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Cubic convergence order yielding iterative regularization methods for ill-posed Hammerstein type operator equations

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Abstract For the solution of nonlinear ill-posed problems, a Two Step Newton-Tikhonov methodology is proposed. Two implementations are discussed and applied to nonlinear ill-posed Hammerstein type operator equations $KF(x) = y$, where K defines the integral operator and F the function of the solution x on which K operates. In the first case, the Fre\'chet derivative of F is invertible in a neighbourhood which includes the initial guess x_0 and the solution $\hat{x}$. In the second case, F is monotone. For both cases, local cubic convergence is established and order optimal error bounds are obtained by choosing the regularization parameter according to the the balancing principle of Pereverzev and Schock (2005).We also present the results of computational experiments giving the evidence of the reliability of our approach.

Keywords Two Step Newton Tikhonov method · Ill-posed Hammerstein operator · Balancing principle · Monotone operator · Regularization

Mathematics Subject Classification 47J06 · 47A52 · 65N20 · 65J20

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1 Introduction

The study of inverse (ill-posed) problems is an active area of research both theoretically and numerically as these problems arise from important physical and engineering applications (see [2,13,15,18]). It can be quite challenging to solve such problems because of their ill-posed nature. Many of these problems can be characterized abstractly as

$$A(x) = y$$

where y denotes the data, A an abstract (ill-posed) operator and x the unknown solution. However, in practice, because of modelling, experimental and computational errors, y is only available as an approximation y^δ. Consequently, it is necessary to solve

$$A(x^\delta) = y^\delta$$

instead of

$$A(x) = y,$$

and, for given classes of operators A, examine how the errors $x^\delta - x$ depend on $y^\delta - y$.

Tikhonov's regularization (e.g., [3]) method has been used extensively to stabilize the approximate solution of nonlinear ill-posed problems. In recent years, increased emphasis has been placed on iterative regularization procedures [6,11] for the approximate solution of such problems. In this paper, we examine the use of iterative regularization procedures for Hammerstein-type integral equations of the form

$$(Ax)(t) := \int_0^1 k(s,t) f(s, x(s)) ds$$

where $k(s,t) \in L^2([0,1] \times [0,1])$, $x \in L^2[0,1]$ and $t \in [0,1]$. A method and associated algorithm are proposed for which local-cubic convergence is established theoretically and validated numerically.

Formally a Hammerstein operator A takes the form $A = KF$ where $K : L^2[0,1] \to L^2[0,1]$ is a linear integral operator with kernel $k(s,t)$:

$$Kx(t) = \int_0^1 k(s,t) x(s) ds$$

and $F : D(F) \subseteq L^2[0,1] \to L^2[0,1]$ is the nonlinear superposition operator (cf. [12], Page 430)

$$Fx(s) = f(s, x(s)). \tag{1.1}$$

More generally, an equation of the form

$$KF(x) = y \tag{1.2}$$

where $F : D(F) \subseteq X \to Z$, is nonlinear and $K : Z \to Y$ is a bounded linear operator is called a (nonlinear) Hammerstein equation [5,8]. In this paper X, Z, Y are Hilbert spaces with inner product $\langle ., . \rangle$ and norm $\|.\|$ respectively.

In [5], George studied an iterative Newton-Tikhonov regularization (NTR) method for approximating a x_0-minimum norm solution $\hat{x}$ of (1.2), where $\hat{x}$ is called an x_0-minimum norm solution, if

$$\|\hat{x} - x_0\| := \min\{\|x - x_0\| : KF(x) = y, x \in D(F)\}.$$

The element x_0 plays the role of a selection criterion (cf. [4]). Because of the nonlinearity of F, the solution $\hat{x}$ may not be unique.

It is assumed throughout that $y^\delta \in Y$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta,$$

F possesses a uniformly bounded Fré́chet derivative for each $x \in D(F)$, i.e.,

$$\|F'(x)\| \leq M, \quad x \in D(F)$$

for some M.

Observe that the solution x of (1.2) with y^δ in place of y can be obtained by first solving

$$Kz = y^\delta \tag{1.3}$$

for z and then solving the non-linear problem

$$F(x) = z. \tag{1.4}$$

In [5,7,8] this was exploited. In [5], z is approximated with z_α^δ;

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*y^\delta, \quad \alpha > 0, \quad \delta > 0,$$

and then solve (1.4) iteratively using the following Newton-type procedure

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_0)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta)$$

to determine the approximations $(x_{n,\alpha}^\delta)$ with $x_{0,\alpha}^\delta := x_0$. Local linear convergence was proved in [5]. In [7], to solve (1.4), George and Kunhanandan used the iteration

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_{n,\alpha}^\delta)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta)$$

with $x_{0,\alpha}^\delta := x_0$ and

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*(y^\delta - KF(x_0)) + F(x_0). \tag{1.5}$$

Local quadratic convergence was established in [7]. As in [7], it is assumed that the solution $\hat{x}$ of (1.2) satisfies

$$\|F(\hat{x}) - F(x_0)\| = \min\{\|F(x) - F(x_0)\| : KF(x) = y, x \in D(F)\}.$$

A sequence (x_n) in X with $\lim x_n = x^*$ is said to be convergent of order $p > 1$, if there exist positive reals β and γ such that, for all $n \in N$,

$$\|x_n - x^*\| \leq \beta e^{-\gamma p^n}. \tag{1.6}$$

If a sequence (x_n) satisfies $\|x_n - x^*\| \leq \beta q^n$, $0 < q < 1$, then (x_n) is said to be linearly convergent.

Recently, George and Shobha [9], considered a dynamical system method for solving (1.2) and obtained optimal order convergence rate. In [5,8,9] it is assumed that $F'(x_0)^{-1}$ exists and in [7] it is assumed that $F'(x)^{-1}$ exists for all $x \in B_r(x_0)$ (the ball of radius r on the centre x_0) for some $r > 0$.

In [1], Argyros and Hilout considered a method called Two Step Directional Newton Method (TSDNM) for approximating a zero x^* of a differentiable function F defined on a convex subset $\mathcal{D}$ of a Hilbert space H with values in $\mathbb{R}$. Motivated by TSDNM we propose, a Two Step Newton-Tikhonov Methods (TSNTM) in this paper for solving (1.2). In particular, its convergence for two different regularity classes of the operator F are examined: Invertible Fr$\acute{e}$chet derivative (IFD) class and Monotone Fr$\acute{e}$chet derivative (MFD) class.

The IFD class $F'(u)^{-1}$ exists and is a bounded operator for all $u \in B_r(x_0)$; i.e., $\|F'(u)^{-1}\| \leq \beta, \quad \forall u \in B_r(x_0)$. Consequently, in this situation, the ill-posedness of (1.2) is essentially due to the nonclosedness of the range of the linear operator K (see [18], page 26).

Remark 1.1 Let the function f in (1.1) be differentiable with respect to the second variable. Then, it follows that the operator F in (1.1) is Fr$\acute{e}$chet differentiable with

$$[F'(x)u](t) = \partial_2 f(t, x(t))u(t), \quad t \in [0, 1],$$

where $\partial_2 f(t, s)$ represents the partial derivative of f with respect to the second variable. If, in addition, the existence of a constant $\kappa_1 > 0$ is assumed such that, for all $x \in B_r(x_0)$ and for all $t \in [0, 1]$, $\partial_2 f(t, x(t)) \geq \kappa_1$, then $F'(u)^{-1}$ exists and is a bounded operator for all $u \in B_r(x_0)$. So F belongs to the IFD class.

The MFD class F is a monotone operator (i.e., $\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in D(F)$) and $F'(.)^{-1}$ does not exists. Consequently, in this situation, the ill-posedness of (1.2) is due to the ill-posedness of F as well as the nonclosedness of the range of the linear operator K.

Example 1.2 ([14, Example 6.1]) Let $F : L^2[0, 1] \to L^2[0, 1]$ be defined by

$$F(x)(t) = K(x)(t) + f(t), \quad x, f \in L^2[0, 1], \quad t \in [0, 1]$$

where $K : L^2[0, 1] \to L^2[0, 1]$ is a compact linear operator such that range of K denoted by $R(K)$ is not closed and $\langle Kh, h\rangle \geq 0$ for $h \in L^2[0, 1]$. Then, $F(x) = y$ is ill-posed as K is a compact operator with non-closed range.The Fr$\acute{e}$chet derivative $F'(x)$ of F is given by

$$F'(x)h = Kh, \quad \forall x, h \in L^2[0, 1].$$

Now, since $\langle Kh, h\rangle \geq 0$ for all $h \in L^2[0, 1]$, F is monotone. Further $F'(u)^{-1}$ does not exists for any $u \in L^2[0, 1]$. Consequently, the operator KF, with K and F as defined above is an example of the MFD Class.

The Preliminaries are given in Sect. 2. Section 3 investigates the convergence of the TSNTM for both the IFD and MFD Classes. Section 4 discusses the algorithm details and Numerical examples are given in Sect. 5.

2 Preliminaries

This section deals with Tikhonov regularized solution z_α^δ of (1.3) and (an a priori and an a posteriori) error estimate for $\|F(\hat{x}) - z_\alpha^\delta\|$. The following assumption is required to obtain the error estimate .

Assumption 2.1 There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|K\|^2$ satisfying;

- $\lim\limits_{\lambda \to 0} \varphi(\lambda) = 0)$
- $$\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \qquad \forall \lambda \in (0, a],$$

  and

- there exists $v \in X$, $\|v\| \leq 1$ such that

$$F(\hat{x}) - F(x_0) = \varphi(K^*K)v.$$

Remark 2.2 The functions

$$\varphi(\lambda) := \lambda^{\nu}, \quad \lambda > 0,$$

for $0 < \nu \leq 1$ and

$$\varphi(\lambda) = \begin{cases} \left[ln\frac{1}{\lambda}\right]^{-p}, & 0 < \lambda \leq e^{-(p+1)} \\ 0 & otherwise \end{cases}$$

for $p \geq 0$ satisfy the above assumption (see [14]).

Let $z_\alpha = (K^*K + \alpha I)^{-1}K^*(y - KF(x_0)) + F(x_0)$. Then by Assumption 2.1 we have,

$$\begin{aligned} \|F(\hat{x}) - z_\alpha\| &\leq \|\alpha(K^*K + \alpha I)^{-1}(F(\hat{x}) - F(x_0))\| \\ &\leq \|\alpha(K^*K + \alpha I)^{-1}\varphi(K^*K)v\| \\ &\leq \sup_{0<\lambda\leq\|K\|^2} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha}\|v\| \leq \varphi(\alpha), \end{aligned}$$

and

$$\|z_\alpha - z_\alpha^\delta\| \leq \|(K^*K + \alpha I)^{-1}K^*(y - y^\delta)\| \leq \frac{\delta}{\sqrt{\alpha}}.$$

The above discussion leads to the following Theorem.

Theorem 2.3 *Let z_α^δ be as in (1.5) and Assumption 2.1 holds. Then*

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}. \tag{2.1}$$

2.1 A priori choice of the parameter

Note that the estimate $\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$ in (2.1) is of optimal order for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \leq a$. Then we have $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$ and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)).$$

So the relation (2.1) leads to $\|F(\hat{x}) - z_\alpha^\delta\| \leq 2\psi^{-1}(\delta)$.

2.2 An adaptive choice of the parameter

In this paper, we propose to choose the parameter α according to the balancing principle established by Pereverzev and Shock [17] for solving ill-posed problems. Let

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\}$$

be the set of possible values of the parameter α.

The selection of numerical value k for the parameter α according to the balancing principle is performed using the rule

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\right\} < N. \tag{2.2}$$

Let

$$k = \max\{i : \alpha_i \in D_N^+\} \tag{2.3}$$

where $D_N^+ = \{\alpha_i \in D_N : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, j = 0, 1, 2, \ldots, i-1\}$.

We will be using the following theorem from [7] for our error analysis.

Theorem 2.4 (cf. [7, Theorem 4.3]) *Let l be as in (2.2), k be as in (2.3) and $z_{\alpha_k}^\delta$ be as in (1.5) with $\alpha = \alpha_k$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z_{\alpha_k}^\delta\| \leq (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta).$$

3 Two step Newton–Tikhonov method

We propose TSNTM for IFD class in Sect. 3.1 and a TSNTM for MFD class in Sect. 3.2.

3.1 TSNTM for IFD class

For an initial guess $x_0 \in X$ the TSNTM for IFD Class is defined as;

$$y_{n,\alpha_k}^\delta = x_{n,\alpha_k}^\delta - F'(x_{n,\alpha_k}^\delta)^{-1}(F(x_{n,\alpha_k}^\delta) - z_{\alpha_k}^\delta), \tag{3.1}$$

$$x_{n+1,\alpha_k}^\delta = y_{n,\alpha_k}^\delta - F'(x_{n,\alpha_k}^\delta)^{-1}(F(y_{n,\alpha_k}^\delta) - z_{\alpha_k}^\delta). \tag{3.2}$$

We need the following assumption for the convergence of TSNTM and to obtain the error estimate.

Assumption 3.1 (cf. [10, Assumption 1.2]) There exist a constant $k_0 > 0$ such that for every $x, u \in B_r(x_0)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v)$, $\|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|$.

Let

$$e_{n,\alpha_k}^\delta := \|y_{n,\alpha_k}^\delta - x_{n,\alpha_k}^\delta\|, \qquad \forall n \geq 0 \tag{3.3}$$

and for $0 < k_0 \leq 1$, let $g : (0, 1) \to (0, 1)$ be the function defined by

$$g(t) = \frac{k_0^2}{8}(4 + 3k_0 t)t^2 \qquad \forall t \in (0, 1). \tag{3.4}$$

For convenience we will use the notation x_n, y_n and e_n for x_{n,α_k}^δ, y_{n,α_k}^δ and e_{n,α_k}^δ respectively.

Hereafter we assume that $\delta \in (0, \delta_0]$ where $\delta_0 < \frac{\sqrt{\alpha_0}}{\beta}$. Let $\|\hat{x} - x_0\| \leq \rho$,

$$\rho < \frac{1}{M}\left(\frac{1}{\beta} - \frac{\delta_0}{\sqrt{\alpha_0}}\right) \tag{3.5}$$

and

$$\gamma_\rho := \beta[M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}].$$

Theorem 3.2 *Let e_n and $g(e_n)$ be as in Eqs. (3.3) and (3.4) respectively, x_n and y_n be as in (3.2) and (3.1) respectively with $\delta \in (0, \delta_0]$. Then under the Assumptions of Theorem 2.4, the following hold:*

(a) $\|x_n - y_{n-1}\| \le \frac{k_0 e_{n-1}}{2} \|y_{n-1} - x_{n-1}\|$;
(b) $\|x_n - x_{n-1}\| \le (1 + \frac{k_0 e_{n-1}}{2}) \|y_{n-1} - x_{n-1}\|$;
(c) $\|y_n - x_n\| \le g(e_{n-1}) \|y_{n-1} - x_{n-1}\|$;
(d) $g(e_n) \le g(\gamma_\rho)^{3^n}, \qquad \forall n \ge 0$;
(e) $e_n \le g(\gamma_\rho)^{(3^n - 1)/2} \gamma_\rho \qquad \forall n \ge 0.$

Proof Observe that

$$\begin{aligned} x_n - y_{n-1} &= y_{n-1} - x_{n-1} - F'(x_{n-1})^{-1}(F(y_{n-1}) - F(x_{n-1})) \\ &= F'(x_{n-1})^{-1}[F'(x_{n-1})(y_{n-1} - x_{n-1}) - (F(y_{n-1}) - F(x_{n-1}))] \\ &= F'(x_{n-1})^{-1} \int_0^1 [F'(x_{n-1}) - F'(x_{n-1} + t(y_{n-1} - x_{n-1}))](y_{n-1} - x_{n-1}) dt \end{aligned}$$

and hence by Assumption 3.1, we have

$$\begin{aligned} \|x_n - y_{n-1}\| &\le \| \int_0^1 \Phi(x_{n-1}, x_{n-1} + t(y_{n-1} - x_{n-1}), y_{n-1} - x_{n-1}) dt \| \\ &\le \frac{k_0}{2} \|y_{n-1} - x_{n-1}\|^2. \end{aligned}$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|x_n - x_{n-1}\| \le \|x_n - y_{n-1}\| + \|y_{n-1} - x_{n-1}\|.$$

To prove (c) we observe that

$$\begin{aligned} e_n = \|y_n - x_n\| &\le \|x_n - y_{n-1} - F'(x_n)^{-1}(F(x_n) - z_\alpha^\delta)\| \\ &\quad + \|F'(x_{n-1})^{-1}(F(y_{n-1}) - z_\alpha^\delta)\| \\ &\le \|x_n - y_{n-1} - F'(x_n)^{-1}(F(x_n) - F(y_{n-1}))\| \\ &\quad + \|[F'(x_{n-1})^{-1} - F'(x_n)^{-1}](F(y_{n-1}) - z_\alpha^\delta)\| \\ &\le \|F'(x_n)^{-1}[F'(x_n)(x_n - y_{n-1}) - (F(x_n) - F(y_{n-1}))]\| \\ &\quad + \|[F'(x_{n-1})^{-1} - F'(x_n)^{-1}](F(y_{n-1}) - z_\alpha^\delta)\| \\ &\le \left\| F'(x_n)^{-1} \int_0^1 [F'(x_n) - F'(y_{n-1} + t(x_n - y_{n-1}))] dt (x_n - y_{n-1}) \right\| \\ &\quad + \|F'(x_n)^{-1}(F'(x_n) - F'(x_{n-1}))F'(x_{n-1})^{-1}(F(y_{n-1}) - z_\alpha^\delta)\| \\ &\le \left\| F'(x_n)^{-1} \int_0^1 [F'(x_n) - F'(y_{n-1} + t(x_n - y_{n-1}))] dt (x_n - y_{n-1}) \right\| \\ &\quad + \|F'(x_n)^{-1}(F'(x_n) - F'(x_{n-1}))(y_{n-1} - x_n)\| \\ &\le \left\| \int_0^1 \Phi(x_n, y_{n-1} + t(x_n - y_{n-1}), x_n - y_{n-1}) dt \right\| \\ &\quad + \|\Phi(x_n, x_{n-1}, y_{n-1} - x_n)\| \\ &\le \frac{k_0}{2} \|x_n - y_{n-1}\|^2 + k_0 \|x_n - x_{n-1}\| \|x_n - y_{n-1}\|. \end{aligned}$$

Therefore by (a) and (b) we have

$$\begin{aligned} e_n &\le \left(\frac{k_0^2}{2} + \frac{3k_0^3}{8}\|y_{n-1} - x_{n-1}\|\right)\|y_{n-1} - x_{n-1}\|^3 \\ &\le g(e_{n-1})e_{n-1}. \end{aligned} \tag{3.6}$$

This completes the proof of (c). □

Since for $\mu \in (0, 1)$, $g(\mu t) \le \mu^2 g(t)$, for all $t \in (0, 1)$, by (3.6) we have,

$$g(e_n) \le g(e_0)^{3^n}$$

and

$$\begin{aligned} e_n \le g^3(e_{n-2})e_{n-1} &\le g^3(e_{n-2})g^3(e_{n-3})e_{n-2}\cdots g(e_0)e_0 \\ &\le g(e_0)^{3^{n-1}+3^{n-2}+\cdots+1}e_0 \\ &\le g(e_0)^{(3^n-1)/2}e_0, \end{aligned} \tag{3.7}$$

provided $e_n < 1, \forall n \ge 0$. From (3.6) it is clear that, $e_n \le 1$ if $e_0 \le 1$, but

$$\begin{aligned} e_0 = \|y_0 - x_0\| &= \|F'(x_0)^{-1}(F(x_0) - z_{\alpha_k}^{\delta})\| \\ &\le \|F'(x_0)^{-1}\|\|(F(x_0) - z_{\alpha_k}^{\delta})\| \\ &\le \beta\|F(x_0) - z_{\alpha_k} + z_{\alpha_k} - z_{\alpha_k}^{\delta}\| \\ &\le \beta\left[\|F(x_0) - F(\hat{x})\| + \|z_{\alpha_k} - z_{\alpha_k}^{\delta}\|\right] \\ &\le \beta\left[\left\|\int_0^1 F'(\hat{x} + t(x_0 - \hat{x}))(x_0 - \hat{x})dt\right\| + \frac{\delta}{\sqrt{\alpha_k}}\right] \\ &\le \beta\left[M\rho + \frac{\delta}{\sqrt{\alpha_k}}\right] \\ &\le \beta\left[M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}\right] \\ &= \gamma_\rho < 1. \end{aligned}$$

As g is monotonic increasing and $e_0 \le \gamma_\rho$, we have $g(e_0) \le g(\gamma_\rho)$. This completes the proof of the Theorem.

Theorem 3.3 *Let* $r = (\frac{1}{1-g(\gamma_\rho)} + \frac{k_0}{2}\frac{\gamma_\rho}{1-g(\gamma_\rho)^2})\gamma_\rho$ *and let the hypothesis of Theorem 3.2 holds. Then* $x_n, y_n \in B_r(x_0)$, *for all* $n \ge 0$.

Proof Note that by (b) of Theorem 3.2 we have

$$\begin{aligned} \|x_1 - x_0\| &\le \left[1 + \frac{k_0}{2}e_0\right]e_0 \\ &\le \left[1 + \frac{k_0}{2}\gamma_\rho\right]\gamma_\rho \\ &\le r \end{aligned} \tag{3.8}$$

i.e., $x_1 \in B_r(x_0)$. Again note that by (3.8) and (c) of Theorem 3.2 we have

$$\begin{aligned}\|y_1 - x_0\| &\le \|y_1 - x_1\| + \|x_1 - x_0\| \\ &\le \left(1 + g(e_0) + \frac{k_0}{2}e_0\right)e_0 \\ &\le \left(1 + g(\gamma_\rho) + \frac{k_0}{2}\gamma_\rho\right)\gamma_\rho \\ &\le r\end{aligned}$$

i.e., $y_1 \in B_r(x_0)$. Further by (3.8) and (b) of Theorem 3.2 we have

$$\begin{aligned}\|x_2 - x_0\| &\le \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\le \left(1 + \frac{k_0}{2}e_1\right)e_1 + \left(1 + \frac{k_0}{2}e_0\right)e_0 \\ &\le \left(1 + \frac{k_0}{2}g(e_0)e_0\right)g(e_0)e_0 + \left(1 + \frac{k_0}{2}e_0\right)e_0 \\ &\le \left(1 + g(e_0) + \frac{k_0}{2}e_0(1 + g(e_0)^2)\right)e_0 \\ &\le \left(1 + g(\gamma_\rho) + \frac{k_0}{2}\gamma_\rho(1 + g(\gamma_\rho)^2)\right)\gamma_\rho \\ &\le r \end{aligned} \tag{3.9}$$

and by (3.9) and (c) of Theorem 3.2 we have

$$\begin{aligned}\|y_2 - x_0\| &\le \|y_2 - x_2\| + \|x_2 - x_0\| \\ &\le g(e_1)e_1 + (1 + g(e_0) + \frac{k_0}{2}e_0(1 + g(e_0)^2))e_0 \\ &\le g(e_0)^4 e_0 + (1 + g(e_0) + \frac{k_0}{2}e_0(1 + g(e_0)^2))e_0 \\ &\le (1 + g(e_0) + g(e_0)^4 + \frac{k_0}{2}e_0(1 + g(e_0)^2))e_0 \\ &\le (1 + g(e_0) + g(e_0)^2 + \frac{k_0}{2}e_0(1 + g(e_0)^2))e_0 \\ &\le (1 + g(\gamma_\rho) + g(\gamma_\rho)^2 + \frac{k_0}{2}\gamma_\rho(1 + g(\gamma_\rho)^2))\gamma_\rho \\ &\le r.\end{aligned}$$

i.e., $x_2, y_2 \in B_r(x_0)$. Continuing this way one can prove that $x_n, y_n \in B_r(x_0), \forall n \ge 0$. This completes the proof. □

The main result of this section is the following Theorem.

Theorem 3.4 *Let y_n and x_n be as in (3.1) and (3.2) respectively, Assumptions of Theorem 3.3 hold and let $0 < g(\gamma_\rho) < 1$. Then (x_n) is a Cauchy sequence in $B_r(x_0)$ and converges to $x_{\alpha_k}^\delta \in \overline{B_r(x_0)}$. Further $F(x_{\alpha_k}^\delta) = z_{\alpha_k}^\delta$ and*

$$\|x_n - x_{\alpha_k}^\delta\| \le Ce^{-\gamma 3^n}$$

where $C = (\frac{1}{1-g(\gamma_\rho)^3} + \frac{k_0\gamma_\rho}{2}\frac{1}{1-(g(\gamma_\rho)^2)^3}g(\gamma_\rho)^{3^n})\gamma_\rho$ and $\gamma = -\log g(\gamma_\rho)$.

Proof Using the relation (b) and (e) of Theorem 3.2, we obtain

$$\begin{aligned}
\|x_{n+m}-x_n\| &\le \sum_{i=0}^{i=m-1} \|x_{n+i+1}-x_{n+i}\|\\
&\le \sum_{i=0}^{i=m-1}\left(1+\frac{k_0 e_{n+i}}{2}\right) e_{n+i}\\
&\le \sum_{i=0}^{i=m-1}\left(1+\frac{k_0 e_0}{2} g(e_0)^{3^{n+i}}\right) g(e_0)^{3^{n+i}} e_0\\
&= \left(1+\frac{k_0 e_0}{2} g(e_0)^{3^n}\right) g(e_0)^{3^n} e_0\\
&\quad+\left(1+\frac{k_0 e_0}{2} g(e_0)^{3^{n+1}}\right) g(e_0)^{3^{n+1}} e_0+\cdots\\
&\quad+\left(1+\frac{k_0 e_0}{2} g(e_0)^{3^{n+m}}\right) g(e_0)^{3^{n+m}} e_0\\
&\le \Bigg[\left(1+g(e_0)^3+g(e_0)^{3^2}+\cdots+g(e_0)^{3^m}\right)\\
&\quad+\frac{k_0 e_0}{2}\left(1+\left(g(e_0)^2\right)^3+\left(g(e_0)^2\right)^{3^2}+\cdots+\left(g(e_0)^2\right)^{3^m}\right)g(e_0)^{3^n}\Bigg] g(e_0\Big)^{3^n} e_0\\
&\le \Bigg[\left(1+g(\gamma_\rho)^3+g(\gamma_\rho)^{3^2}+\cdots+g(\gamma_\rho)^{3^m}\right)\\
&\quad+\frac{k_0 \gamma_\rho}{2}\left(1+\left(g(\gamma_\rho)^2\right)^3+\left(g(\gamma_\rho)^2\right)^{3^2}+\cdots+\left(g(\gamma_\rho)^2\right)^{3^m}\right)g(\gamma_\rho)^{3^n}\Bigg] g(\gamma_\rho)^{3^n}\gamma_\rho\\
&\le C g(\gamma_\rho)^{3^n}\\
&\le C e^{-\gamma 3^n}.
\end{aligned}$$

Thus x_n is a Cauchy sequence in $B_r(x_0)$ and hence it converges, say to $x_{\alpha_k}^\delta \in \overline{B_r(x_0)}$. Observe that

$$\begin{aligned}
\|F(x_n)-z_{\alpha_k}^\delta\| &= \|F'(x_n)(x_n-y_n)\|\\
&\le \|F'(x_n)\|\|(x_n-y_n)\|\\
&\le Me_n \le Mg(\gamma_\rho)^{3^n}\gamma_\rho. \qquad (3.10)
\end{aligned}$$

Now by letting $n\to\infty$ in (3.10) we obtain $F(x_{\alpha_k}^\delta)=z_{\alpha_k}^\delta$. This completes the proof. □

Remark 3.5 Note that $0<g(\gamma_\rho)<1$ and hence $\gamma>0$. So by (1.6), sequence (x_n) converges cubically to $x_{\alpha_k}^\delta$.

Hereafter we assume that

$$\rho \le r < \frac{1}{k_0}.$$

Remark 3.6 Note that the above assumption is satisfied, if

$$k_0 \le \min\left\{1, \frac{1-g(\gamma_\rho)^2}{\gamma_\rho}\left[\frac{-1}{1-g(\gamma_\rho)}+\sqrt{\frac{1}{(1-g(\gamma_\rho))^2}+\frac{2}{1-g(\gamma_\rho)^2}}\right]\right\}.$$

Theorem 3.7 *Suppose that Assumption 2.1 holds. If in addition $k_0 r<1$, then*

$$\|\hat{x}-x_{\alpha_k}^\delta\| \le \frac{\beta}{1-k_0 r}\|F(\hat{x})-z_{\alpha_k}^\delta\|.$$

Proof Observe that

$$\begin{aligned}
\|\hat{x} - x^{\delta}_{\alpha_k}\| &= \|\hat{x} - x^{\delta}_{\alpha_k} + F'(x_0)^{-1}[F(x^{\delta}_{\alpha_k}) - F(\hat{x}) + F(\hat{x}) - z^{\delta}_{\alpha_k}]\| \\
&\le \|F'(x_0)^{-1}[F'(x_0)(\hat{x} - x^{\delta}_{\alpha_k}) + F(x^{\delta}_{\alpha_k}) - F(\hat{x})]\| \\
&\quad + \|F'(x_0)^{-1}(F(\hat{x}) - z^{\delta}_{\alpha_k})\| \\
&\le \left\| F'(x_0)^{-1} \int_0^1 [F'(x_0) - F'(\hat{x} + t(x^{\delta}_{\alpha_k} - \hat{x})](\hat{x} - x^{\delta}_{\alpha_k})dt \right\| \\
&\quad + \|F'(x_0)^{-1}(F(\hat{x}) - z^{\delta}_{\alpha_k})\| \\
&\le \left\| \int_0^1 \Phi(x_0, \hat{x} + t(x^{\delta}_{\alpha_k} - \hat{x}), \hat{x} - x^{\delta}_{\alpha_k})dt \right\| \\
&\quad + \|F'(x_0)^{-1}(F(\hat{x}) - z^{\delta}_{\alpha_k})\| \\
&\le k_0 r \|\hat{x} - x^{\delta}_{\alpha_k}\| + \beta \|F(\hat{x}) - z^{\delta}_{\alpha_k}\|.
\end{aligned}$$

The last step follows from Assumption 3.1 and the relation $\|x_0 - \hat{x} - t(x^{\delta}_{\alpha_k} - \hat{x})\| \le r$. This completes the proof. The following Theorem is a consequence of Theorems 3.4 and 3.7. □

Theorem 3.8 *Let x_n be as in (3.2), assumptions in Theorems 3.4 and 3.7 hold. Then*

$$\|\hat{x} - x_n\| \le Ce^{-\gamma 3^n} + \frac{\beta}{1 - k_0 r}\|F(\hat{x}) - z^{\delta}_{\alpha_k}\|$$

where C and γ are as in Theorem 3.4.

Now since $l \le k$ and $\alpha_\delta \le \alpha_{l+1} \le \mu\alpha_l$, we have

$$\frac{\delta}{\sqrt{\alpha_k}} \le \frac{\delta}{\sqrt{\alpha_l}} \le \mu\frac{\delta}{\sqrt{\alpha_\delta}} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

This leads to the following theorem.

Theorem 3.9 *Let x_n be as in (3.2), assumptions in Theorems 2.4, 3.4 and 3.7 hold. Let*

$$n_k := \min\left\{n : e^{-\gamma 3^n} \le \frac{\delta}{\sqrt{\alpha_k}}\right\}.$$

Then

$$\|\hat{x} - x_{n_k}\| = \bigcirc(\psi^{-1}(\delta)).$$

3.2 TSNTM for MFD class

We need the following assumption in addition to the earlier assumptions for our convergence analysis.

Assumption 3.10 There exists a continuous, strictly monotonically increasing function $\varphi_1 : (0, b] \to (0, \infty)$ with $b \ge \|F'(\hat{x})\|$ satisfying;

- $\lim\limits_{\lambda \to 0} \varphi_1(\lambda) = 0$,
- $$\sup_{\lambda \ge 0} \frac{\alpha\varphi_1(\lambda)}{\lambda + \alpha} \le \varphi_1(\alpha) \qquad \forall \lambda \in (0, b]$$

  and

- there exists $v \in X$ with $\|v\| \leq 1$ (cf. [16]) such that

$$x_0 - \hat{x} = \varphi_1(F'(\hat{x}))v.$$

- for each $x \in B_{\tilde{r}}(\hat{x})$ there exists a bounded linear operator $G(x, \hat{x})$ (cf. [19]) such that

$$F'(x) = F'(\hat{x})G(x, \hat{x})$$

with $\|G(x, \hat{x})\| \leq K_1$.

Let Assumption 3.1 holds with $\tilde{r}$ in place of r, $\rho \leq \tilde{r} < \frac{1}{k_0}$ and let $c \leq \alpha_k$.

First we consider a TSNTM for approximating the zero $x_{c\alpha_k}^{\delta}$ of

$$F(x) + \frac{\alpha_k}{c}(x - x_0) = z_{\alpha_k}^{\delta}. \tag{3.11}$$

and then we show that $x_{c\alpha_k}^{\delta}$ is an approximation to the solution $\hat{x}$ of (1.2). For an initial guess $x_0 \in X$ and for $R(x) := F'(x) + \frac{\alpha_k}{c}I$, the TSNTM for MFD Class is defined as:

$$\tilde{y}_{n,\alpha}^{\delta} = \tilde{x}_{n,\alpha}^{\delta} - R(\tilde{x}_{n,\alpha}^{\delta})^{-1}[F(\tilde{x}_{n,\alpha}^{\delta}) - z_{\alpha_k}^{\delta} + \frac{\alpha_k}{c}(\tilde{x}_{n,\alpha}^{\delta} - x_0)] \tag{3.12}$$

and

$$\tilde{x}_{n+1,\alpha}^{\delta} = \tilde{y}_{n,\alpha}^{\delta} - R(\tilde{x}_{n,\alpha}^{\delta})^{-1}[F(\tilde{y}_{n,\alpha}^{\delta}) - z_{\alpha_k}^{\delta} + \frac{\alpha_k}{c}(\tilde{y}_{n,\alpha}^{\delta} - x_0)]. \tag{3.13}$$

where $\tilde{x}_{0,\alpha} := x_0$. Note that with the above notation

$$\|R(x)^{-1}F'(x)\| \leq 1.$$

Let

$$\tilde{e}_{n,\alpha}^{\delta} := \|\tilde{y}_{n,\alpha}^{\delta} - \tilde{x}_{n,\alpha}^{\delta}\|, \qquad \forall n \geq 0. \tag{3.14}$$

Here also for convenience we use the notation $\tilde{x}_n$, $\tilde{y}_n$ and $\tilde{e}_n$ for $\tilde{x}_{n,\alpha}^{\delta}$, $\tilde{y}_{n,\alpha}^{\delta}$ and $\tilde{e}_{n,\alpha}^{\delta}$ respectively.

Let

$$\rho \leq \frac{1}{M}\left(1 - \frac{\delta_0}{\sqrt{\alpha_0}}\right)$$

with $\delta_0 < \sqrt{\alpha_0}$
and

$$\tilde{\gamma}_{\rho} := M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}.$$

Theorem 3.11 *Let $\tilde{e}_n$ and g be as in Eqs. (3.14) and (3.4) respectively, $\tilde{x}_n$ and $\tilde{y}_n$ be as in (3.13) and (3.12) respectively with $\delta \in (0, \delta_0]$. Then the following hold:*

(a) $\|\tilde{x}_n - \tilde{y}_{n-1}\| \leq \frac{k_0\tilde{e}_{n-1}}{2}\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;
(b) $\|\tilde{x}_n - \tilde{x}_{n-1}\| \leq (1 + \frac{k_0\tilde{e}_{n-1}}{2})\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;
(c) $\|\tilde{y}_n - \tilde{x}_n\| \leq g(\tilde{e}_{n-1})\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;
(d) $g(\tilde{e}_n) \leq g(\tilde{\gamma}_{\rho})^{3^n}, \qquad \forall n \geq 0$;
(e) $\tilde{e}_n \leq g(\tilde{\gamma}_{\rho})^{(3^n-1)/2}\tilde{\gamma}_{\rho} \qquad \forall n \geq 0$.

Proof Observe that

$$\begin{aligned}
\tilde{x}_n - \tilde{y}_{n-1} &= \tilde{y}_{n-1} - \tilde{x}_{n-1} - R(\tilde{x}_{n-1})^{-1}\Big(F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1}) \\
&\quad + \frac{\alpha_k}{c}(\tilde{y}_{n-1} - \tilde{x}_{n-1})\Big) \\
&= R(\tilde{x}_{n-1})^{-1}\Big[R(\tilde{x}_{n-1})(\tilde{y}_{n-1} - \tilde{x}_{n-1}) \\
&\quad - \Big(F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1}) - \frac{\alpha_k}{c}(\tilde{y}_{n-1} - \tilde{x}_{n-1})\Big)\Big] \\
&= R(\tilde{x}_{n-1})^{-1}\int_0^1 [F'(\tilde{x}_{n-1}) - F'(\tilde{x}_{n-1} + t(\tilde{y}_{n-1} - \tilde{x}_{n-1}))] \\
&\quad \times (\tilde{y}_{n-1} - \tilde{x}_{n-1})dt.
\end{aligned}$$

Now since $\|R(\tilde{x}_{n-1})^{-1}F'(\tilde{x}_{n-1})\| \le 1$, the proof of (a) and (b) follows as in Theorem 3.2. To prove (c) we observe that

$$\begin{aligned}
\tilde{e}_n &\le \left\|\tilde{x}_n - \tilde{y}_{n-1} - R(\tilde{x}_n)^{-1}\left(F(\tilde{x}_n) - z_\alpha^\delta + \frac{\alpha_k}{c}(\tilde{x}_n - x_0)\right)\right\| \\
&\quad + \left\|R(\tilde{x}_{n-1})^{-1}\left(F(\tilde{y}_{n-1}) - z_\alpha^\delta + \frac{\alpha_k}{c}(\tilde{y}_{n-1} - x_0)\right)\right\| \\
&\le \left\|\tilde{x}_n - \tilde{y}_{n-1} - R(\tilde{x}_n)^{-1}\left(F(\tilde{x}_n) - F(\tilde{y}_{n-1}) + \frac{\alpha_k}{c}(\tilde{x}_n - \tilde{y}_{n-1})\right)\right\| \\
&\quad + \left\|[R(\tilde{x}_{n-1})^{-1} - R(\tilde{x}_n)^{-1}]\left(F(\tilde{y}_{n-1}) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(\tilde{y}_{n-1} - x_0)\right)\right\| \\
&\le \Big\|R(\tilde{x}_n)^{-1}\Big[R(\tilde{x}_n)(\tilde{x}_n - \tilde{y}_{n-1}) - (F(\tilde{x}_n) - F(\tilde{y}_{n-1})) \\
&\quad - \frac{\alpha_k}{c}(\tilde{x}_n - \tilde{y}_{n-1})\Big]\Big\| \\
&\quad + \left\|[R(\tilde{x}_{n-1})^{-1} - R(\tilde{x}_n)^{-1}]\left(F(\tilde{y}_{n-1}) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(\tilde{y}_{n-1} - x_0)\right)\right\| \\
&\le \left\|R(\tilde{x}_n)^{-1}\int_0^1 [F'(\tilde{x}_n) - F'(\tilde{y}_{n-1} + t(\tilde{x}_n - \tilde{y}_{n-1})]dt(\tilde{x}_n - \tilde{y}_{n-1})\right\| \\
&\quad + \Big\|R(\tilde{x}_n)^{-1}(F'(\tilde{x}_n) - F'(\tilde{x}_{n-1}))R(\tilde{x}_{n-1})^{-1}\Big(F(\tilde{y}_{n-1}) - z_{\alpha_k}^\delta \\
&\quad + \frac{\alpha_k}{c}(\tilde{y}_{n-1} - x_0)\Big)\Big\|.
\end{aligned}$$

The remaining part of the proof is analogous to the proof of Theorem 3.2. □

Theorem 3.12 *Let $\tilde{r} = (\frac{1}{1-g(\tilde{\gamma}_\rho)} + \frac{k_0}{2}\frac{\tilde{\gamma}_\rho}{1-g(\tilde{\gamma}_\rho)^2})\tilde{\gamma}_\rho$ and the assumptions of Theorem 3.11 hold. Then $\tilde{x}_n, \tilde{y}_n \in B_{\tilde{r}}(x_0)$, for all $n \ge 0$.*

Proof Analogous to the proof of Theorem 3.3. □

Theorem 3.13 *Let $\tilde{y}_n$ and $\tilde{x}_n$ be as in (3.12) and (3.13) respectively and assumptions of Theorem 3.12 hold. Then $(\tilde{x}_n)$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$ and converges to $x_{c\alpha_k}^\delta \in \overline{B_{\tilde{r}}(x_0)}$. Further $F(x_{c\alpha_k}^\delta) + \frac{\alpha_k}{c}(x_{c\alpha_k}^\delta - x_0) = z_{\alpha_k}^\delta$ and*

$$\|\tilde{x}_n - x_{c\alpha_k}^\delta\| \le \tilde{C}e^{-\gamma_1 3^n}$$

where $\tilde{C} = (\frac{1}{1-g(\tilde{\gamma}_\rho)^3} + \frac{k_0\tilde{\gamma}_\rho}{2}\frac{1}{1-(g(\tilde{\gamma}_\rho)^2)^3}g(\tilde{\gamma}_\rho)^{3^n})\tilde{\gamma}_\rho$ and $\gamma_1 = -\log g(\tilde{\gamma}_\rho)$.

Proof Analogous to the proof of Theorem 3.4 one can prove that $\tilde{x}_n$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$ and hence it converges, say to $x^\delta_{c\alpha_k} \in \overline{B_{\tilde{r}}(x_0)}$. Observe that

$$\begin{aligned}\left\| F(\tilde{x}_n) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{x}_n - x_0)\right\| &= \|R(\tilde{x}_n)(\tilde{x}_n - \tilde{y}_n)\| \\ &\le \|R(\tilde{x}_n)\|\|(\tilde{x}_n - \tilde{y}_n)\| \\ &\le \left(\|F'(x_n)\| + \frac{\alpha_k}{c}\right)\tilde{e}_n \\ &\le \left(\|F'(x_n)\| + \frac{\alpha_k}{c}\right) g(\tilde{e}_0)^{3^n}\tilde{e}_0 \\ &\le \left(\|F'(x_n)\| + \frac{\alpha_k}{c}\right) g(\tilde{\gamma}_\rho)^{3^n}\tilde{\gamma}_\rho. \end{aligned} \tag{3.15}$$

Now by letting $n \to \infty$ in (3.15) we obtain $F(x^\delta_{c\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{c\alpha_k} - x_0) = z^\delta_{\alpha_k}$. This completes the proof. □

Assume that $K_1 < \frac{1-k_0\tilde{r}}{1-c}$ and for the sake of simplicity assume that $\varphi_1(\alpha) \le \varphi(\alpha)$ for $\alpha > 0$.

Theorem 3.14 *Suppose* $x^\delta_{c\alpha_k}$ *is the solution of* (3.11) *and Assumptions* 3.1 *and* 3.10 *hold. Then*

$$\|\hat{x} - x^\delta_{c\alpha_k}\| = O(\psi^{-1}(\delta))$$

Proof Note that $c(F(x^\delta_{c\alpha_k}) - z^\delta_{\alpha_k}) + \alpha_k(x^\delta_{c\alpha_k} - x_0) = 0$, so

$$\begin{aligned}(F'(\hat{x}) + \alpha_k I)(x^\delta_{c\alpha_k} - \hat{x}) &= (F'(\hat{x}) + \alpha_k I)(x^\delta_{c\alpha_k} - \hat{x}) \\ &\quad - c(F(x^\delta_{c\alpha_k}) - z^\delta_{\alpha_k}) - \alpha_k(x^\delta_{c\alpha k} - x_0) \\ &= \alpha_k(x_0 - \hat{x}) + F'(\hat{x})(x^\delta_{c\alpha_k} - \hat{x}) \\ &\quad - c[F(x^\delta_{c\alpha_k}) - z^\delta_{\alpha_k}] \\ &= \alpha_k(x_0 - \hat{x}) + F'(\hat{x})(x^\delta_{c\alpha_k} - \hat{x}) \\ &\quad - c[F(x^\delta_{c\alpha_k}) - F(\hat{x}) + F(\hat{x}) - z^\delta_{\alpha_k}] \\ &= \alpha_k(x_0 - \hat{x}) - c(F(\hat{x}) - z^\delta_{\alpha_k}) + F'(\hat{x})(x^\delta_{c\alpha_k} - \hat{x}) \\ &\quad - c[F(x^\delta_{c\alpha_k}) - F(\hat{x})].\end{aligned}$$

Thus

$$\begin{aligned}\|x^\delta_{c\alpha_k} - \hat{x}\| &\le \|\alpha_k(F'(\hat{x} + \alpha_k I)^{-1}(x_0 - \hat{x})\| + \|(F'(\hat{x}) + \alpha_k I)^{-1} \\ &\quad c(F(\hat{x}) - z^\delta_{\alpha_k})\| + \|(F'(\hat{x}) + \alpha_k I)^{-1}[F'(\hat{x})(x^\delta_{c\alpha_k} - \hat{x}) \\ &\quad - c(F(x^\delta_{c\alpha_k}) - F(\hat{x}))]\| \\ &\le \|\alpha_k(F'(\hat{x}) + \alpha_k I)^{-1}(x_0 - \hat{x})\| + \|F(\hat{x}) - z^\delta_{\alpha_k}\| \\ &\quad + \|(F'(\hat{x}) + \alpha_k I)^{-1}\int_0^1 [F'(\hat{x}) - cF'(\hat{x} + t(x^\delta_{c\alpha_k} - \hat{x}))] \\ &\quad (x^\delta_{c\alpha_k} - \hat{x})dt\| \\ &\le \|\alpha_k(F'(\hat{x}) + \alpha_k I)^{-1}(x_0 - \hat{x})\| + \|F(\hat{x}) - z^\delta_{\alpha_k}\| + \Gamma \end{aligned} \tag{3.16}$$

where $\Gamma := \|(F'(\hat{x}) + \alpha_k I)^{-1} \int_0^1 [F'(\hat{x}) - cF'(\hat{x} + t(x^\delta_{c\alpha_k} - \hat{x})](x^\delta_{c\alpha_k} - \hat{x})dt\|$. So by Assumption 3.10, we obtain

$$\begin{aligned}\Gamma &\leq \left\|(F'(\hat{x}) + \alpha_k I)^{-1} \int_0^1 [F'(\hat{x}) - F'(\hat{x} + t(x^\delta_{c\alpha_k} - \hat{x})k)](x^\delta_{c\alpha_k} - \hat{x})dt\right\| \\ &\quad + (1-c)\left\|(F'(\hat{x}) + \alpha_k I)^{-1} F'(\hat{x})\right. \\ &\quad \left.\times \int_0^1 G(\hat{x} + t(x^\delta_{c\alpha_k} - \hat{x}), \hat{x})(x^\delta_{c\alpha_k} - \hat{x})dt\right\| \\ &\leq k_0\tilde{r}\|x^\delta_{c\alpha_k} - \hat{x}\| + (1-c)K_1\|x^\delta_{c\alpha_k} - \hat{x}\| \end{aligned} \tag{3.17}$$

and hence by (3.16) and (3.17) we have

$$\begin{aligned}\|x^\delta_{c\alpha_k} - \hat{x}\| &\leq \frac{\|\alpha_k(F'(\hat{x}) + \alpha_k I)^{-1}(x_0 - \hat{x})\| + \|F(\hat{x}) - z^\delta_{\alpha_k}\|}{1 - (1-c)K_1 - k_0\tilde{r}} \\ &\leq \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1 - (1-c)K_1 - k_0\tilde{r}} \\ &= O(\psi^{-1}(\delta)).\end{aligned}$$

This completes the proof of the Theorem. □

The following Theorem is a consequence of Theorems 3.13 and 3.14.

Theorem 3.15 *Let $\tilde{x}_n$ be as in (3.13), assumptions in Theorems 3.13 and 3.14 hold. Then*

$$\|\hat{x} - \tilde{x}_n\| \leq \tilde{C}e^{-\gamma_1 3^n} + O(\psi^{-1}(\delta))$$

where $\tilde{C}$ and γ_1 are as in Theorem 3.13.

Theorem 3.16 *Let $\tilde{x}_n$ be as in (3.13), assumptions in Theorems 2.4, 3.13 and 3.14 hold. Let*

$$n_k := \min\left\{n : e^{-\gamma_1 3^n} \leq \frac{\delta}{\sqrt{\alpha_k}}\right\}.$$

Then

$$\|\hat{x} - \tilde{x}_{n_k}\| = O(\psi^{-1}(\delta)).$$

4 Algorithm

Note that for $i, j \in \{0, 1, 2, \cdots, N\}$

$$z^\delta_{\alpha_i} - z^\delta_{\alpha_j} = (\alpha_j - \alpha_i)(K^*K + \alpha_j I)^{-1}(K^*K + \alpha_i I)^{-1}[K^*(y^\delta - KF(x_0))].$$

Therefore the balancing principle algorithm associated with the choice of the parameter specified in Sect. 2 involves the following steps.

- $\alpha_0 = \mu^2\delta^2$, $\mu > \max\{1, \beta\}$ for IFD class and $\mu > 1$ for MFD class.
- $\alpha_i = \mu^{2i}\alpha_0$;
- solve for w_i : $(K^*K + \alpha_i I)w_i = K^*(y^\delta - KF(x_0))$;
- solve for $j < i$, z_{ij} : $(K^*K + \alpha_j I)z_{ij} = (\alpha_j - \alpha_i)w_i$;
- if $\|z_{ij}\| > \frac{4}{\mu^{j+1}}$, then take $k = i - 1$;

- otherwise, repeat with $i+1$ in place of i.
- choose $n_k = \min\{n : e^{-\gamma 3^n} \leq \frac{\delta}{\sqrt{\alpha_k}}\}$ in IFD Class and $n_k = \min\{n : e^{-\gamma_1 3^n} \leq \frac{\delta}{\sqrt{\alpha_k}}\}$ in MFD Class
- solve x_{n_k} using the iteration (3.2) or $\tilde{x}_{n_k}$ using the iteration (3.13).

5 Numerical examples

In this section we consider an example for illustrating the algorithm considered in the above section. We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with $dim V_n = n+1$. Precisely we choose V_n as the space of linear splines in a uniform grid of $n+1$ points in [0, 1].

Example 5.1 We consider the operator $KF : L^2(0,1) \longrightarrow L^2(0,1)$ where $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$F(u) := u^3$$

and $K : L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1 \\ (1-s)t, & 0 \leq t \leq s \leq 1 \end{cases}.$$

The Fréchet derivative of F is given by

$$F'(u)w = 3(u^2)w.$$

Observe that

$$\begin{aligned}[F'(v) - F'(u)]w &= 3(v^2 - u^2)w \\ &= 3u^2\left(\frac{v^2}{u^2} - 1\right)w \\ &= F'(u)\Phi(u,v,w),\end{aligned}$$

where $\Phi(u,v,w) = (\frac{v^2}{u^2} - 1)w = \frac{(v+u)(v-u)}{u^2}w$. Thus Φ satisfies the Assumption 3.1 (cf. [20, Example 2.7]).

We take $f(t) = \frac{6\sin\pi t + \sin^3(\pi t)}{9\pi^2}$ and $f^\delta = f + \delta$. Then the exact solution

$$\hat{x}(t) = \sin \pi t.$$

We use

$$x_0(t) = \sin \pi t + 1/10$$

as our initial guess, so that the function $F(x_0) - F(\hat{x})$ satisfies the source condition

$$F(x_0) - F(\hat{x}) = \varphi(F'(\hat{x}))\left(\frac{3\sin^2(\pi t) + 3.3\sin(\pi t) + 0.91}{30(1/2 + \sin \pi t)^2}\right)$$

where $\varphi(\lambda) = \lambda$. Thus we expect to have an accuracy of order at least $O(\delta^{\frac{1}{2}})$.

Table 1 Iterations and corresponding error estimates of Example 5.1

n	k	α_k	$\lVert u_k^h - \hat{x}\rVert$	$\frac{\lVert u_k^h - \hat{x}\rVert}{(\delta+\varepsilon_h)^{1/2}}$
32	4	0.1714	0.0246	0.0953
64	4	0.1710	0.0248	0.0960
128	4	0.1709	0.0249	0.0964
256	4	0.1709	0.0250	0.0966
512	4	0.1709	0.0250	0.0967
1024	4	0.1709	0.0250	0.0968

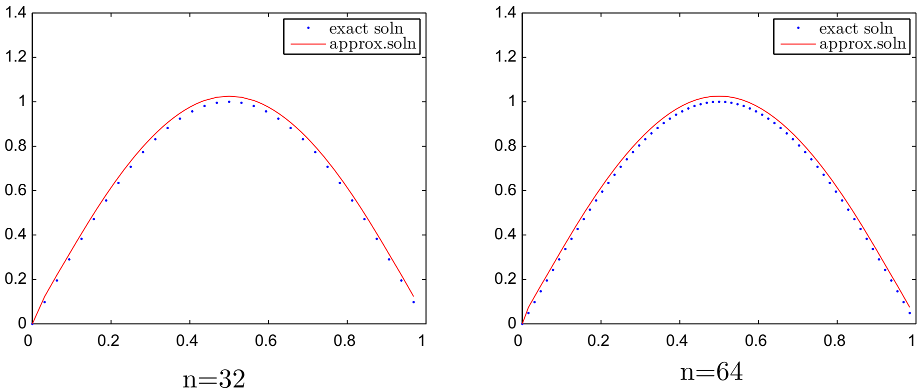

Fig. 1 Curve of the exact and approximate solutions of Example 5.1

We choose $\alpha_0 = (1.5))\delta + \varepsilon_h)^2$, $\mu = 1.5$, $\delta = 0.0667$, $\beta = 0.925$, $\rho = 0.1$, $\gamma_\rho = 0.8212$ and $g_h(\gamma_\rho) = 0.54$ approximately. In this example, for all n, the number of iteration $n_k = 2$. The results of the computation are presented in Table 1. The plots of the exact and the approximate solution obtained are given in Figs. 1 and 2.

Example 5.2 (cf. [21, section 4.3]) To illustrate the method for MFD class, we consider the space $X = Y = L^2[0, 1]$ and the Fredholm integral operator $K : L^2(0, 1) \to L^2(0, 1)$. Then for all $x(t), y(t) : x(t) > y(t)$:

$$\langle F(x) - F(y), x - y\rangle = \int_0^1 \left[\int_0^1 k(t, s)(x^3 - y^3)(s)ds\right](x - y)(t)dt \geq 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t, s)(u(s))^2 w(s)ds.$$

So for any $u \in B_r(x_0)$, $x_0{}^2(s) \geq k_3 > 0$, $\forall s \in (0, 1)$, we have

$$F'(u)w = F'(x_0)G(u, x_0)w,$$

where $G(u, x_0) = (\frac{u}{x_0})^2$.

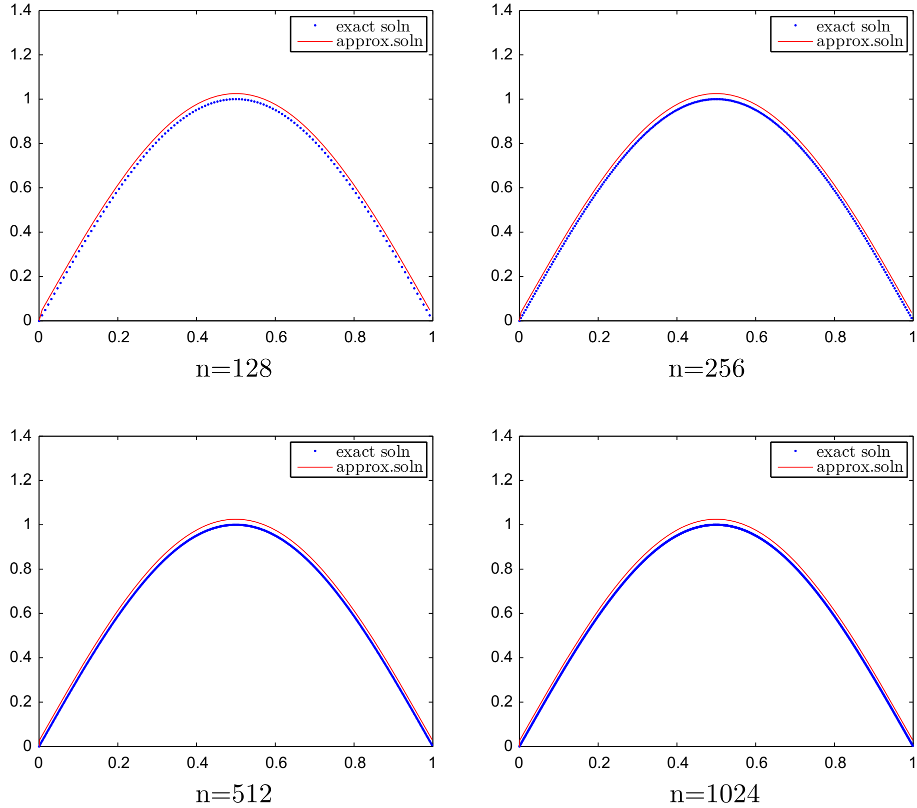

Fig. 2 Curve of the exact and approximate solutions of Example 5.1

Further observe that

$$\begin{aligned}[F'(v) - F'(u)]w(s) &= 3\int_0^1 k(t,s)(v^2(s) - u^2(s))w(s)ds \\ &:= F'(u)\Phi(u,v,w),\end{aligned}$$

where $\Phi(u, v, w) = [\frac{v^2}{u^2} - 1]w$.

In our computation, we take

$$\begin{aligned} f(t) = \frac{1}{36\pi^2}(27\sin\pi t - \sin 3\pi t) + \frac{1}{36\pi}(27t^2\cos\pi t - 3t^2\cos 3\pi t \\ + 6t\cos 3\pi t - 3\cos 3\pi t - 27t\cos\pi t)\end{aligned}$$

and $f^\delta = f + \delta$. Then the exact solution is

$$\hat{x}(t) = \sin\pi t.$$

We use

$$x_0(t) = \sin\pi t + \frac{3}{4\pi^2}(1 + t\pi^2 - t^2\pi^2 - \cos^2(\pi t))$$

Table 2 Iterations and corresponding error estimates of Example 5.2

n	k	α_k	$\|u_k^h - \hat{x}\|$	$\frac{\|u_k^h - \hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
8	4	0.1790	0.0363	0.1388
16	4	0.1729	0.0432	0.1669
32	4	0.1714	0.0450	0.1742
64	4	0.1710	0.0455	0.1761
128	4	0.1709	0.0456	0.1765
256	4	0.1709	0.0456	0.1767
512	4	0.1709	0.0456	0.1767
1024	4	0.1709	0.0456	0.1767

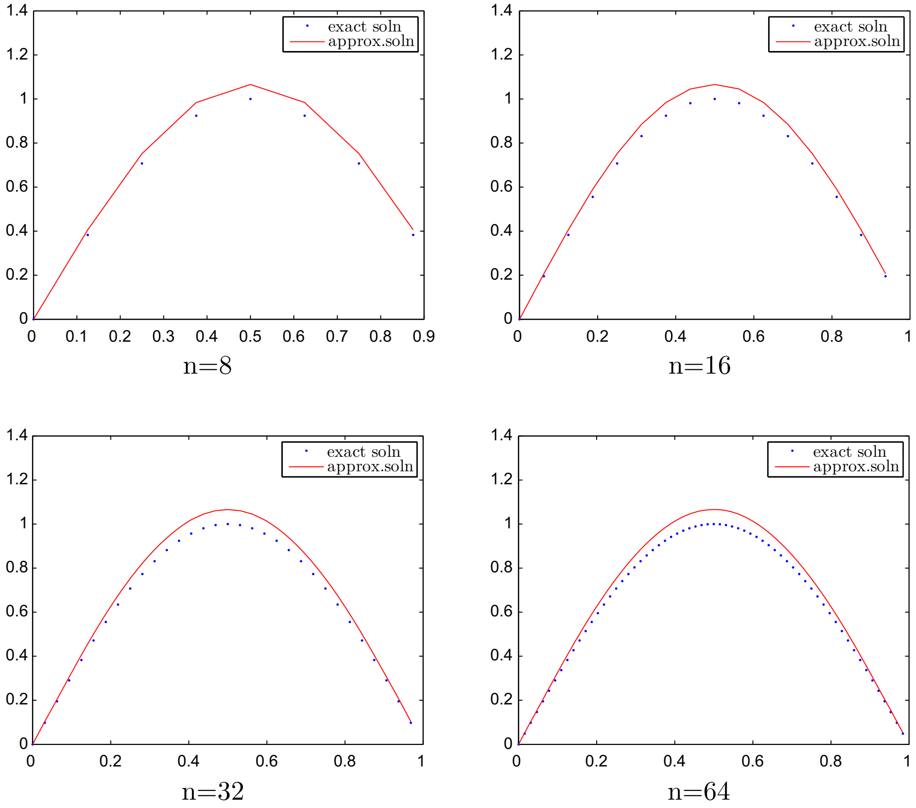

Fig. 3 Curve of the exact and approximate solutions of Example 5.2

as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition

$$x_0 - \hat{x} = F'(\hat{x})1 = \varphi_1(F'(x_0))G(x_0, \hat{u})$$

where $\varphi_1(\lambda) = \lambda$. Thus we expect to have an accuracy of order at least $O((\delta + \varepsilon_h)^{\frac{1}{2}})$.

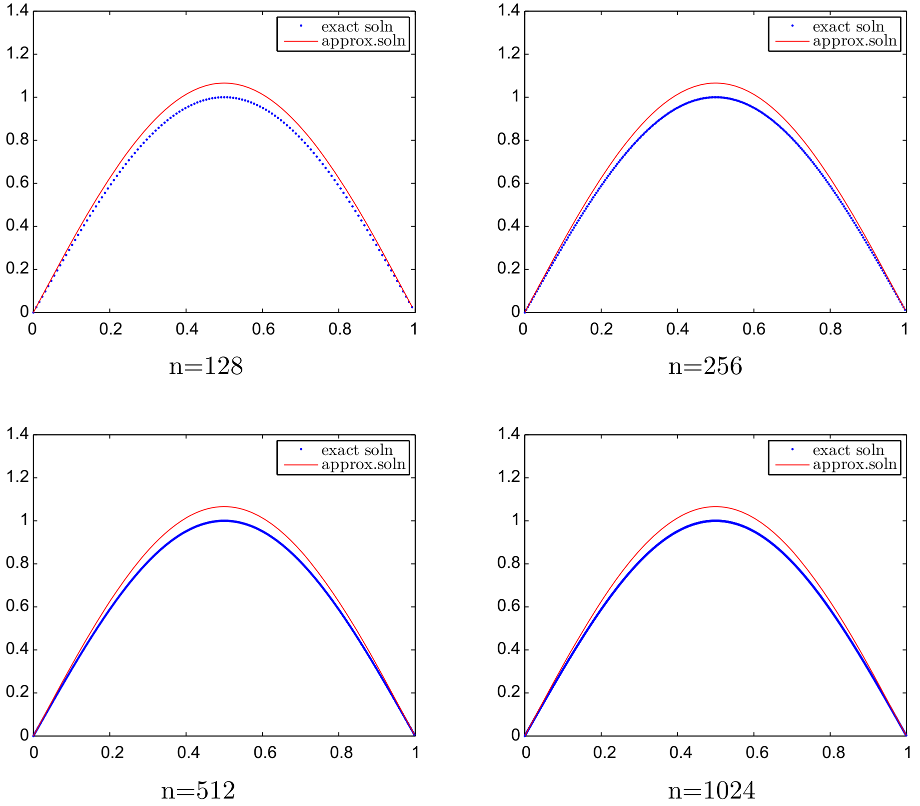

Fig. 4 Curve of the exact and approximate solutions of Example 5.2

We choose $\alpha_0 = (1.5)\delta^2$, $\mu = 1.5$, $\delta = 0.0667 = c$, $\varepsilon_h = \frac{1}{10n^2}$, $\rho = 0.19$, $\tilde{\gamma}_\rho = 0.8173$ and $\tilde{g_h}(\gamma_\rho) = 0.54$ approximately. For all n, the number of iteration $n_k = 3$ in this example. The results of the computation are presented in Table 2. The plots of the exact and the approximate solution obtained are given in Figs. 3 and 4.

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