ON EXCEPTIONAL VALUES OF ENTIRE AND MEROMORPHIC FUNCTIONS

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ABSTRACT

Let $f(z)$ be meromorphic function of finite nonzero order ρ. Assuming certain growth estimates on f by comparing it with $r^\rho L(r)$ where $L(r)$ is a slowly changing function we have obtained the bounds for the zeros of $f(z) - g(z)$ where $g(z)$ is a meromorphic function satisfying $T(r, g) = o\{T(r, f)\}$ as $r \to \infty$. These bounds are satisfied but for some exceptional functions. Examples are given to show that such exceptional functions exist.

1. Let $f(z)$ be a meromorphic function of order $\rho\,(0 < \rho < \infty)$. If $f(z)$ is an entire function let $M(r, f) = \max |f(z)|$ on $|z| = r$. Let $T(r, f)$ be the Nevanlinna's characteristic function for $f(z)$ and $g_1(z)$, $g_2(z), \cdots$ be any set of functions satisfying

$$T\big(r, g_i(z)\big) = o\big(T(r, f)\big) \quad \text{as} \quad r \to \infty\,(i = 1, 2, \ldots). \tag{1.1}$$

Let $n(r, x)$, $\bar{n}(r, x)$ be the number of zeros and the number of distinct zeros respectively of $f(z) - x$ and $\bar{n}(r, f - g)$ the number of distinct zeros of $f(z) - g(z)$ in $|z| \leqq r$. Define

$$\bar{N}\left(r, \frac{1}{f-g}\right) = \int_0^r \frac{\bar{n}(t, f-g)}{t}\, dt.$$

If g is an infinite constant let $\bar{n}(r, f - g) = \bar{n}(r, f)$ the number of distinct poles of $f(z)$ in $|z| \leqq r$.

In this paper we study the exceptional values of the function $f(z)$ by making use of the comparison function $r^\rho L(r)$ where $L(r)$ is a slowly increasing function satisfying

$L(Ct) \sim L(t)$ as $t \to \infty$ for every fixed positive C. Let k denote any constant $\geqq 1$ and

$$h(\rho) = \{\rho + (1+\rho^2)^{\frac{1}{2}}\} \left\{\frac{1+(1+\rho^2)^{\frac{1}{2}}}{\rho}\right\}^{\rho} \quad (\rho > 0). \tag{1.2}$$

Let A be a constant not necessarily the same at each occurrence.

THEOREM 1.—*If* f (z) *is an entire function of order* $\rho\,(o < \rho < \infty)$ *satisfying*

$$\limsup_{r\to\infty} \frac{\log M(kr,f)}{r^{\rho} L(r)} = a \qquad (0 \leqq a \leqq \infty) \tag{1.3}$$

then

$$\limsup_{r\to\infty} \frac{\bar{n}(r, f-g)}{r^{\rho} L(r)} \geqq \frac{a\rho}{2k^{\rho} h(\rho)} \tag{1.4}$$

and

$$\limsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{f-g}\right)}{r^{\rho} L(r)} \geqq \frac{a}{2k^{\rho} h(\rho)} \tag{1.5}$$

for every entire function $g(z)$ (including a polynomial or a finite constant) satisfying (1.1) with one possible exception.

Remark.—The exceptional function may actually exist. Consider for example

$$f(z) = \prod_{n=2}^{\infty} \left(1 + \frac{z}{n(\log n)^2}\right).$$

Here

$$\bar{n}(r, 0) \sim \{r/(\log r)^2\}; \quad \log M(r,f) \sim (r/\log r).$$

Set

$$r^{\rho} L(r) = r^{\rho(r)}$$

where

$$\rho(r) = 1 - \frac{\log\log r}{\log r}$$

Then $\rho(r)$ is a proximate order relative to $\log M(r, f)$ and $r^{\rho(r)-\rho}$ is a slowly increasing function [*see* Levin[3] (p. 32)]. Also

$$\lim_{r\to\infty} \frac{\log M(r,f)}{r^{\rho(r)}} = 1,$$

but

$$\frac{\bar{n}(r, 0)}{r^{\rho(r)}} \to 0 \quad \text{as} \quad r \to \infty.$$

Proof.—First take $0 < a < \infty$. Set

$$B = \frac{a\rho}{2} \frac{\lambda - 1}{\lambda + 1} (\lambda k)^{-\rho} \, (\lambda > 1). \tag{1.6}$$

Let us suppose, if possible, that there are two functions $g_1(z)$ and $g_2(z)$ for which

$$\limsup_{r\to\infty} \frac{\bar{n}(r, f - g)}{r^\rho L(r)} \leqq C < B.$$

Let $C < C_1 < B$, then

$$\frac{\bar{n}(r, f - g_1)}{r^\rho L(r)} < C_1, \quad \text{for all} \quad r \geqq r_0$$

and

$$\bar{N}\left(r, \frac{1}{f - g_1}\right) = A + \int_{r_0}^{r} \frac{\bar{n}(t, f - g_1)}{t} dt$$

$$< A + C_1 \int_{r_0}^{r} t^{\rho-1} L(t) \, dt$$

We have by [1, Lemma 5]

$$\int_{r_0}^{r} t^{\rho-1} L(t) \, dt \sim \frac{L(r)}{\rho} r^\rho.$$

Hence

$$\bar{N}\left(r, \frac{1}{f - g_1}\right) < \frac{C_1}{\rho} r^\rho L(r) \left(1 + o(1)\right).$$

Similarly for

$$\bar{N}\left(r, \frac{1}{f-g_2}\right).$$

Further by a result of Nevanlinna[2] (p. 47)

$$\{1+o(1)\}\, T(r,f) < \bar{N}\left(r, \frac{1}{f-g_1}\right) + \bar{N}\left(r, \frac{1}{f-g_2}\right) + O(\log r)$$

Hence

$$T(r,f) < \frac{2C_1}{\rho} r^\rho L(r)\{1+o(1)\} \quad \text{for all} \quad r \geqq r_0.$$

Also

$$\log M(r,f) > (a-\epsilon)\frac{r^\rho}{k^\rho} L\left(\frac{r}{k}\right)$$

for arbitrarily large r and from [2, p. 18] for all large r

$$\log M(r,f) < \frac{\lambda+1}{\lambda-1} T(\lambda r, f) \qquad (\lambda > 1).$$

Thus

$$(a-\epsilon)\frac{r^\rho}{k^\rho} L\left(\frac{r}{k}\right) < \frac{\lambda+1}{\lambda-1}\frac{2C_1}{\rho}(\lambda r)^\rho L(\lambda r)\{1+o(1)\}$$

for arbitrarily large r.

Since $L(Ct) \sim L(t)$ for every fixed positive C we have

$$C_1 \geqq \frac{a\rho}{2}\frac{\lambda-1}{\lambda+1}(\lambda k)^{-\rho} = B.$$

This gives a contradiction. Hence

$$\limsup_{r\to\infty} \frac{\bar{n}(r, f-g)}{r^\rho L(r)} \geqq B$$

except possibly for one $g(z)$.

The best choice of λ in (1.6) can be easily seen to be

$$\lambda = \frac{\left(1+(1+\rho^2)^{\frac{1}{2}}\right)}{\rho}$$

and we get (1.3) for $0 < a < \infty$. The argument for $a = \infty$ is similar. We need take an arbitrary large number in place of a. The case $a = 0$ is obvious.

The proof of (1.5) is similar. We need take

$$B = \frac{a}{2}\frac{\lambda - 1}{\lambda + 1}(\lambda k)^{-\rho}\,(\lambda > 1).$$

COROLLARY 1.—*If* f (z) *is an entire function of order* ρ $(0 < \rho < \infty)$ *satisfying*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho} L(r)} = a \qquad (0 \leqq a \leqq \infty)$$

then

$$\limsup_{r\to\infty} \frac{n(r,x)}{r^{\rho} L(r)} \geqq \frac{a\rho}{2h(\rho)} \tag{1.7}$$

except possibly for one value of x.

This is got by putting $k = 1$ and $g(z) = x$ in (1.4) and observing $n \geqq \bar{n}$. This result is due to S. K. Singh[6], (Thm. 1).

COROLLARY 2.—*If* f (z) *is an entire function of order* ρ $(0 < \rho < \infty)$ then

$$\liminf_{r\to\infty} \frac{\log M(kr,f)}{\bar{n}(r,f-g)} \leqq \frac{2k^{\rho}h(\rho)}{\rho} \tag{1.8}$$

and

$$\liminf_{r\to\infty} \frac{\log M(kr,f)}{\bar{N}\left(r, \frac{1}{f-g}\right)} \leqq 2k^{\rho}\,h(\rho) \tag{1.9}$$

for every entire function $g(z)$ with one possible exception. We can choose a comparison function $L(r)$ in (1.3) such that $o < a < \infty$, for example, if $L(r) = r^{\rho(r)-\rho}$ where $\rho(r)$ is the proximate order relative to $\log M(r,f)$ then

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho(r)}}$$

is different from zero and infinity *see* B. Ja. Levin[3] (p. 32). Then (1.8) immediately follows from the relation

$$\liminf_{r\to\infty} \frac{f(r)}{g(r)} \leq \frac{\limsup_{r\to\infty} f(r)}{\limsup_{r\to\infty} g(r)}$$

by taking

$$f(r) = \frac{\log M(kr, f)}{r^\rho L(r)}$$

and

$$g(r) = \frac{\bar{n}(r, f-g)}{r^\rho L(r)}.$$

Proof of (1.9) is similar.

For an alternate proof of Corollary 2 *see* S. M. Shah[5], (Thm. 3).

THEOREM 2.—*If* f (z) *is a meromorphic function of order* $\rho\,(0 < \rho < \infty)$ *satisfying*

$$\limsup_{r\to\infty} \frac{T(kr, f)}{r^\rho L(r)} = a \qquad (0 \leqq a \leqq \infty) \tag{2.1}$$

then

$$\limsup_{r\to\infty} \frac{\bar{n}(r, f-g)}{r^\rho L(r)} \geqq \frac{\rho a}{3k^\rho} \tag{2.2}$$

and

$$\limsup_{r\to\infty} \frac{\bar{N}\left(r, \dfrac{1}{f-g}\right)}{r^\rho L(r)} \geqq \frac{a}{3k^\rho} \tag{2.3}$$

except possibly for two meromorphic functions $g(z)$ (including a constant, finite or infinite) satisfying (1.1)

COROLLARY 3.—*Under the same conditions of the above theorem*

$$\liminf_{r\to\infty} \frac{T(kr, f)}{\bar{n}(r, f-g)} \leqq \frac{3k^\rho}{\rho} \tag{2.4}$$

and

$$\liminf_{r\to\infty} \frac{T(kr, f)}{\bar{N}\left(r, \dfrac{1}{f-g}\right)} \leqq 3k^\rho \tag{2.5}$$

Proof.—Let $0 < a < \infty$. Let us suppose that there are three functions $g_i(z)$ $(i = 1, 2, 3)$ for which

$$\limsup_{r\to\infty} \frac{\bar{n}(r, f-g)}{r^\rho L(r)} = C_i$$

where

$$C_i < \frac{\rho a}{3k^\rho}. \quad \text{Let} \quad C = \max(C_1, C_2, C_3) \quad \text{and}$$

$$C < D < \frac{\rho a}{3k^\rho}.$$

Hence

$$\bar{n}(r, f - g_i) < D r^\rho L(r) \quad \text{for all} \quad r \geqq r_0$$

and

$$\bar{N}\left(r, \frac{1}{f - g_i}\right) = A + D \int_{r_0}^{r} \frac{\bar{n}(t, f - g_i)}{t} dt \; (i = 1, 2, 3)$$

$$< A + D \int_{r_0}^{r} t^{\rho-1} L(t) \, dt$$

$$\sim A + D \frac{r^\rho L(r)}{\rho}.$$

Also from Nevanlinna[2], (p. 47) we have

$$\{1 + o(1)\} T(r, f) < \sum_{i=1}^{3} \bar{N}\left(r, \frac{1}{f - g_i}\right) + O(\log r).$$

Hence

$$\{1 + o(1)\} T(r, f) < \frac{3D}{\rho} r^\rho L(r) \{1 + o(1)\}$$

Also from (2.1) for arbitrarily large values of r we have

$$T(r, f) > (a - \epsilon)(r/k)^\rho L(r/k)$$

and hence

$$(a - \epsilon)\left(\frac{r}{k}\right)^\rho L\left(\frac{r}{k}\right) < \frac{3D}{\rho} r^\rho L(r) \{1 + o(1)\}$$

for a sequence of $r \to \infty$. Since $L(r/k) \sim L(r)$ we have

$$D \geqq \frac{\rho a}{3k^\rho}.$$

This gives a contradiction and the result is proved for $0 < a < \infty$. The case $a = \infty$ is similar if we take arbitrarily large number in place of a. If $a = 0$ the result is obvious.

Proof of (2.3) is similar. Corollary 3 follows as in Theorem 1 if we take the comparison function $r^\rho L(r)$ such that

$$\limsup_{r\to\infty} \frac{T(kr, f)}{r^\rho L(r)}$$

is finite and non-zero which is always possible.

For an alternate proof of Corollary 3 with $k = 1$ and $g(z) = x$ *see* [5]. In the general case $k \geqq 1$ *see* [6].

THEOREM 3.—*Let* f (z) *be a meromorphic function of order* ρ $(0 < \rho < \infty)$. Let

$$\lim_{r\to\infty} \frac{T(kr, f)}{r^\rho L(r)} = a \qquad (0 < a < \infty) \tag{3.1}$$

and

$$\lim_{r\to\infty} \frac{\bar{n}(r, f - g_i)}{r^\rho L(r)} = 0 \qquad (i = 1, 2) \tag{3.2}$$

for any two different meromorphic functions $g_i(z)\,(g_i(z) \not\equiv \infty)$ $(i = 1, 2)$ and satisfying (1.1), then for all meromorphic functions $g(z)$ satisfying (1.1) including an infinite constant

$$\lim_{r\to\infty} \frac{\bar{n}(r, f - g)}{r^\rho L(r)} = \frac{a\rho}{k^\rho} \tag{3.3}$$

and

$$T(r, f') \sim 2T(r, f) \tag{3.4}$$

where $T(r, f')$ is the characteristic function for $f'(z)$. We need the following lemma [7, p. 30].

LEMMA.—*If* $\int_{r_0}^{r} \phi(t)\,dt \sim Ar^\rho L(r)$, *where* $\phi(t)$ *is a non-decreasing function, then* $\phi(r) \sim A\rho r^\rho L(r)$.

Proof of Theorem 3.—We have from (3.2)

$$\bar{n}(r, f - g_i) = o\{r^\rho L(r)\} \quad \text{as} \quad r \to \infty \quad \text{and}$$

hence

$$\bar{N}\left(r, \frac{1}{f-g_i}\right) = o\,\{r^{\rho} L(r)\} \qquad \text{as} \quad r \to \infty.$$

Also from [2, p. 47]

$$\{1 + o(1)\}\, T(r, f) < \sum_{i=1}^{2} \bar{N}\left(r, \frac{1}{f-g_i}\right) + \bar{N}\left(r, \frac{1}{f-g}\right)$$
$$+ O(\log r).$$

Using (3.1) we get for all $r \geqq r_0$

$$(a-\epsilon)\left(\frac{r}{k}\right)^{\rho} L\left(\frac{r}{k}\right) < o\,\{r^{\rho} L(r)\} + \bar{N}\left(r, \frac{1}{f-g}\right) + O(\log r).$$

Hence

$$\liminf_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{f-g}\right)}{r^{\rho} L(r)} \geqq \frac{a}{k^{\rho}}. \tag{3.5}$$

Also, since $g(z)$ satisfies (1.1)

$$\bar{N}\left(r, \frac{1}{f-g}\right) < \{1 + o(1)\}\, T(r, f)$$
$$< \{1 + o(1)\}\, (a+\epsilon)\, (r/k)^{\rho} L(r/k).$$

Hence

$$\limsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{f-g}\right)}{r^{\rho} L(r)} \leqq \frac{a}{k^{\rho}}. \tag{3.6}$$

From (3.5) and (3.6) we get

$$\lim_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{f-g}\right)}{r^{\rho} L(r)} = \frac{a}{k^{\rho}} \tag{3.7}$$

(3.3) follows from (3.7) immediately by the lemma when $\phi(t) = \bar{n}(t, f-g)$. To prove (3.4) we take $g(z) \equiv \infty$ in (3.7). We have then on using (3.1)

$$\frac{T(r, f')}{T(r, f)} \geqq \frac{N(r, f) + \bar{N}(r, f)}{T(r, f)} \geqq \frac{2\bar{N}(r, f)}{T(r, f)}$$

$$\geqq 2\left(\frac{a-\epsilon}{a+\epsilon}\right)\frac{L(r)}{L(r/k)} \qquad \text{for all} \quad r \geqq r_0$$

$$\sim 2\left(\frac{a-\epsilon}{a+\epsilon}\right).$$

Hence

$$\liminf_{r\to\infty} \frac{T(r, f')}{T(r, f)} \geqq 2. \tag{3.8}$$

Also from Nevanlinna[4] (p. 104), we have

$$\limsup_{r\to\infty} \frac{T(r, f')}{T(r, f)} \leqq 2 \tag{3.9}$$

(3.4) follows from (3.8) and (3.9).

This completes the proof of Theorem 3.

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