FURTHER RESULTS ON SUPER EDGE-MAGIC DEFICIENCY OF GRAPHS

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Abstract

Acharya and Hegde have introduced the notion of strongly k-indexable graphs: A (p,q)-graph G is said to be *strongly k-indexable* if its vertices can be assigned distinct integers $0,1,2,...,p-1$ so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices can be arranged as an arithmetic progression $k,k+1,k+2,...,k+(q-1)$. Such an assignment is called a strongly k-indexable labeling of G. Figueroa-Centeno et.al, have introduced the concept of super edge-magic deficiency of graphs: Super edge-magic deficiency of graph G is the minimum number of isolated vertices added to G so that the resulting graph is super edge-magic. They conjectured that super edge-magic deficiency of complete bipartite graph $K_{m,n}$ is $(m-1)(n-1)$ and proved it for the case $m=2$. In this paper we prove that the conjuctre is true for $m=3$, 4 and 5, using the concept of strongly k-indexable labelings [1].

1 Introduction

For all terminology and notation in graph theory we follow Harary [6] and West [7].

Graph labelings, where the *vertices* and *edges* are assigned *real values* or *subsets of a set* are subject to certain conditions, have often been motivated by their utility to various applied fields and their intrinsic mathematical interest (logico-mathematical). An enormous body of literature has grown around the subject, especially in the last forty years or so, and is still getting embellished due to increasing number of application driven concepts [5].

Acharya and Hegde [1, 2] have introduced the the concept of strongly k-indexable graphs.

Given a graph $G=(V,E)$, the set $\mathcal{N}$ of nonnegative integers, a finite subset $\mathcal{A}$ of $\mathcal{N}$ and a commutative binary operation $+:\mathcal{N}\times\mathcal{N}\rightarrow\mathcal{N}$, every vertex function $f:V(G)\rightarrow\mathcal{A}$ induces an edge function $f^+:E(G)\rightarrow\mathcal{N}$ such that $f^+(uv)=f(u)+f(v),\forall\ uv\in E(G)$. Such vertex functions are called ***additive vertex functions***. An **additive labeling** of a graph G is an injective additive vertex function f such that the induced edge function f^+ is injective.

For the given (p,q)-graph $G=(V,E)$.

1. $f(V)=\{f(u):u\in V(G)\}$.
2. $f^+(E)=\{f^+(e):e\in E(G)\}$.

Definition 1.1 *An indexable labeling of a (p,q)-graph G with $f^+(E)=\{k,\ k+d,\ ...,\ k+(q-1)d\}$ is called **strongly** $(k,\ d)$**-indexable labeling** of G.*

[1]Key Words: Strongly k-indexable graphs, Super edge-magic deficiency of graphs

Definition 1.2 *A strongly (k,d)-indexable labeling of a (p,q) graph G with $d = 1$ is called a **strongly k-indexable labeling**. A graph which admits such a labeling for atleast one value of k is called strongly k-indexable graph.*

Enomoto et.al.,[3] have introduced the the concept of super edge-magic graph.

Definition 1.3 *A graph G is said to be super edge-magic if it admits a bijection $f : V \cup E \rightarrow \{1, 2, ..., p+q\}$ with $f(V) = \{1, 2, ..., p\}$ and $f(E) = \{p+1, p+2, ..., p+q\}$ such that $f(u) + f(v) + f(uv) = c(f)$, $uv \in V$ where $c(f)$ is a constant.*

From the above definition one can see that a graph is super edge-magic if and only if it is strongly k-indexable for some k.

R. M. Figueroa-Centenoa et.al.,[4] have introduced the concept of super edge-magic deficiency of graphs.

Definition 1.4 *Super edge-magic deficiency of a graph G is the minimum number of isolated vertices added to G so that the resulting graph is super edge-magic. and is denoted by $\mu_s(G)$.*

From the above definitions one can see that $0 \leq \mu_s(G) \leq \infty$.

As, a graph is super edge-magic if and only if it is strongly k-indexable, super edge-magic deficiency can be equvialently defined as the minimum number of isolated vertices added to a graph G so that the resulting graph is strongly k-indexable for some k. For the sake of convenience we call this parameter as **vertex dependent characteristic** and is denote it by $d_c(G)$. Figueroa-Centenoa et.al.,[4] have proved that

Theorem 1.5 : *The vertex dependent characteristic of complete bipartite graph $K_{m,n}$ is at most $\leq (m-1)(n-1)$.*

They conjuctured that

Conjecture 1.6 *: The vertex dependent characteristic of complete bipartite graph $K_{m,n}$ is equal to $(m-1)(n-1)$.*

Also, they proved that

Theorem 1.7 *The vertex dependent characteristic of complete bipartite graph $K_{2,n}$ is $(n-1)$.*

2 Results

In this section we prove the above mentioned conjucture for $m = 3$, 4 and 5, using the concept of strongly k-indexable labelings.

Theorem 2.1 : *The vertex dependent characteristic of complete bipartite graph $K_{3,n}$ is $2(n-1)$.*

Proof: From Theorem 1.5, clearly

$$d_c(K_{3,\,n}) \leq 2(n-1). \tag{1}$$

From Theorem 1.7, $d_c(K_{3,2})$= 2.
Suppose $d_c(K_{3,\,n}) < 2(n-1)$ for some integer $n \geq 3$ then there exists a strongly k-indexable labeling $f : V(K_{3,\,n} \cup (2n-2-j)K_1) \rightarrow \{0,\ 1, ..., 3n-j\}$ for some integer $j \geq 1$ such that

$$f^+(K_{3,\,n}) = f^+(K_{3,\,n} \cup (2n-2-j)K_1) = \{k,\ k+2, ..., k+3n-1\}.$$

Let $A = \{x_i : x_i \in V(K_{3,n}),\ \deg(x_i) = n \text{ and } f(x_i) < f(x_{i+1}),\ i = 1,\ 2\}$.

$B = \{y_i : y_i \in V(K_{3,\,n}),\ \deg(y_i) = 3 \text{ and } f(y_i) < f(y_{i+1}); 1 \leq i \leq n-1\}$.

$C = \{z_i : z_i \in V((2n-2-j)K_1),\ \deg(z_i) = 0,\ 1 \leq i \leq 2n-2-j\}$.
Let $f(x_1) = a$ then $f(x_2) = a + b$ and $f(x_3) = a + b + c$ where b, c are positive integers.
Consider the following mutually exclusive subsets of $f^+(K_{3,n})$.

$$\begin{aligned}
A_1 &= \{a + f(y_1),\ a + b + f(y_1),\ a + b + c + f(y_1)\} \\
A_2 &= \{a + f(y_2),\ a + b + f(y_2),\ a + b + c + f(y_2)\} \\
A_3 &= \{a + f(y_3),\ a + b + f(y_3),\ a + b + c + f(y_3)\} \\
&\cdots \\
A_n &= \{a + f(y_n),\ a + b + f(y_n),\ a + b + c + f(y_n)\}
\end{aligned} \tag{2}$$

Since f is strongly k-indexable,

$$f^+(K_{3,\,n}) = A_1 \cup A_2 \cup A_3 \cup ... \cup A_n.$$

Therefore $a + f(y_1) = k$ and $a + b + c + f(y_n) = k + 3n - 1$. There are ($b$ - 1) edge values between each $a+f(y_i)$ and $a+b+f(y_i)$, $1 \leq i \leq n$ in $f^+(K_{3,\,n})$ and (c - 1) edge values between each $a+b+f(y_i)$ and $a+b+c+f(y_i)$, $1 \leq i \leq n$ in $f^+(K_{3,\,n})$. As there are only $3n$ elements in $f^+(K_{3,\,n})$, we must have $(b-1)n + (c-1)n + 2 \leq 3n$ which implies

$$(b-1)n + (c-1)n \leq 3n - 2 < 3n \Rightarrow b + c < 5.$$

Therefore possible values of b and c are one among the following.
(1). $b = 1$ and $c = 3$.
(2). $b = 3$ and $c = 1$.
(3). $b = 1$ and $c = 2$.
(4). $b = 2$ and $c = 1$.
(5). $b = 2$ and $c = 2$.
(6). $b = 1$ and $c = 1$.
Case 1: $b = 1$ and $c = 3$.

From equation (2)

$$\begin{aligned}
A_1 &= \{a + f(y_1),\ a + 1 + f(y_1),\ a + 4 + f(y_1)\} \\
A_2 &= \{a + f(y_2),\ a + 1 + f(y_2),\ a + 4 + f(y_2)\} \\
A_3 &= \{a + f(y_3),\ a + 1 + f(y_3),\ a + 4 + f(y_3)\} \\
&\cdots \\
A_n &= \{a + f(y_n),\ a + 1 + f(y_n),\ a + 4 + f(y_n)\}
\end{aligned}$$

One can observe that , the increasing order of edge values of $K_{3,\,n}$ are

$$a + f(y_1),\ a + 1 + f(y_1),\ a + f(y_2),$$

$$a + 1 + f(y_2),\ a + 4 + f(y_1),\ a + f(y_3),\ ......$$

From this increasing order we get,

$$f(y_2) = 2 + f(y_1) \ \text{ and } \ f(y_3) = 5 + f(y_1)$$

But then

$$\begin{aligned} f(x_2) + f(y_3) &= a + 1 + 5 + f(y_1) \\ &= a + 6 + f(y_1) \\ &= f(x_3) + f(y_2) - \text{a contradiction (because } f^+ \text{ is injective).} \end{aligned}$$

Case 2: $b = 3$ and $c = 1$.
By similar arguments as in **Case 1**, we get a contradiction.
Case 3: $b = 1$ and $c = 2$.

From equation (2)

$$\begin{aligned} &A_1 = \{a + f(y_1),\ a + 1 + f(y_1),\ a + 3 + f(y_1)\} \\ &A_2 = \{a + f(y_2),\ a + 1 + f(y_2),\ a + 3 + f(y_2)\} \\ &A_3 = \{a + f(y_3),\ a + 1 + f(y_3),\ a + 3 + f(y_3)\} \\ &\cdots \\ &A_n = \{a + f(y_n),\ a + 1 + f(y_n),\ a + 3 + f(y_n)\} \end{aligned}$$

One can easily observe that

$$\begin{aligned} f(x_2) + f(y_2) &= a + 1 + f(y_2) \\ &= a + 3 + f(y_1) \\ &= f(x_3) + f(y_1) - \text{a contradiction.} \end{aligned}$$

Case 4: $b = 2$ and $c = 1$.
By similar arguments as in **Case 3**, we get a contradiction.
Case 5: $b = 1$ and $c = 1$.
If $b = c = 1$ and then

$$\begin{aligned} k &= a + f(y_1) \\ k + 1 &= a + 1 + f(y_1) \\ k + 2 &= a + 2 + f(y_1) \\ &\cdots \\ k + 3n - 1 &= a + 2 + f(y_n) \end{aligned} \tag{3}$$

From equation (3) we get,

$$\begin{aligned} f(y_2) &= 3 + f(y_1) \\ f(y_3) &= 6 + f(y_1) \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ f(y_n) &= 3(n-1) + f(y_1) \end{aligned} \tag{4}$$

From equation (3),

$$\begin{aligned} f(y_n) \quad &= k + 3n - 1 - 2 - a \\ &= k + 3n - 3 - (k - f(y_1)) \\ &= 3n - 3 + f(y_1)) \\ &\leq 3n - j \text{ (} \because 3n - j \text{ is the maximum vertex value.)} \\ \Rightarrow f(y_1) \quad &\leq 3 - j. \\ \text{But } f(y_1) \quad &\geq 0 \ \Rightarrow 3 - j \geq 0 \ \Rightarrow j \in \{1, \ 2, \ 3\} \end{aligned}$$

Note that

$$\begin{aligned} f(A) = \quad & \{a, \ a+1, \ a+2\}. \\ f(B) = \quad & \{f(y_1), \ f(y_1)+3, \ f(y_1)+6, ..., f(y_1)+3(n-1)\}. \text{ (From (4))} \\ f(C) = \quad & \{f(z_1), \ f(z_2), \ f(z_3), ..., f(z_{2n-2-j})\}. \end{aligned}$$

Let $F = \{f(y_1)+1, \ f(y_1)+2, \ f(y_1)+4, \ f(y_1)+5, \ f(y_1)+7, \ ..., f(y_1)+3n-4\}$.

Clearly $F \subseteq f(K_{3,\,n} \cup (2n-2-j)K_1)$ and F contains 2(n-1) vertex values. Also note that $F \cap f(B) = \phi$.

Sub Case (5.1): j = 1.

Then $f(C)$ contains $2n-3$ vertex values and therefore one element of F must be in $f(A)$. Let $f(y_1)+3s-5 \in f(A)$ for some integer $s, \ 2 \leq s \leq n$. Then

$$\begin{aligned} a \quad &= f(y_1)+3s-5 \ \Rightarrow a+1 \in f(B) \quad \text{—a contradiction.} \\ a+1 \quad &= f(y_1)+3s-5 \ \Rightarrow a \in f(B) \quad \text{—a contradiction.} \\ a+2 \quad &= f(y_1)+3s-5 \ \Rightarrow a+1 \in f(B) \quad \text{—a contradiction.} \end{aligned}$$

Let $f(y_1)+3r-4 \in f(A)$ for some integer $s, \ 2 \leq r \leq n$. Then

$$\begin{aligned} a \quad &= f(y_1)+3r-4 \ \Rightarrow a+1 \in f(B) \quad \text{—a contradiction.} \\ a+1 \quad &= f(y_1)+3r-4 \ \Rightarrow a+2 \in f(B) \quad \text{—a contradiction.} \\ a+2 \quad &= f(y_1)+3r-4 \ \Rightarrow a \in f(B) \quad \text{—a contradiction.} \end{aligned}$$

Therefore $j \neq 1$.
Sub Case (5.2): $j = 2$.
Then $f(C)$ contains $2n-4$ vertex values and therefore two elements of F must be in $f(A)$. Let $f(y_1)+3t-2,\ f(y_1)+3t-4 \in f(A)$ for some integer $t,\ 1 \leq t \leq n$. Then, $a+1 = f(y_1)+3t-3 = f(y_1)+3(t-1) \in f(B)$-a contradiction.
Let $f(y_1)+3m-5,\ f(y_1)+3m-4 \in f(A)$ for some integer $m,\ 1 \leq m \leq n$. Since these two values are consecutive, either $a \in f(B)$ or $a+2 \in f(B)$-a contradiction.
Therefore $j \neq 2$.
Sub Case (5.3): $j = 3$.
Then $f(C)$ contains $2n-5$ elements and therefore three elements of F must be in $f(A)$, which is impossible because elements of $f(A)$ are consecutive. Clearly $j \neq 3$. Thus for $j \geq 1$, $(K_{3,\,n}) \cup (2n-2-j)K_1$ is not strongly k-indexable.
Case 6: $b = 2$ and $c = 2$.

From equation (2)

$$A_1 = \{a+f(y_1),\ a+2+f(y_1),\ a+4+f(y_1)\}$$

$$A_2 = \{a+f(y_2),\ a+2+f(y_2),\ a+4+f(y_2)\}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$A_n = \{a+f(y_n),\ a+2+f(y_n),\ a+4+f(y_n)\}$$

Then the increasing order of edge values of $K_{3,\,n}$ are

$$a+f(y_1),\ a+f(y_2),\ a+2+f(y_1),\ a+2+f(y_2),$$

$$a+4+f(y_1),\ a+4+f(y_2),\ a+f(y_3),\ ......$$

$$\Longrightarrow f(y_2) = 1+f(y_1), f(y_3) = 6+f(y_1) \text{ and } f(y_4) = 7+f(y_1).$$

If n is odd, that is $n = 2r+1$ then there are $4r$ vertex values which are not used between $f(y_1)$ and $f(y_{2r+1})$. Therefore $2n-2-j = 4r-j \geq 4r \Longrightarrow j \leq 0$ -a contradiction to $j \geq 1$.
If n is even integer then,

$$f(y_n) = 3n-5+f(y_1), f(y_{n-1}) = f(y_n)-1.$$

$$k = a+f(y_1), k+3n-1 = a+4+f(y_n)$$

$$\Longrightarrow f(y_n) = k+3n-5-a$$

$$\Longrightarrow f(y_n) = k+3n-5-(k-f(y_1))$$

$$\Longrightarrow f(y_n) = 3n-5+f(y_1)) \leq 3n-j$$

$$\Longrightarrow j \in \{1, 2, , 3, 4, 5\}$$

Threrefore

$$f(A) = \{a,\ a+2,\ a+4\}.$$

$$f(B) = \{f(y_1),\ f(y_1)+1,\ f(y_1)+6,\ f(y_1)+7, ..., f(y_1)+3n-5\}.$$

$$f(C) = \{f(z_1),\ f(z_2),\ f(z_3), ..., f(z_{2n-2-j})\}.$$

Again, let $R = \{f(y_1)+2,\ f(y_1)+3,\ f(y_1)+4,\ f(y_1)+5,\ f(y_1)+8,\ ...,\}$.

Clearly $R \subseteq f(K_{3,\,n} \cup (2n-2-j)K_1)$ and R contains (2n-4) vertex values and $R \cap f(B) = \phi$. Similar to the arguments used for Sub Cases (5.1), (5.2) and (5.3) we can show that $j \neq 1, 2, , 3, 4, 5$. Hence from (1) $d_c(K_{3,\,n}) = 2(n-1)$. $\diamond$

Theorem 2.2 : *The vertex dependent characteristic of complete bipartite graph* $K_{4,\,n}$ *is* $3(n-1)$.

Proof: From Theorem 1.5, clearly

$$d_c(K_{4,\,n}) \leq 3(n-1). \tag{5}$$

From Theorem 1.7 and 2.1, $d_c(K_{4,\,2}) = 3$ and $d_c(K_{4,\,3}) = 6$. Assume that $d_c(K_{4,\,n}) < 3(n-1)$ for some integer $n \geq 4$ then there exists a strongly k-indexable labeling $f : V(K_{4,\,n} \cup (3n-3-j)K_1) \rightarrow \{0,\ 1, ..., 4n-j\}$ for some integer $j \geq 1$ such that

$$f^+(K_{4,\,n}) = f^+(K_{4,\,n} \cup (3n-3-j)K_1) = \{k,\ k+2, ..., k+4n-1\}.$$

Let $A = \{x_i : x_i \in V(K_{4,\,n}),\ \deg(x_i) = n$ and $f(x_i) < f(x_{i+1}),\ i = 1,\ 2,\ 3\}$.

$B = \{y_i : y_i \in V(K_{4,\,n}),\ \deg(y_i) = 4$ and $f(y_i) < f(y_{i+1}); 1 \leq i \leq n-1\}$.

$C = \{z_i : z_i \in V((3n-3-j)K_1),\ \deg(z_i) = 0,\ 1 \leq i \leq 3n-3-j\}$.
Let $f(x_1) = a$ then $f(x_2) = a+b$, $f(x_3) = a+b+c$ and $f(x_3) = a+b+c+d$ where b, c, d are positive integers.
Similar to pevious theorems consider the mutually exclusive subsets of $f^+(K_{4,\,n})$.

$$\begin{aligned}
A_1 &= \{a+f(y_1),\ a+b+f(y_1),\ a+b+c+f(y_1),\ a+b+c+d+f(y_1)\}\\
A_2 &= \{a+f(y_2),\ a+b+f(y_2),\ a+b+c+f(y_2),\ a+b+c+d+f(y_2)\}\\
A_3 &= \{a+f(y_3),\ a+b+f(y_3),\ a+b+c+f(y_3),\ a+b+c+d+f(y_3)\}\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
A_n &= \{a+f(y_n),\ a+b+f(y_n),\ a+b+c+f(y_n),\ a+b+c+d+f(y_n)\}
\end{aligned} \tag{6}$$

There are (b - 1), (c - 1) and (d - 1) distinct edge values between each $a+f(y_i)$ and $a+b+f(y_i)$, $a+b+f(y_i)$ and $a+b+c+f(y_i)$ and $a+b+c+f(y_i)$ and $a+b+c+d+f(y_i)$, $1 \leq i \leq n$ in $f^+(K_{4,\,n})$ respectively. As there are only $4n$ elements in $f^+(K_{4,\,n})$, we must have $(b-1)n+(c-1)n+(d-1)n+2 \leq 4n$. Therefore we get $b+c+d < 7$.
There are many possible values of b, c and d but it is enough if we consider the following seven cases.
(1). $b = 1$, $c = 1$ and $d = 2$.

(2). $b = 1$, $c = 1$ and $d = 3$.
(3). $b = 1$, $c = 1$ and $d = 4$.
(4). $b = 2$, $c = 1$ and $d = 2$.
(5). $b = 2$, $c = 1$ and $d = 3$.
(6). $b = 1$, $c = 1$ and $d = 1$.
(7). $b = 2$, $c = 2$ and $d = 2$.

Case 1: $b = 1$, $c = 1$ and $d = 2$.
In this case, note that $f(y_2) = 3 + f(y_1)$ and therefore we get

$$f(x_4) + f(y_1) \;= f(x_2) + f(y_2) - \text{a contradiction (because } f^+ \text{ is injective).}$$

Case 2: $b = 1$, $c = 1$ and $d = 3$.
In this case also, note that $f(y_2) = 3 + f(y_1)$ and therefore we get

$$f(x_4) + f(y_1) \;= f(x_3) + f(y_2) - \text{a contradiction.}$$

Case 3: $b = 1$, $c = 1$ and $d = 4$.
Similarly, in this case $f(y_3) = 4 + f(y_2)$. Therefore,

$$f(x_3) + f(y_3) \;= f(x_4) + f(y_2) - \text{a contradiction.}$$

Case 4: $b = 2$, $c = 1$ and $d = 2$.
Note that $f(y_2) = 1 + f(y_1)$

$$f(x_3) + f(y_1) \;= f(x_2) + f(y_2) - \text{a contradiction.}$$

Case 5: $b = 2$, $c = 1$ and $d = 3$.
Note that in this case also $f(y_2) = 1 + f(y_1)$

$$f(x_3) + f(y_1) \;= f(x_2) + f(y_2) - \text{a contradiction.}$$

Case 6: $b = 1$, $c = 1$ and $d = 1$. and
Case 7: $b = 2$, $c = 2$ and $d = 2$. also arrive at contradiction using analogous arguments of Theorem 2.1 Case-5 and Case-6. Therefore from all these seven cases, clearly $j \not\geq 1$. Hence from (5) $d_c(K_{4,\,n}) = 3(n-1)$. $\diamond$

Theorem 2.3 . *The vertex dependent characteristic of a complete bipartite graph $K_{5,n}$ is $4(n-1)$.*

Proof. Consider the complete bipartite graph $K_{5,n}$. From Theorem 1.5, we have

$$d_c(K_{5,n}) \leq 4(n-1) \tag{7}$$

Also, we see that $d_c(K_{5,2}) = 4$, $d_c(K_{5,3}) = 8$ and $d_c(K_{5,4}) = 12$. Assume that $d_c(K_{5,n}) < 4(n-1)$ for some positive integer $n \geq 5$. Then, there exists a strongly k-indexable labeling $f : V(K_{5,n} \cup (4n-4-j)K_1) \to \{0, 1, 2, \ldots, 5n-j\}$ for some positive integer $j \geq 1$ such that $f^+(K_{5,n}) = f^+(K_{5,n} \cup (4n-4-j)K_1) = \{k, k+2, \ldots, k+5n-1\}$.

$$A = \{x_i : x_i \in V(K_{5,n}), deg(x_i) = n, f(x_i) < f(x_{i+1}), i = 1, 2, 3, 4\}$$

$$B = \{y_i : y_i \in V(K_{5,n}), deg(y_i) = 5, f(y_i) < f(y_{i+1}), 1 \leq i \leq n-1\}$$

$$C = \{z_i : z_i \in V((4n-4-j)K_1), deg(z_i) = 0, 1 \leq i \leq 4n-4-j\}.$$

$f(x_1) = a$, then $f(x_2) = a+b$, $f(x_3) = a+b+c$, $f(x_4) = a+b+c+d$ and $f(x_5) = a+b+c+d+e$, where b, c, d, e are positive integers. Consider the following mutually exclusive subsets of $f^+(K_{5,n})$.

$A_1 = \{a+f(y_1), a+b+f(y_1), a+b+c+f(y_1), a+b+c+d+f(y_1), a+b+c+d+e+f(y_1)\}$

$A_2 = \{a+f(y_2), a+b+f(y_2), a+b+c+f(y_2), a+b+c+d+f(y_2), a+b+c+d+e+f(y_2)\}$

$A_3 = \{a+f(y_3), a+b+f(y_3), a+b+c+f(y_3), a+b+c+d+f(y_3), a+b+c+d+e+f(y_3)\}$

. . . .

$$A_n = \{a+f(y_n), a+b+f(y_n), a+b+c+f(y_n), a+b+c+d+f(y_n), a+b+c+d+e+f(y_n)\} \quad (8)$$

Since f is strongly k-indexable,

$$f^+(K_{5,n}) = A_1 \cup A_2 \cup \cdots \cup A_n.$$

Therefore, $a + f(y_1) = k$ and $a + +b + c + d + e + f(y_n) = k + 5n - 1$. Note that there are $(b-1)$ edge values between $a + f(y_i)$ and $a + b + f(y_i), 1 \leq i \leq n$, $(c-1)$ edge values between $a + b + f(y_i)$ and $a + b + c + f(y_i), 1 \leq i \leq n$, $(d-1)$ edge values between $a + b + c + f(y_i)$ and $a + b + c + d + f(y_i), 1 \leq i \leq n$, $(e-1)$ edge values between $a + b + c + d + f(y_i)$ and $a + b + c + d + e + f(y_i), 1 \leq i \leq n$ in $f^+(K_{5,n})$. As there are only $5n$ elements in $f^+(K_{5,n})$, we must have $(b-1)n + (c-1)n + (d-1)n + (e-1)n + 2 \leq 5n$, from which we get,$(b-1)n + (c-1)n + (d-1)n + (e-1)n \leq 5n - 2 < 5n$
$\Rightarrow\ b + c + d + e < 9$.

Even though there are many possible values of b, c, d, e satisfying $b + c + d + e < 9$, it is enough to consider the following twelve cases.

Case 1: $b = 1, c = 1, d = 1, e = 5$.
From equation (8)

$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 2 + f(y_1), a + 3 + f(y_1), a + 8 + f(y_1)\}$

$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 2 + f(y_2), a + 3 + f(y_2), a + 8 + f(y_2)\}$

$A_3 = \{a + f(y_3), a + 1 + f(y_3), a + 2 + f(y_3), a + 3 + f(y_3), a + 8 + f(y_3)\}$

$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$

$$A_n = \{a + f(y_n), a + 1 + f(y_n), a + 2 + f(y_n), a + 3 + f(y_n), a + 8 + f(y_n)\}$$

Then, the increasing order of edge values of $K_{5,n}$ are $a+f(y_1), a+1+f(y_1), a+2+f(y_1), a+3+f(y_1), a+f(y_2), a+1+f(y_2), a+2+f(y_2), a+3+f(y_2), a+8+f(y_1), a+f(y_3), \ldots, a+8+f(y_n)$. From this increasing order, we get
$a+f(y_2) = a+3+f(y_1)$ and $a+8+f(y_1) = a+f(y_3)$

$\Rightarrow \ f(y_3) = 9+f(y_1)$ and $f(y_2) = 4+f(y_1)$.

But $f(x_4)+f(y_3) = a+3+9+f(y_1) = (a+8)+(4+f(y_1)) = f(x_5)+f(y_2)$.

This is a contradiction as f is injective.

Case 2: $b=1, c=1, d=1, e=4$.
From equation (8)
$A_1 = \{a+f(y_1), a+1+f(y_1), a+2+f(y_1), a+3+f(y_1), a+7+f(y_1)\}$

$A_2 = \{a+f(y_2), a+1+f(y_2), a+2+f(y_2), a+3+f(y_2), a+7+f(y_2)\}$

$A_3 = \{a+f(y_3), a+1+f(y_3), a+2+f(y_3), a+3+f(y_3), a+7+f(y_3)\}$

$\ldots \quad \ldots \quad \ldots \quad \ldots$
$A_n = \{a+f(y_n), a+1+f(y_n), a+2+f(y_n), a+3+f(y_n), a+7+f(y_n)\}$

Then, one can easily observe that
$a+3+f(y_1) = a+f(y_2)$ and $a+7+f(y_1) = a+f(y_3)$

$\Rightarrow \ f(y_2) = 4+f(y_1)$ and $f(y_3) = 8+f(y_1)$.

But $f(x_4)+f(y_2) = a+3+f(y_2) = (a+3)+(4+f(y_1) = f(x_5)+f(y_1))$.

This is again a contradiction.
Case 3: $b=1, c=1, d=1, e=3$.
From equation (8)
$A_1 = \{a+f(y_1), a+1+f(y_1), a+2+f(y_1), a+3+f(y_1), a+6+f(y_1)\}$

$A_2 = \{a+f(y_2), a+1+f(y_2), a+2+f(y_2), a+3+f(y_2), a+6+f(y_2)\}$

$A_3 = \{a+f(y_3), a+1+f(y_3), a+2+f(y_3), a+3+f(y_3), a+6+f(y_3)\}$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$A_n = \{a+f(y_n), a+1+f(y_n), a+2+f(y_n), a+3+f(y_n), a+6+f(y_n)\}$

Then, one can easily observe that
$a+3+f(y_1) = a+f(y_2)$ and $a+6+f(y_1) = a+f(y_3)$
$\Rightarrow \ f(y_2) = 4+f(y_1)$ and $f(y_3) = 7+f(y_1)$.
But $f(x_3)+f(y_2) = a+2+4+f(y_2) = (a+6)+f(y_1) = f(x_5)+f(y_1)$.
This is again a contradiction.

Case 4: $b = 1, c = 1, d = 1, e = 2$.
From equation (8)

$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 2 + f(y_1), a + 3 + f(y_1), a + 5 + f(y_1)\}$

$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 2 + f(y_2), a + 3 + f(y_2), a + 5 + f(y_2)\}$

$A_3 = \{a + f(y_3), a + 1 + f(y_3), a + 2 + f(y_3), a + 3 + f(y_3), a + 5 + f(y_3)\}$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$A_n = \{a + f(y_n), a + 1 + f(y_n), a + 2 + f(y_n), a + 3 + f(y_n), a + 5 + f(y_n)\}$

Then, one can easily observe that

$a + 3 + f(y_1) = a + f(y_2)$ and $a + 5 + f(y_1) = a + f(y_3)$

$\Rightarrow \ f(y_2) = 4 + f(y_1)$ and $f(y_3) = 6 + f(y_1)$.

But $f(x_2) + f(y_2) = a + 1 + 4 + f(y_1) = (a + 5) + f(y_1) = f(x_5) + f(y_1)$.

This is again a contradiction.

Case 5: $b = 3, c = 3, d = 1, e = 1$.
From equation (8)

$A_1 = \{a + f(y_1), a + 3 + f(y_1), a + 6 + f(y_1), a + 7 + f(y_1), a + 8 + f(y_1)\}$

$A_2 = \{a + f(y_2), a + 3 + f(y_2), a + 6 + f(y_2), a + 7 + f(y_2), a + 8 + f(y_2)\}$

$A_3 = \{a + f(y_3), a + 3 + f(y_3), a + 6 + f(y_3), a + 7 + f(y_3), a + 8 + f(y_3)\}$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$A_n = \{a + f(y_n), a + 3 + f(y_n), a + 6 + f(y_n), a + 7 + f(y_n), a + 8 + f(y_n)\}$

Then, one can easily observe that

$a + 3 + f(y_1) = a + f(y_2)$ and $a + 8 + f(y_1) = a + f(y_3)$

$\Rightarrow \ f(y_2) = 4 + f(y_1)$ and $f(y_3) = 9 + f(y_1)$.

But $f(x_2) + f(y_2) = a + 3 + 4 + f(y_1) = (a + 7) + f(y_1) = f(x_4) + f(y_1)$.

This is again a contradiction.

Case 6: $b = 2, c = 2, d = 2, e = 1$.
From equation (8)

$A_1 = \{a + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 6 + f(y_1), a + 7 + f(y_1)\}$

$A_2 = \{a + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 6 + f(y_2), a + 7 + f(y_2)\}$

$A_3 = \{a + f(y_3), a + 2 + f(y_3), a + 4 + f(y_3), a + 6 + f(y_3), a + 7 + f(y_3)\}$

$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$

$A_n = \{a + f(y_n), a + 2 + f(y_n), a + 4 + f(y_n), a + 6 + f(y_n), a + 7 + f(y_n)\}$

Then, one can easily observe that
$a + f(y_2) = a + 2 + f(y_1)$ and $a + f(y_3) = a + 7 + f(y_1)$

$\Rightarrow \; f(y_2) = 3 + f(y_1)$ and $f(y_3) = 8 + f(y_1)$.

But $f(x_3) + f(y_2) = a + 4 + 3 + f(y_2) = (a + 7) + f(y_1) = f(x_5) + f(y_1)$.

This is again a contradiction.

Case 7: $b = 2, c = 2, d = 1, e = 1$.
From equation (8)

$A_1 = \{a + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 5 + f(y_1), a + 6 + f(y_1)\}$

$A_2 = \{a + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 5 + f(y_2), a + 6 + f(y_2)\}$

$A_3 = \{a + f(y_3), a + 2 + f(y_3), a + 4 + f(y_3), a + 5 + f(y_3), a + 6 + f(y_3)\}$

$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$

$A_n = \{a + f(y_n), a + 2 + f(y_n), a + 4 + f(y_n), a + 5 + f(y_n), a + 6 + f(y_n)\}$

Then, one can easily observe that
$a + f(y_2) = a + 2 + f(y_1)$ and $a + f(y_3) = a + 6 + f(y_1)$
$\Rightarrow \; f(y_2) = 3 + f(y_1)$ and $f(y_3) = 7 + f(y_1)$.

But $f(x_2) + f(y_2) = a + 2 + 3 + f(y_2) = (a + 5) + f(y_1) = f(x_4) + f(y_1)$.

This is again a contradiction.

Case 8: $b = 1, c = 2, d = 2, e = 3$.
From equation (8)

$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 3 + f(y_1), a + 5 + f(y_1), a + 8 + f(y_1)\}$

$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 3 + f(y_2), a + 5 + f(y_2), a + 8 + f(y_2)\}$

$A_3 = \{a + f(y_3), a + 1 + f(y_3), a + 3 + f(y_3), a + 5 + f(y_3), a + 8 + f(y_3)\}$

$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$

$A_n = \{a + f(y_n), a + 1 + f(y_n), a + 3 + f(y_n), a + 5 + f(y_n), a + 8 + f(y_n)\}$

Then, one can easily observe that

$$a + f(y_2) = a + 3 + f(y_1) \text{ and } a + f(y_3) = a + 8 + f(y_1)$$

$$\Rightarrow \quad f(y_2) = 4 + f(y_1) \text{ and } f(y_3) = 9 + f(y_1).$$

But $f(x_2) + f(y_2) = a + 1 + 4 + f(y_1) = (a + 5) + f(y_1) = f(x_4) + f(y_1)$.

This is again a contradiction.

Case 9: $b = 1, c = 1, d = 2, e = 4$.
From equation (8)

$$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 8 + f(y_1)\}$$

$$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 8 + f(y_2)\}$$

$$A_3 = \{a + f(y_3), a + 1 + f(y_3), a + 2 + f(y_3), a + 4 + f(y_3), a + 8 + f(y_3)\}$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$A_n = \{a + f(y_n), a + 1 + f(y_n), a + 2 + f(y_n), a + 4 + f(y_n), a + 8 + f(y_n)\}$$

Then, one can easily observe that

$$a + f(y_2) = a + 4 + f(y_1) \text{ and } a + f(y_3) = a + 8 + f(y_1)$$

$$\Rightarrow f(y_2) = 5 + f(y_1) \text{ and } f(y_3) = 9 + f(y_1).$$

But $f(x_1) + f(y_3) = a + 9 + f(y_2) = (a + 4) + (5 + f(y_1)) = f(x_4) + f(y_2)$.

This is again a contradiction.

Case 10: $b = 1, c = 1, d = 2, e = 3$.
From equation (8)

$$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 7 + f(y_1)\}$$

$$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 7 + f(y_2)\}$$

$$A_3 = \{a + f(y_3), a + 1 + f(y_3), a + 2 + f(y_3), a + 4 + f(y_3), a + 7 + f(y_3)\}$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$A_n = \{a + f(y_n), a + 1 + f(y_n), a + 2 + f(y_n), a + 4 + f(y_n), a + 7 + f(y_n)\}$$

Then, one can easily observe that

$$a + f(y_2) = a + 4 + f(y_1) \text{ and } a + f(y_3) = a + 7 + f(y_1)$$

$$\Rightarrow \quad f(y_2) = 5 + f(y_1) \text{ and } f(y_3) = 8 + f(y_1).$$

But $f(x_3) + f(y_2) = a + 2 + 5 + f(y_2) = (a + 7) + f(y_1) = f(x_5) + f(y_1)$.

This is again a contradiction.

Case 11: $b = 2, c = 2, d = 2, e = 2$.
From equation (8)

$$A_1 = \{a + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 6 + f(y_1), a + 8 + f(y_1)\}$$

$$A_2 = \{a + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 6 + f(y_2), a + 8 + f(y_2)\}$$

$$A_3 = \{a + f(y_3), a + 2 + f(y_3), a + 4 + f(y_3), a + 6 + f(y_3), a + 8 + f(y_3)\}$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$A_n = \{a + f(y_n), a + 2 + f(y_n), a + 4 + f(y_n), a + 6 + f(y_n), a + 8 + f(y_n)\}$$

Then, one can easily observe that

$$a + 2 + f(y_1) = a + f(y_2) \text{ and } a + 8 + f(y_1) = a + f(y_3)$$

$$\Rightarrow f(y_2) = 3 + f(y_1) \text{ and } f(y_3) = 9 + f(y_1).$$

But $f(x_1) + f(y_3) = a + 9 + f(y_1) = (a + 6) + (3 + f(y_1)) = f(x_4) + f(y_2)$.

This is again a contradiction.

Case 12: $b = 1, c = 1, d = 1, e = 1$.
Then

$$k = a + f(y_1)$$

$$k + 1 = a + 1 + f(y_1)$$

$$k + 2 = a + 2 + f(y_1)$$

$$k + 3 = a + 3 + f(y_1)$$

$$k + 4 = a + 4 + f(y_1)$$

$$k + 5 = a + f(y_2)$$

$$k + 6 = a + 1 + f(y_2)$$

$$\dots \qquad \dots$$

$$k + 5n - 1 = a + 4 + f(y_n) \tag{9}$$

From equation (9), we get

$$f(y_2) = 5 + f(y_1)$$

$$f(y_3) = 10 + f(y_1)$$

$$f(y_4) = 15 + f(y_1)$$

$$\dots \qquad \dots$$

$$f(y_n) = 5(n-1) + f(y_1) \tag{10}$$

From equation (9),

$$f(y_n) = k + 5n - 1 - a - 4$$

$= k + 5n - 5 - (k - f(y_1)$
$= 5n - 5 + f(y_1$
$\leq 5n - j$ (since $5n - j$ is the maximum vertex value)
$\Rightarrow f(y_1) \leq 5 - j$
But$f(y_1) \geq 0 \Rightarrow 5 - j \geq 0$
$\Rightarrow j \in \{1, 2, 3, 4, 5\}$.
Note that $f(A) = \{a, a+1, a+2, a+3, a+4\}$,
$f(B) = \{f(y_1), 5 + f(y_1), 10 + f(y_1), \dots, 5(n-1) + f(y_1)\}$,
$f(C) = \{f(z_1), f(z_2), \dots, f(z_{(4n-4-j)})\}$.
Let $F = \{f(y_1)+1, f(y_1)+2, f(y_1)+3, f(y_1)+4, f(y_1)+6, f(y_1)+7, f(y_1)+8, f(y_1)+9, \dots, f(y_1)+5n-6\}$.
Clearly $F \subseteq f(K_{5,n} \cup (4n-4-j)K_1)$ and F contains $4(n-1)$ vertex values. Also $F \cap f(B) = \emptyset$.
We have three sub cases.
Case 12.1: $j = 1$.

Then, $f(C)$ contains $4n-5$ vertex values and hence one element of F must be in $f(A)$. Let $f(y_1) + 5m - 7 \in f(A)$ for some positive integer m, $2 \leq m \leq n$. Then $a = f(y_1) + 5m - 7$

$$\Rightarrow \ a + 2 \in f(B), \text{ a contradiction.}$$

$$a + 1 = f(y_1) + 5m - 7 \Rightarrow a + 3 \in f(B)\text{- a contradiction}$$

$$a + 2 = f(y_1) + 5m - 7 \Rightarrow a + 4 \in f(B)\text{- a contradiction}$$

$$a + 3 = f(y_1) + 5m - 7 \Rightarrow a + 4 \in f(B)\text{- a contradiction}$$

$$a + 4 = f(y_1) + 5m - 7 \Rightarrow a + 1 \in f(B)\text{- a contradiction}$$

Let $f(y_1) + 5r - 6 \in f(A)$ for some integer $r, 2 \leq r \leq n$. Then,

$$a = f(y_1) + 5r - 6 \Rightarrow a + 1 \in f(B)\text{- a contradiction}$$

$$a + 1 = f(y_1) + 5r - 6 \Rightarrow a + 2 \in f(B)\text{- a contradiction}$$

$$a + 2 = f(y_1) + 5r - 6 \Rightarrow a + 3 \in f(B)\text{- a contradiction}$$

$$a + 3 = f(y_1) + 5r - 6 \Rightarrow a + 4 \in f(B)\text{- a contradiction}$$

$$a + 4 = f(y_1) + 5r - 6 \Rightarrow a + 3 \in f(B)\text{- a contradiction}$$

Therefore $j \neq 1$.
Case 12.2: $j = 2$.

Then, $f(C)$ contains $4n - 6$ vertex values and therefore two elements of F must be in $f(A)$. Let $f(y_1) + 5t - 4, f(y_1) + 5t - 6 \in f(A)$ for some positive integer $t, 1 \leq t \leq n$. Then, $a + 1 = f(y_1) + 5t - 5 = f(y_1) + 5(t - 1) \in f(B)$- a contradiction. Let $f(y_1) + 5w - 7, f(y_1) + 5w - 6 \in f(A)$ for some positive integer $w, 1 \leq w \leq n$. Since these two values are consecutive, either $a \in f(B)$ or $a + 2 \in f(B)$- a contradiction. Therefore, $j \neq 2$.
Case 12.3: $j = 3$.

Then, $f(C)$ contains $4n - 7$ vertex values and therefore three elements of F must be in $f(A)$. This is impossible because elements of $f(A)$ are consecutive. Clearly $j \neq 3$.

Proceeding on similar lines to case 12.3 above, we get contradictions when $j = 4, 5$. Thus for $j \geq 1$, $K_{5,n} \cup (4n - 4 - j)K_1$ is not strongly k-indexable. Hence from equation (7), we get $d_c(K_{5,n}) = 4(n - 1)$. This completes the proof. $\diamond$

Remark 1. In strongly k-indexable labelings it is enough to consider only vertex labelings(as vertex labelings induces edge labelings) whereas in super edge-magic labelings one has to deal with two functions. From the proof of theorem 1.7 mentioned in Figueroa-Conteno et.al., one can see that it is easier to prove the results on super edge-magic deficiency of graphs using the concept of strongly k-indexable labelings rather than super edge-magic labelings.

References

[1] B. D. Acharya and S. M. Hegde, *Arithmetic graphs*, J. Graph Theory, 14(3) (1990), 275-299.

[2] B. D. Acharya and S. M. Hegde,, *Strongly indexable graphs*, Discrete Mathematics, 93(2-3) (1991), 123-129

[3] H. Enomoto, A.S. Llado, T. Nakamigwa and G. Ringel, *Super edge-magic graphs*, SUT Journal of Mathematics 34(2)(1998),105-109.

[4] R.M. Figueroa-Centeno, R. Ichishima and F. A. Muntanar-Batle, *On the super edge-magic deficincy*, ARS Combinatoria 78(2006)33-45.

[5] J.A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics #DS* 6 (2005), pp. 1-148.

[6] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Massachusetts, 1972.

[7] D. B. West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.

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