is not same as the original signal y, but rather a distortion of y. This distortion is often modeled by an additive noise or error term $e \in Y$ (cf, [2])i.e.,

$$y^e = y + e = Kf + e. \tag{1.2}$$

Here we assume the noise to be independent of data or a random fluctuation.

In [2], Daubechies et.al. assumed that the "size" of the noise can be measured by its norm $\|e\|$ and further no information on $e = y^e - y$ is available beyond an upper bound on its norm in Y. Further in [8], it is assumed that the noisy data y^e can be represented as $y^e = y + \delta\xi$, with $\|\xi\|_Y \leq 1$.

In [2] and [8], the authors assumed that an upper bound on the norm of the noise $\|e\|$ is available. In this paper instead of the upper bound on $\|e\|$, we assume that, an approximation e^{δ} of e are available with

$$\|e^{\delta} - e\| \leq \delta. \tag{1.3}$$

Therefore instead of equation (1.2) one has to deal with the equation

$$K(f) = y^e - e^{\delta}. \tag{1.4}$$

We assume throughout that, equation (1.1) has a solution $\hat{f}$ (not necessarily unique), which in general does not depend continuously on the right hand side data y.

The focus in [2] and [8] was to recover $\hat{f} = K^{\dagger}y$ from (1.1) under the assumption that it has a sparse expansion

$$\hat{f} = \sum_i \lambda_i v_i \tag{1.5}$$

on the given system $\{v_i\}$. It is known ([2]) that a sparse structure of $K^{\dagger}y$ with respect to $\{v_i\}$ can be recovered by minimizing the functional

$$D_{\alpha,p} = \|Kx - y^e\|_Y^2 + \alpha\|x\|_p^p, \qquad 1 \le p < 2. \tag{1.6}$$

where α is a regularization parameter and x is the input signal.

It is worth to note that the reconstruction of a sparse structure is essentially the evaluation of coefficients λ_i in (1.5). For a system $\{v_i\}$ of linearly independent elements in X, each such coefficient can be seen as a value of some linear functional $\lambda_i(\hat{f})$ of the element $\hat{f}$, i.e., $\lambda_i(\hat{f}) = \langle l_i, \hat{f}\rangle$, where l_i is the generalized Ritz representation of $\hat{f}$.

From this point of view, the sparsity reconstruction can be seen as the problem of direct functional estimation. Further from the Corollary 3.2 of [1] it follows that by estimating $\langle l_i, \hat{f}\rangle$, by the standard Tikhonov regularization method; i.e.,

$$\langle l_i, z_\alpha^\delta\rangle, \tag{1.7}$$

where z_α^δ is the Tikhonov approximation, is of optimal order for a wide range of functionals l_i and elements $\hat{f}$, provided the regularization parameter α is chosen properly. Further note that the construction of a Tikhonov approximation z_α^δ, and a calculation of estimation (1.7) for each individual l_i, are less computationally demanding than a minimization of (1.6).

Thus in this paper we present a procedure for reconstruction of the signal $\hat{f}$, based on the standard Tikhonov regularization method. In this method a regularized approximation x_α^δ of (1.4) is obtained by solving the minimization problem;

$$\underset{x \in X}{min} J_\alpha^\delta(x), \qquad J_\alpha^\delta(x) = \|K(x) - y^e + e^\delta\|^2 + \alpha\|x\|^2. \tag{1.8}$$

The minimizer x_α^δ of the Tikhonov functional $J_\alpha^\delta(x)$ satisfies the Euler equation

$$K^*Kx_\alpha^\delta + \alpha x_\alpha^\delta = K^*(y^e - e^\delta). \tag{1.9}$$

In this paper we take x_α^δ as an approximation for $\hat{f}$. The paper is organized as follows. In section 2 we derive the error estimate $\|\hat{f} - x_\alpha^\delta\|$ and in section 3 we consider the discretized Tikhonov regularization. Section 4 deals with the error analysis and parameter choice strategy for discretized Tikhonov regularization. Computed examples are shown in section 5. We conclude the work in section 6.

2 Preliminaries

We have at our disposal the element x_α^δ depending on the parameter α. Now it is important to find the distance $\|\hat{f} - x_\alpha^\delta\|$ between $\hat{f}$ and x_α^δ. We observed that

$$\begin{aligned}
\|\hat{f} - x_\alpha^\delta\| &= \|\hat{f} - (K^*K + \alpha I)^{-1}K^*(y^e - e^\delta)\| \\
&= \|\hat{f} - (K^*K + \alpha I)^{-1}K^*(y^e - e + e - e^\delta)\| \\
&= \|\hat{f} - (K^*K + \alpha I)^{-1}K^*(y + e - e^\delta)\| \\
&\leq \|\hat{f} - (K^*K + \alpha I)^{-1}K^*y\| + \|(K^*K + \alpha I)^{-1}K^*(e - e^\delta)\| \\
&\leq \|\alpha(K^*K + \alpha I)^{-1}\hat{f}\| + \|(K^*K + \alpha I)^{-1}K^*(e - e^\delta)\| \\
&\leq \|\alpha(K^*K + \alpha I)^{-1}\hat{f}\| + \frac{\delta}{\sqrt{\alpha}}. \qquad (2.10)
\end{aligned}$$

Thus in order to obtain an error estimate for $\|\hat{f} - x_\alpha^\delta\|$ one has to obtain an error estimate for $\|\alpha(K^*K + \alpha I)^{-1}\hat{f}\|$.

We will use the following assumption to obtain an error estimate for $\|\alpha(K^*K + \alpha I)^{-1}\hat{f}\|$.

Assumption 2.1 *There exists a continuous, strictly monotonically increasing function* $\varphi : (0, a] \to (0, \infty)$ *with* $a \geq \|K^*K\|$ *satisfying;*

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0$
- $$\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \qquad \forall \lambda \in (0, a].$$
- *there exists* $v \in X$ *such that*
$$\hat{f} = \varphi(K^*K)v. \qquad (2.11)$$

The proof of the following theorem is analogous to the proof of Theorem 2.4 in [5].

Theorem 2.2 *Let the Assumptions 2.1 holds. Then*

$$\|\alpha(K^*K + \alpha I)^{-1}\hat{f}\| \leq \|v\|\varphi(\alpha). \qquad (2.12)$$

Thus by (2.10) and (2.12) we have the following theorem.

Theorem 2.3 *Let the Assumptions 2.1 holds and* x_α^δ *be as in (1.9). Then*

$$\|\hat{f} - x_\alpha^\delta\| = \max\{\|v\|, 1\}(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}).$$

The error estimate in Theorem 2.3 is of optimal order with respect to δ for the choice of $\alpha := \alpha(\delta)$ such that $\sqrt{\alpha}\varphi(\alpha) = \delta$. Note that in this case $\alpha = \varphi^{-1}(\psi^{-1}(\delta))$ where $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$.

The point is that the function φ is usually unknown. Therefore in practical applications different parameters $\alpha = \alpha_i$ are often selected from some finite set

$$D_N = \{\alpha_i : 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\},$$

and corresponding elements $x^\delta_{\alpha_i}, i = 1, 2 \cdots, N$ are studied on-line. We will discuss about such a parameter choice rule, called the balancing principle in section 4.

3 Discretized Tikhonov regularization

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X. We assume that

$$\varepsilon_h := \|K(I - P_h)\| \to 0.$$

as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ pointwise and if K is a compact operator. The discretized Tikhonov regularization method for the regularized equation (1.9) consists of solving the equation

$$(P_hK^*KP_h + \alpha I)x^\delta_{\alpha,h} = P_hK^*(y^\delta - e^\delta). \tag{3.13}$$

Theorem 3.1 *Let $x^\delta_{\alpha,h}$ be as in (3.13) and Assumption 2.1 holds. Then*

$$\|\hat{f} - x^\delta_{\alpha,h}\| \leq C(\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\sqrt{\alpha}}). \tag{3.14}$$

where $C = \max\{1, \|v\|, \|\hat{f}\|\}$.

Proof. Since

$$x^\delta_{\alpha,h} = (P_hK^*KP_h + \alpha I)^{-1}P_hK^*(y^e - e^\delta)$$

we have

$$\begin{aligned}
\hat{f} - x^\delta_{\alpha,h} &= \hat{f} - (P_hK^*KP_h + \alpha I)^{-1}P_hK^*(y^e - e^\delta) \\
&= \hat{f} - (P_hK^*KP_h + \alpha I)^{-1}P_hK^*(y + e - e^\delta) \\
&= \hat{f} - (P_hK^*KP_h + \alpha I)^{-1}P_hK^*y \\
&\quad -(P_hK^*KP_h + \alpha I)^{-1}P_hK^*(e - e^\delta) \\
&= \hat{f} - (K^*K + \alpha I)^{-1}K^*y + (K^*K + \alpha I)^{-1}K^*y \\
&\quad -(P_hK^*KP_h + \alpha I)^{-1}P_hK^*y \\
&\quad -(P_hK^*KP_h + \alpha I)^{-1}P_hK^*(e - e^\delta) \\
&= \alpha(K^*K + \alpha I)^{-1}\hat{f} \\
&\quad +(K^*K + \alpha I)^{-1}K^*y - (P_hK^*KP_h + \alpha I)^{-1}P_hK^*y \\
&\quad -(P_hK^*KP_h + \alpha I)^{-1}P_hK^*(e - e^\delta).
\end{aligned}$$

Now since

$$\|[(P_hK^*KP_h+\alpha I)^{-1}P_hK^*(e-e^\delta)\| \leq \frac{\delta}{\sqrt{\alpha}} \tag{3.15}$$

by Theorem 2.2 we have

$$\|\hat{f}-x^\delta_{\alpha,h}\| \leq \|v\|\varphi(\alpha)+\frac{\delta}{\sqrt{\alpha}}+\Lambda \tag{3.16}$$

where $\Lambda = \|(K^*K+\alpha I)^{-1}K^*y-(P_hK^*KP_h+\alpha I)^{-1}P_hK^*y\|$. Observe that

$$\begin{aligned}
\|\Lambda\| &= \|(P_hK^*KP_h+\alpha I)^{-1}[(P_hK^*KP_h+\alpha I)K^* \\
&\quad -P_hK^*(KK^*+\alpha I)](KK^*+\alpha I)^{-1}y \\
&\leq \|(P_hK^*KP_h+\alpha I)^{-1}P_hK^*K(P_h-I)K^*(KK^*+\alpha I)^{-1}y\| \\
&\quad +\|(P_hK^*KP_h+\alpha I)^{-1}\alpha(P_h-I)K^*(KK^*+\alpha I)^{-1}y\| \\
&\leq \frac{\varepsilon_h\|\hat{f}\|}{\sqrt{\alpha}}. \qquad (3.17)
\end{aligned}$$

Thus (3.14) follows from (3.16) and (3.17). This completes the proof.

4 Error Analysis

The error estimate in (3.14) is of optimal order with respect to $\delta+\varepsilon_h$ for the choice of $\alpha := \alpha_{\delta,h}$ such that $\sqrt{\alpha_{\delta,h}}\varphi(\alpha_{\delta,h}) = \delta+\varepsilon_h$. Note that in this case $\alpha_{\delta,h} = \varphi^{-1}(\psi^{-1}(\delta+\varepsilon_h))$ where $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$.

But, such an a priori parameter choice α_δ cannot be used in practice since the smoothness properties of the unknown solution $\hat{f}$ reflected in the function φ are generally unknown. In an a posteriori choice, one finds a parameter $\alpha_{\delta,h}$ without making use of the unknown source function φ. There exist many parameter choice strategies in the literature, for example see [4, 7, 9, 10, 11]. In this paper we consider the balancing principle considered in [10] for choosing the parameter $\alpha_{\delta,h}$.

4.1 The balancing principle

The balancing principle considered in [10] starts with a finite number of real numbers $\alpha_0, \alpha_1, \cdots, \alpha_N$, such that

$$\alpha_0 < \alpha_1 < \cdots < \alpha_N.$$

In this paper we consider a particular case where $\alpha_i = \mu^{2i}\alpha_0$, $\mu > 1$, $\alpha_0 > 0$ and $i = 1, 2, ...N$. Let

$$l := max\{i : \varphi(\alpha_i) \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}\} < N \tag{4.18}$$

and

$$k := max\{i : \|x^{\delta}_{\alpha_i,h} - x^{\delta}_{\alpha_j,h}\| \leq \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}, j = 0, 1, 2, \cdots, i\}. \tag{4.19}$$

The proof of the next theorem is analogous to the proof of Theorem 1.2 in [10], but for the sake of completeness, we supply its proof as well.

Theorem 4.1 *Let l be as in (4.18), k be as in (4.19) $x^{\delta}_{\alpha_k,h}$ be as in (3.13) with $\alpha = \alpha_k$. Then $l \leq k$,*
and

$$\|\hat{f} - x^{\delta}_{\alpha_k,h}\| \leq (1 + \frac{2\mu}{\mu-1})C\mu\psi^{-1}(\delta+\varepsilon_h). \tag{4.20}$$

Proof. Note that, to prove $l \leq k$, it is enough to prove that, for $i = 1, 2, \cdots, N$

$$\varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}} \Longrightarrow \|x^{\delta}_{\alpha_i,h} - x^{\delta}_{\alpha_j,h}\| \leq \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}, \forall j = 0, 1, 2, \cdots, i.$$

For $j \leq i$,

$$\begin{aligned} \|x^{\delta}_{\alpha_i,h} - x^{\delta}_{\alpha_j,h}\| &\leq \|x^{\delta}_{\alpha_i,h} - \hat{f}\| + \|\hat{f} - x^{\delta}_{\alpha_j,h}\| \\ &\leq C[\varphi(\alpha_i) + \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}} + \varphi(\alpha_j) + \frac{\delta}{\sqrt{\alpha_j}}] \\ &\leq C[\frac{2(\delta+\varepsilon_h)}{\sqrt{\alpha_i}} + \frac{2(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}] \\ &\leq \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}. \end{aligned}$$

This proves the relation $l \leq k$. Now since $\sqrt{\alpha_{l+m}} = \mu^m\sqrt{\alpha_l}$, by using triangle inequality successively, we obtain

$$\begin{aligned} \|\hat{f} - x^{\delta}_{\alpha_k,h}\| &\leq \|\hat{f} - x^{\delta}_{\alpha_l,h}\| + \sum_{j=l+1}^{k} \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_{j-1}}} \\ &\leq \|\hat{f} - x^{\delta}_{\alpha_l,h}\| + \sum_{m=0}^{k-l-1} \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_l}\mu^m} \\ &\leq \|\hat{f} - x^{\delta}_{\alpha_l,h}\| + (\frac{\mu}{\mu-1})\frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_l}} \end{aligned}$$

Therefore by (3.14) and (4.18) we have

$$\begin{aligned} \|\hat{f} - x^{\delta}_{\alpha_k,h}\| &\leq C[\varphi(\alpha_l) + \frac{\delta+\varepsilon_h}{\sqrt{\alpha_l}}] + (\frac{\mu}{\mu-1})\frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_l}} \\ &\leq (2 + \frac{4\mu}{\mu-1})C\mu\psi^{-1}(\delta+\varepsilon_h). \end{aligned}$$

The last step follows from the inequality $\sqrt{\alpha_\delta} \leq \sqrt{\alpha_{l+1}} \leq \mu\sqrt{\alpha_l}$ and $\frac{\delta}{\sqrt{\alpha_\delta}} = \psi^{-1}(\delta + \varepsilon_h)$.

This completes the proof.

5 Computed examples

The algorithm associated with the choice of the parameter specified in Theorem 4.1 involves the following steps:

```
begin
   i=0
```

$\alpha_0 = (\delta + \varepsilon_h)^2.$

```
    repeat
      i=i+1
```

$\alpha_i = \mu^{2i}\alpha_0$

Solve for $z_i : (K^*K + \alpha_i I)z_i = K^*(y^e - e^\delta)$

```
          j=-1
          repeat
             j=j+1
```

Solve for $w_{i,j} : (K^*K + \alpha_j I)w_{i,j} = (\alpha_j - \alpha_i)z_i$

until($\|w_{i,j}\| \leq 4\mu^{-j}$AND $j < i$)

k = i-1.

Solve for $e_k : (K^*K + \alpha_k I)e_k = K^*(y^e - e^\delta)$

We illustrate the performance of the method considered in the previous sections with two example. The computations are carried out in Matlab.

Example 5.1 *In this example we consider the space $X = Y = L[0, 1]$ and consider*

$$K(x)(s) = \int_0^1 k(s,t)x(t)dt \tag{5.21}$$

with

$$k(s,t) = \begin{cases} 0, & s \leq t \\ s-t, & s > t. \end{cases} \tag{5.22}$$

We apply the above Algorithm by choosing V_n as the space of linear splines in a uniform grid of $n+1$ points in $[0, 1]$. Specifically for fixed n we consider $t_i = \frac{i-1}{n}, i = 1, 2, \cdots, n+1$ as the grid points.

In this example we take $y^e = \frac{1}{12}(6s^2 - 4s^3 + s^4)$, $e = \frac{1}{24}(6s^2 - 4s^3 + s^4)$. *Then the exact solution is* $\hat{f} = \frac{1}{2}(s-1)^2$. *Since* $\hat{f} = K^*K(2) = R(K^*K))$, $\varphi(\lambda) = \lambda$ *and hence* $\psi^{-1}(\delta + \varepsilon_h) = \varphi(\alpha_\delta) = (\delta + \varepsilon_h)^{2/3}$. *Here e is randomly perturbed by* $e^\delta = e + \delta$. *According to the theory,*

$$\|\hat{f} - x^\delta_{\alpha_k,h}\| = \bigcirc(\psi^{-1}(\delta + \varepsilon_h)).$$

The result are given in Table 1. Here and below $e_k := \|\hat{f} - x^\delta_{\alpha_k,h}\|$.

Example 5.2 *In this example we take* $X = Y = L^2[0,\pi]$ *and* $K : X \to Y$ *is given by*

$$K(x)(s) = \int_0^\pi k(s,t)x(t)dt \tag{5.23}$$

with $k(s,t)$ *as in (5.22). In this example also we choose* V_n *as the space of linear splines in a uniform grid of* $n+1$ *points in* $[0,\pi]$ *with the grid point* $t_i = \left(\frac{i-1}{n}\right)\pi,\ i = 1, 2, ...n+1$. *Here we take* $y^e = -2(s + \sin(s))$, $e = -(s + \sin(s))$ *and* $e^\delta = e + \delta$. *Then the exact solution is* $\hat{f} = \sin(s)$. *The result are given in Table 2.*

Table 1: $\delta = 0.0096\mu = 2.06$

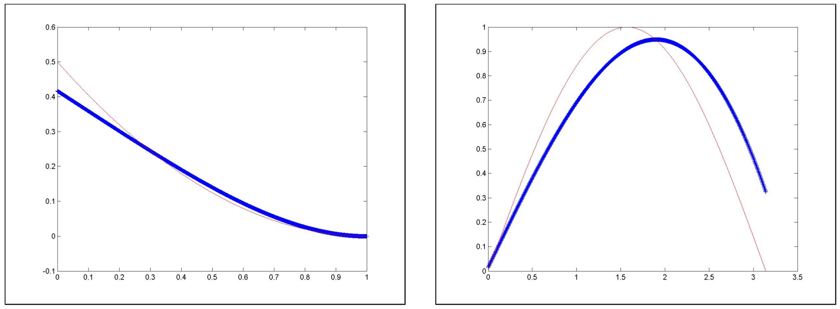
n	k	e_k	$\frac{e_k}{\psi^{-1}(\delta+\varepsilon_h)}$
32	2	0.3023	3.0916
64	2	0.3027	3.0968
128	2	0.3028	3.0984
256	2	0.3028	3.0990
512	2	0.3029	3.0992
1024	2	0.3029	3.0992

Table 2: $\delta = 0.0667\mu = 1.5$

n	k	e_k	$\frac{e_k}{\psi^{-1}(\delta+\varepsilon_h)}$
32	7	0.9189	2.2650
64	8	0.6398	1.5777
128	9	0.7886	1.9449
256	9	0.7887	1.9450
512	9	0.7887	1.9451
1024	9	0.7887	1.9451

6 Conclusion

In this paper we have proposed a method to reconstruct the signal from a distorted signal. The computational results shows the reliability of the method. Though we have only shown the results of reconstruction of 1-D signals, the method can be extended to 2-D signals (Images) as well, with a minor modification in the basis.



a b

Figure 1: The original and reconstructed signal: (a) The dotted line shows the original signal and the dark line shows the reconstructed one for Example 5.1. (b)The dotted line shows the original signal and the dark line shows the reconstructed one for Example 5.2.

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