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ORIGINAL PAPER

Construction of Mercedes–Benz Frame in $\mathbb{R}^n$

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Abstract In this article, Mercedes–Benz (MB) frame having 3 vectors in $\mathbb{R}^2$ is generalized to the space $\mathbb{R}^n$ with $n+1$ vectors through a complete concrete method. A necessary and sufficient condition for a normed tight frame to be an MB frame is given and MB frame is explored with the help of diagram vectors. In a new approach, it has been proved that there is no MB frame in $\mathbb{R}^n$ with more than $n+1$ vectors and there is always an equiangular tight frame for every $n \geq 2$, using MB frame.

Keywords Frame · Tight frame · Mercedes–Benz frame · Equiangular frame · Diagram vector

Mathematics Subject Classification Primary 42C15 · 15A03

Introduction

The study of frames began in 1952 by Duffin and Schaeffer [4] and popularized by Daubachies [3]. Two notable survey papers [9,10] on frames were published in 2007 by motivating the redundant set of vectors which span the vector space and have similar properties of orthonormal basis. A good introduction and motivation to the frames in finite dimensional Hilbert spaces can be found in [5]. If $\{\varphi_i\}_{i=1}^n$ is an orthonormal basis for an n-dimensional Hilbert space then each vector x has a unique representation as $x = \sum_{i=1}^n \langle x, \varphi_i \rangle \varphi_i$. Suppose a signal is represented as a vector and transmitted by sending the sequence of coefficients of

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its representation. Using an orthonormal basis for analyzing and reconstructing this signal is not always possible because the loss of any coefficient during transmission means that the original signal cannot be recovered. Redundancy is introduced in frames so that it might be possible to reconstruct a signal if some of the coefficients are lost. A tight frame is a special case of frames, which has a reconstruction formula similar to that of an orthonormal basis. Because of this simple formulation of reconstruction, tight frames are employed in a variety of applications such as sampling, signal processing, filtering, smoothing, denoising, compression, image processing, and in other areas.

Mercedes–Benz (MB) frame [6,8,11–14] is a tight and an equiangular frame. The construction of the MB frame in the space $\mathbb{R}^n$ was first described in [6,8], later found in [12] which provides a good survey on equiangular tight frames. Finite tight frames are of interest due to their applications in diverse areas and physical interpretations.

The results in this paper are organized as follows. In "Preliminaries" section, basic definitions, known results on equiangular tight frames, diagram vectors and MB frames are recalled. "The MB Frame in the Space $\mathbb{R}^n$" section provides a method to the construction of MB frame. In "Concrete Construction of MB Frame in $\mathbb{R}^n$" section, a concrete method is proposed to construct the MB frame in the space $\mathbb{R}^n$ with $n+1$ vectors. In the final section, a necessary and sufficient condition on a normed tight frame to be an MB frame is given. An MB frame in $\mathbb{R}^{n-1}$ is discussed in $\mathbb{R}^n$ as an equiangular tight frame which need not be an MB frame. A necessary and sufficient condition on a collection of normed vectors to be an MB frame is discussed with the help of diagram vectors. Throughout the paper, H is an n-dimensional Hilbert space, specially the space $\mathbb{R}^n$.

Preliminaries

Definition 1 A system of vectors $\{\varphi_i\}_{i=1}^m$ in an n-dimensional Hilbert space $H, m \geq n$, is called a **frame** if there exist constants A, B such that $0 < A \leq B < \infty$ and

$$A\|x\|^2 \leqslant \sum_{j=1}^{m} |\langle x, \varphi_j\rangle|^2 \leqslant B\|x\|^2, \text{ for all } x \in H.$$

Here, A is the greatest lower frame bound and B is the least upper frame bound. A frame is called a **tight frame** if $A = B$ and is a **Parseval frame** if $A = B = 1$.

Let $\{\varphi_i\}_{i=1}^m$ be a frame for H and $I = \{1, 2, \ldots, m\}$ a finite index set. The linear map $\Phi : \ell^2(I) \to H$ defined by $\Phi(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i\varphi_i$ is called the **synthesis operator**. Its adjoint $\Phi^* : H \to \ell^2(I)$ defined by $\Phi^*(x) = \{\langle x, \varphi_i\rangle\}_{i\in I}$ is called the **analysis operator**. The **frame operator** is defined by $\Phi\Phi^*$. Using an orthonormal basis for H, the synthesis operator associated with a system of vectors $\{\varphi_i\}_{i=1}^m$ can be written as $n \times m$ matrix

$$\Phi = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \varphi_1 & \varphi_2 & \cdots & \varphi_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

and the analysis operator Φ is given by $m \times n$ matrix

$$\Phi^* = \begin{bmatrix} \leftarrow \varphi_1^* \rightarrow \\ \leftarrow \varphi_2^* \rightarrow \\ \vdots \\ \leftarrow \varphi_m^* \rightarrow \end{bmatrix} \qquad \text{where } \varphi_i^* = \overline{\varphi_i}^T$$

and the frame operator is an $n \times n$ matrix $\Phi\Phi^*$. The following characterization for tight frame is used in the paper.

Proposition 1 [7] *A system of vectors* $\{\varphi_i\}_{i=1}^m$, $m \geq n$, *is a* **tight frame** *in* $\mathbb{R}^n$ *iff* $\Phi\Phi^T = AI_n$, *where* Φ *is the synthesis matrix corresponding to the vectors* $\{\varphi_i\}_{i=1}^m$ *and* I_n *is the identity matrix of order n.*

Definition 2 A system of nonzero vectors $\{\varphi_i\}_{i=1}^{n+1}$ in $\mathbb{R}^n$ is called a **Mercedes–Benz system** if $\langle\varphi_i, \varphi_j\rangle = -\frac{1}{n}$ for $i \neq j$.

Definition 3 A system of nonzero vectors $\{\varphi_i\}_{i=1}^m$ in $\mathbb{R}^n$, $m \geq n$, is said to be **normed** if $\|\varphi_i\| = 1$, for all $i = 1, 2, \ldots, m$. A normed system of vectors $\{\varphi_i\}_{i=1}^m$ in $\mathbb{R}^n$, $m \geq n$, is said to be **equiangular** if there exists $c > 0$ such that $|\langle\varphi_i, \varphi_j\rangle| = c$ for $i \neq j$. A finite normed tight frame with the frame constant A is called an **A-FNTF**.

Theorem 1 [1] *If a system of nonzero vectors* $\{\varphi_i\}_{i=1}^m$, $m \geq n$, *is an A-FNTF for an n-dimensional Hilbert space H, then* $A = \frac{m}{n}$.

Theorem 2 [12] *A system of nonzero vectors* $\{\varphi_i\}_{i=1}^{n+1}$ *in* $\mathbb{R}^n$ *is a normed tight frame if and only if it is an equiangular system with* $c = \frac{1}{n}$.

A frame $\{\varphi_i\}_{i=1}^m$, $m \geq 2$, in $\mathbb{R}^2$ can be represented in polar coordinates as $\varphi_i = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \end{bmatrix}$ where a_i is the length of φ_i and θ_i is the angle that the vector makes with x-axis. If $\{\varphi_i\}_{i=1}^m$ is a tight frame then $\Phi\Phi^T = AI_2$. Thus

$$\Phi\Phi^T = \begin{bmatrix} \sum a_i^2 \cos^2\theta_i & \sum a_i^2 \cos\theta_i \sin\theta_i \\ \sum a_i^2 \cos\theta_i \sin\theta_i & \sum a_i^2 \sin^2\theta_i \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

implies that $\sum_{i=1}^{n} \begin{bmatrix} a_i^2 \cos 2\theta_i \\ a_i^2 \sin 2\theta_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using double-angle formulas. Here, the vector $\tilde{\varphi}_i = \begin{bmatrix} a_i^2 \cos 2\theta_i \\ a_i^2 \sin 2\theta_i \end{bmatrix}$ is called the **diagram vector** [5] associated with the frame vector φ_i.

Theorem 3 [5] *A frame* $\{\varphi_i\}_{i=1}^m$, $m \geq 2$, *in* $\mathbb{R}^2$ *is a tight frame if and only if* $\sum_{i=1}^m \tilde{\varphi}_i = \mathbb{O}$, *where* $\mathbb{O}$ *is the zero column vector.*

The following description of the diagram vector is useful algebraically, while the original definition is well-suited to geometric reasoning : If $\varphi = \begin{bmatrix} a\cos\theta \\ a\sin\theta \end{bmatrix} = \begin{bmatrix} \varphi(1) \\ \varphi(2) \end{bmatrix}$, then $\tilde{\varphi} = \begin{bmatrix} \varphi^2(1) - \varphi^2(2) \\ 2\varphi(1)\varphi(2) \end{bmatrix}$ where $(\varphi(i))^2 = \varphi^2(i)$. The notion of an associated diagram vector is generalized to an arbitrary vector in $\mathbb{R}^n$ as shown below.

Definition 4 [2] For any vector $\varphi \in \mathbb{R}^n$, define the diagram vector associated with φ, denoted $\tilde{\varphi}$, by

$$\tilde{\varphi} = \frac{1}{\sqrt{n-1}} \begin{bmatrix} \varphi^2(1) - \varphi^2(2) \\ \vdots \\ \varphi^2(n-1) - \varphi^2(n) \\ \sqrt{2n}\varphi(1)\varphi(2) \\ \vdots \\ \sqrt{2n}\varphi(n-1)\varphi(n) \end{bmatrix} \in \mathbb{R}^{n(n-1)}$$

where the difference of squares $\varphi^2(i) - \varphi^2(j)$ and the product $\varphi(i)\varphi(j)$ occur exactly once for $i < j, i = 1, 2, \ldots, n-1$.

Note that for $n = 2$ the above definition agrees with the standard notion of diagram vectors in $\mathbb{R}^2$.

Theorem 4 [2] *Let $\{\varphi_i\}_{i=1}^k$ be a sequence of vectors in $\mathbb{R}^n$, not all of which are zero. Then $\{\varphi_i\}_{i=1}^k$ is a tight frame if and only if $\sum_{i=1}^k \tilde{\varphi}_i = \mathbb{O}$.*

The MB Frame in the Space $\mathbb{R}^n$

Consider the following three unit vectors in $\mathbb{R}^2$:

$$\varphi_1^2 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)^T, \quad \varphi_2^2 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)^T, \quad \varphi_3^2 = (0, 1)^T$$

where the superscript indicates the dimension of vectors. The synthesis matrix corresponding to these vectors is

$$\Phi_2 = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

It is easy to check that $\Phi_2\Phi_2^T = \frac{3}{2}I_2$. Hence, the system $\{\varphi_1^2, \varphi_2^2, \varphi_3^2\}$ is a *tight frame* with frame constant $\frac{3}{2}$, known as the **Mercedes–Benz (MB) frame** [8] in $\mathbb{R}^2$. Note that

$$\sum_{k=1}^{3} \varphi_k^2 = \mathbb{O}, \quad \langle \varphi_k, \varphi_j \rangle = -\frac{1}{2} \quad \text{for} \quad k \neq j. \tag{1}$$

Hence, the angle between distinct φ_k^2's is equal to $120°$. In $\mathbb{R}^1$, there are only two normalized vectors $\varphi_1^1 = (-1)$ and $\varphi_2^1 = (1)$. From the formal point of view, they form a tight frame. These vectors possess a property similar to (1) : $\varphi_1^1 + \varphi_2^1 = 0$ and $\langle \varphi_1^1, \varphi_2^1 \rangle = -1$. It is natural to call the system $\{\varphi_1^1, \varphi_2^1\}$ an MB frame in $\mathbb{R}^1$. Figure 1 presents the MB frame in $\mathbb{R}^1$ and $\mathbb{R}^2$.

We emphasize that the system $\{\varphi_1^2, \varphi_2^2, \varphi_3^2\}$ is obtained from the system $\{\varphi_1^1, \varphi_2^1\}$ in the following way: The vectors φ_1^1 and φ_2^1 rotate downward about the origin by the same angle and are transformed to the vectors φ_1^2 and φ_2^2. We add $\varphi_3^2 = (0, 1)^T$ to the vectors φ_1^2 and φ_2^2. This observation can be useful for constructing MB frames in the space $\mathbb{R}^n$ by induction.

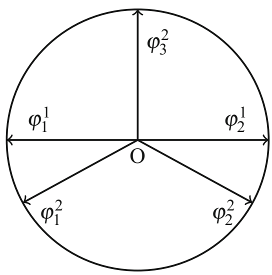

Fig. 1 [12] Mercedes–Benz frame in $\mathbb{R}^1$ and $\mathbb{R}^2$

Assume that the system of unit vectors $\{\varphi_1^{n-1}, \varphi_2^{n-1}, \ldots, \varphi_n^{n-1}\}$ has been constructed and

$$\sum_{k=1}^{n} \varphi_k^{n-1} = \mathbb{O}, \quad \left\langle \varphi_k^{n-1}, \varphi_j^{n-1} \right\rangle = -\frac{1}{n-1} \quad \text{for} \quad k \neq j. \tag{2}$$

In order to construct MB frame in the space $\mathbb{R}^n$, we set $\varphi_{n+1}^n = (0, ..., 0, 1)^T$ and for $j = 1, 2, \ldots, n$,

$$\varphi_j^n = c_n \begin{pmatrix} \varphi_j^{n-1} \\ -h_n \end{pmatrix}, \text{where} - h_n \text{ is an nth coordinate of } \varphi_j^n.$$

The parameters c_n and h_n can be obtained from the normalization condition $||\varphi_j^n|| = 1$ and the condition $\sum_{j=1}^{n+1} \varphi_j^n = \mathbb{O}$. From the normalization condition, we have $1 = ||\varphi_j^n||^2 = c_n^2(1 + h_n^2)$. Hence

$$c_n = \frac{1}{\sqrt{1 + h_n^2}}.$$

The equality $\sum_{j=1}^{n+1} \varphi_j^n = \mathbb{O}$ results $c_n h_n = \frac{1}{n}$. For the parameters c_n and h_n we obtain the explicit formulas

$$c_n = \frac{\sqrt{n^2 - 1}}{n}, \ h_n = \frac{1}{\sqrt{n^2 - 1}}.$$

Under this choices of c_n and h_n, the condition $\langle \varphi_j^n, \varphi_k^n \rangle = -\frac{1}{n}$ for $k \neq j$ is automatically satisfied. Indeed, let $k, j \in \{1, 2, \ldots, n\}, \ k \neq j$. Then, by (2), we have

$$\left\langle \varphi_j^n, \varphi_k^n \right\rangle = c_n^2 \left(\left\langle \varphi_j^{n-1}, \varphi_k^{n-1} \right\rangle + h_n^2 \right) = -\frac{n+1}{n^2} + \frac{1}{n^2} = -\frac{1}{n}.$$

For $j = 1, 2, \ldots, n$ and $k = n+1$ we have $\langle \varphi_j^n, \varphi_{n+1}^n \rangle = -c_n h_n = -\frac{1}{n}$. Thus, for all natural numbers n, one can construct a system of unit vectors $\{\varphi_1^n, \varphi_2^n, \ldots, \varphi_{n+1}^n\}$ in $\mathbb{R}^n$ such that

$$\sum_{j=1}^{n+1} \varphi_j^n = \mathbb{O}, \quad \left\langle \varphi_j^n, \varphi_k^n \right\rangle = -\frac{1}{n} \quad \text{for} \quad k \neq j.$$

Note that the above formula is valid for $n = 2$. The construction of the system $\{\varphi_j^n\}_{j=1}^{n+1}$ was first described in [6,8], later found in [11].

Theorem 5 ([11,12]) *The MB frame $\{\varphi_i^n\}_{i=1}^{n+1}$ in $\mathbb{R}^n$ is a tight frame.*

Proof We apply the induction on n. For $n = 1$ and $n = 2$ the assertion holds. By induction assume that $\{\varphi_i^{n-1}\}_{i=1}^n$ in $\mathbb{R}^{n-1}$ is a tight frame. In this case, together with (2), the following equality holds:

$$\sum_{k=1}^{n} \left| \left\langle x, \varphi_k^{n-1} \right\rangle \right|^2 = \frac{n}{n-1} \|x\|^2 \quad \forall \quad x \in \mathbb{R}^{n-1}.$$

Consider a vector $x \in \mathbb{R}^n$ and set $x = \begin{pmatrix} x^{n-1} \\ x_n \end{pmatrix}$. We find

$$\sum_{k=1}^{n+1} \left|\langle x, \varphi_k^n \rangle\right|^2 = c_n^2 \sum_{k=1}^{n} \left|\left\langle x^{n-1}, \varphi_k^{n-1} \right\rangle - x_n h_n \right|^2 + x_n^2$$
$$= c_n^2 \sum_{k=1}^{n} \left|\left\langle x^{n-1}, \varphi_k^{n-1} \right\rangle\right|^2 + \left(n c_n^2 h_n^2 + 1\right) x_n^2 = \frac{n+1}{n} \|x\|^2.$$

Hence, the MB frame $\{\varphi_i^n\}_{i=1}^{n+1}$ in $\mathbb{R}^n$ is a tight frame. □

Concrete Construction of MB Frame in $\mathbb{R}^n$

In this section, we generalize the MB frame in $\mathbb{R}^n$ through a concrete method. Consider the MB frame in $\mathbb{R}^2$

$$\varphi_1^2 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)^T, \quad \varphi_2^2 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)^T, \quad \varphi_3^2 = (0, 1)^T.$$

Observe that the system $\{\varphi_1^2, \varphi_2^2, \varphi_3^2\}$ satisfies the following properties:

$$\sum_{k=1}^{3} \varphi_k^2 = \mathbb{O}, \quad \langle \varphi_k, \varphi_j \rangle = -\frac{1}{2} \quad \text{for} \quad k \neq j, \quad \|\varphi_k\| = 1, \tag{3}$$

we denote jth row of the matrix Φ_2 by $r_j \Phi_2$. Hence for each $k = 1, 2, 3$, we have

$$\|r_k \Phi_2\|^2 = \frac{3}{2} = \text{frame constant and } \langle r_k \Phi_2, r_j \Phi_2 \rangle = 0 \quad \text{for} \quad k \neq j. \tag{4}$$

Thus, the system $\{\varphi_1^2, \varphi_2^2, \varphi_3^2\}$ is an A-FNTF and, of course an equiangular frame by Theorem 2. We emphasize that the system $\{\varphi_1^2, \varphi_2^2, \varphi_3^2\}$ can be obtained from the system $\{\varphi_1^1, \varphi_2^1\}$ in the following way: Translate $\sqrt{x}$ times of this system by adding $-\sqrt{1-x}$ to each vector to get the vectors in $\mathbb{R}^2$. This ensures the normed vectors. Now, add another vector $(y_1, y_2)^T$. Therefore, with these vectors the synthesis matrix is

$$\Phi_2 = \begin{bmatrix} -\sqrt{x} & \sqrt{x} & y_1 \\ -\sqrt{1-x} & -\sqrt{1-x} & y_2 \end{bmatrix}.$$

In order to get the MB frame in $\mathbb{R}^2$, we should have $\Phi_2 \Phi_2^T = \frac{3}{2} I_2$. For this to happen, the following equations should hold:

$$2x + y_1^2 = \frac{3}{2}, \quad 2 - 2x + y_2^2 = \frac{3}{2}, \quad y_1 y_2 = 0$$

$y_1 y_2 = 0$ implies that $y_1 = 0$ or $y_2 = 0$. Choosing $y_1 = 0$, we get $x = \frac{3}{4}$ and $y_2^2 = 1$. Now, by choosing $y_2 = 1$, we get the following vectors

$$\varphi_1^2 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)^T, \quad \varphi_2^2 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)^T, \quad \varphi_3^2 = (0, 1)^T$$

which is an MB frame in $\mathbb{R}^2$. The similar construction can be seen in [15] in order to prove the vertices of the tetrahedron form an A-FNTF. This observation can be useful for constructing

MB frame in the space $\mathbb{R}^n$ by induction. Assume that the system of normed vectors $\{\varphi_i^{n-1}\}_{i=1}^n$ has been constructed in $\mathbb{R}^{n-1}$ such that they satisfy the following properties:

$$\sum_{k=1}^{n} \varphi_k^{n-1} = \mathbb{O}, \quad \langle \varphi_k, \varphi_j \rangle = -\frac{1}{n-1} \quad \text{for} \quad k \neq j, \quad \|\varphi_k\| = 1.$$

In the matrix Φ_{n-1},

$$\|r_k \Phi_{n-1}\|^2 = \frac{n}{n-1} = \text{frame constant and } \langle r_k \Phi_{n-1}, r_j \Phi_{n-1} \rangle = 0 \text{ for } k \neq j.$$

Thus, the system $\{\varphi_i^{n-1}\}_{i=1}^n$ is an A-FNTF and an equiangular frame. In the process of above construction from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$, we get the following synthesis matrix of order $n \times (n+1)$:

$$\Phi_n = \begin{bmatrix} \varphi_1^{n-1}\sqrt{x} & \varphi_2^{n-1}\sqrt{x} & \cdots & \varphi_n^{n-1}\sqrt{x} & y^{n-1} \\ -\sqrt{1-x} & -\sqrt{1-x} & \cdots & -\sqrt{1-x} & y_n \end{bmatrix}_{n\times(n+1)}.$$

Here, $y^{n-1} = (y_1, y_2, \ldots, y_{n-1})^T$. In order to get MB frame in $\mathbb{R}^n$, we should have

$$\Phi_n \Phi_n^T = \frac{n+1}{n} I_n.$$

For this to happen, the following n equations should hold:

$$\|r_j \Phi_n\|^2 = \frac{n+1}{n}, \quad \text{for } j = 1, 2, \ldots, n.$$

$$\text{But } \|r_j \Phi_n\|^2 = x\|r_j \Phi_{n-1}\|^2 + y_j^2 = x\frac{n}{n-1} + y_j^2 = \frac{n+1}{n} \text{ for } j = 1, 2, \ldots, n-1. \tag{5}$$

$$\text{And for } j = n,\ \|r_j \Phi_n\|^2 = n(1-x) + y_n^2 = \frac{n+1}{n}. \tag{6}$$

$$\text{Equations in (5) imply that } y_j^2 = y_k^2 \text{ for } j, k \in \{1, 2, \ldots, n-1\}. \tag{7}$$

Also from $\Phi_n \Phi_n^T = \frac{n+1}{n} I_n$, by comparing non diagonal elements, we have, $y_j y_k = 0$ for $j, k \in \{1, 2, \ldots, n\}$. These equations together with the equations in (7) imply that $y_k = 0$ for $k = 1, 2, \ldots, n-1$. Hence, from (5), we get $x = \frac{n^2-1}{n^2}$ and from (6) $y_n^2 = 1$. Now, by choosing $y_n = 1$, we get the following system of $n+1$ vectors in $\mathbb{R}^n$:

$$\varphi_k^n = \left(\varphi_k^{n-1}\frac{\sqrt{n^2-1}}{n}, -\frac{1}{n}\right)^T \text{ for } k = 1, 2, \ldots, n \text{ and } \varphi_{n+1}^n = (0, 0, \ldots, 0, 1)^T.$$

It is easy to check that these vectors satisfy the properties (3) and (4) in the space $\mathbb{R}^n$. Hence these vectors form an MB frame in the space $\mathbb{R}^n$. Note that the above construction leads to get the equiangular condition $\langle \varphi_i^n, \varphi_j^n \rangle = -\frac{1}{n}$ automatically without even considering it. We emphasize that the above construction is different from the proposed construction which is given in [6,8]. Our construction depends on the basic assumption of tight frame together with the observations of properties (3) and (4). We consider the general vector $(y_1, y_2, \ldots, y_n)^T$ instead of a particular vector $(0, 0, \ldots, 1)^T$ in the construction of MB frame from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$ which leads to get the vector $(0, 0, \ldots, 1)^T$.

Main Results

We present a necessary and sufficient condition on a normed tight frame to be an MB frame.

Theorem 6 *A normed tight frame* $\{\varphi_i\}_{i=1}^{n+1}$ *in* $\mathbb{R}^n$ *is an MB frame if and only if* $\sum_{i=1}^{n+1}\varphi_i = \mathbb{O}$.

Proof Let the normed tight frame $\{\varphi_i\}_{i=1}^{n+1}$ be an MB frame in $\mathbb{R}^n$. To prove the condition $\sum_{i=1}^{n+1}\varphi_i = \mathbb{O}$, let us assume that $\sum_{i=1}^{n+1}\varphi_i \neq \mathbb{O}$. Then for $j = 1, 2, \ldots, n+1$, we have, $\langle\sum_{i=1}^{n+1}\varphi_j, \varphi_i\rangle = \sum_{i=1}^{n+1}\langle\varphi_j, \varphi_i\rangle = \|\varphi_j\|^2 - n\frac{1}{n} \neq 0$. This implies that $\|\varphi_j\|^2 \neq 1$, which is a contradiction. Hence, the vectors sum $\sum_{i=1}^{n+1}\varphi_i = \mathbb{O}$.

To prove the converse, let $\sum_{i=1}^{n+1}\varphi_i = \mathbb{O}$ be given. Then by Theorem 2, the normed tight frame $\{\varphi_i\}_{i=1}^{n+1}$ in $\mathbb{R}^n$ is an equiangular frame with $c = |\langle\varphi_i, \varphi_j\rangle| = \frac{1}{n}$ for $i \neq j$. But $|\langle\varphi_i, \varphi_j\rangle| = \frac{1}{n}$ implies that $\langle\varphi_i, \varphi_j\rangle = \frac{1}{n}$(or) $-\frac{1}{n}$ for $i \neq j$. If $\langle\varphi_i, \varphi_j\rangle = \frac{1}{n}$, for $j = 1, 2, \ldots, n+1$, we have, $0 = \langle\sum_{i=1}^{n+1}\varphi_j, \varphi_i\rangle = \sum_{i=1}^{n+1}\langle\varphi_j, \varphi_i\rangle = \|\varphi_j\|^2 + n\frac{1}{n}$. This implies that $\|\varphi_j\|^2 = -1$, which is not true. Therefore $\langle\varphi_i, \varphi_j\rangle = -\frac{1}{n}$ for $i \neq j$. Hence $\{\varphi_i\}_{i=1}^{n+1}$ in $\mathbb{R}^n$ is an MB frame. □

The condition $\sum_{i=1}^{n+1}\varphi_i = \mathbb{O}$ in an MB frame can be used to prove in a simple way that there is no MB frame in $\mathbb{R}^n$ with more than $n+1$ vectors.

Corollary 1 [12] *There is no MB frame* $\{\varphi_i\}_{i=1}^{m}$ *in* $\mathbb{R}^n$ *with* $n \geq 1$ *and* $m > n+1$.

Proof If $\{\varphi_i\}_{i=1}^{m}$ is an MB frame in $\mathbb{R}^n$ then $\langle\varphi_i, \varphi_j\rangle = -\frac{1}{n}$ for $i \neq j$, $\|\varphi_j\| = 1$ and $\sum_{i=1}^{m}\varphi_i = \mathbb{O}$. But for $j = 1, 2, \ldots, m$, we have, $0 = \langle\sum_{i=1}^{m}\varphi_i, \varphi_j\rangle = \sum_{i=1}^{m}\langle\varphi_i, \varphi_j\rangle = \|\varphi_i\|^2 - (m-1)\frac{1}{n}$. This implies that $\|\varphi_i\|^2 = \frac{m-1}{n}$. But $\frac{m-1}{n} = 1$ if $m - 1 = n$ or $m = n+1$. Hence, there is no MB frame $\{\varphi_i\}_{i=1}^{m}$ in $\mathbb{R}^n$ with $n \geq 1$ and $m > n+1$. □

Now, we prove that there exists an equiangular tight frame for every $n \geq 2$.

Corollary 2 *For every* $n \geq 2$*, there exists an equiangular tight frame with* $n+1$ *vectors in* $\mathbb{R}^n$.

Proof For every $n \geq 2$, in the construction of MB frame from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$ it is assumed as an A-FNTF. Therefore, from Theorem 2, there is an equiangular tight frame in $\mathbb{R}^n$ with $n+1$ vectors. □

Now, we discuss a necessary and sufficient condition on a collection of unit vectors to be an MB frame with the help of diagram vectors. This is interesting due to both vectors sum as well as associated diagram vectors sum to become zero.

Corollary 3 *A normed system of vectors* $\{\varphi_i\}_{i=1}^{n+1}$ *in* $\mathbb{R}^n$ *is an MB frame if and only if* $\sum_{i=1}^{n+1}\varphi_i = \mathbb{O}$ *and* $\sum_{i=1}^{n+1}\tilde{\varphi}_i = \mathbb{O}$.

Proof Let $\{\varphi_i\}_{i=1}^{n+1}$ be an MB frame. Then by Theorem 5, the MB frame $\{\varphi_i\}_{i=1}^{n+1}$ in $\mathbb{R}^n$ is a normed tight frame and from Theorem 6, it is observe that if a normed tight frame $\{\varphi_i\}_{i=1}^{n+1}$ is an MB frame then $\sum_{i=1}^{n+1}\varphi_i = \mathbb{O}$. Now, by Theorem 4, the tight frame $\{\varphi_i\}_{i=1}^{n+1}$ in $\mathbb{R}^n$ implies that $\sum_{i=1}^{n+1}\tilde{\varphi}_i = \mathbb{O}$.
To prove the converse, let $\sum_{i=1}^{n+1}\varphi_i = \mathbb{O}$ and $\sum_{i=1}^{n+1}\tilde{\varphi}_i = \mathbb{O}$ be given. By Theorem 4, the condition $\sum_{i=1}^{n+1}\tilde{\varphi}_i = \mathbb{O}$ in $\mathbb{R}^n$ implies that $\{\varphi_i\}_{i=1}^{n+1}$ is a normed tight frame. Now, by Theorem 6, a normed tight frame with $\sum_{i=1}^{n+1}\varphi_i = \mathbb{O}$ is an MB frame in $\mathbb{R}^n$. □

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References

1. Benedetto, J.J., Fickus, M.: Finite normalized tight frames. Adv. Comput. Math. **18**(2–4), 357–385 (2003)
2. Copenhaver, M.S., Kim, Y.H., Logan, C., Mayfield, K., Narayan, S.K., Petro, M.J., Sheperd, J.: Diagram vectors and tight frame scaling in finite dimensions. Oper. Matrices **8**(1), 73–88 (2014)
3. Daubechies, I.: Ten Lectures on Wavelets, Volume 61 of CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1982)
4. Duffin, R.J., Schaeffer, A.C.: A class of nonharmonic Fourier series. Trans. Am. Math. Soc. **72**, 341–366 (1952)
5. Han, D., Kornelson, K., Larson, D., Weber, E.: Frames for Undergraduates, Volume 40 of Student Mathematical Library. American Mathematical Society, Providence (2007)
6. Istomina, M.N., Pevnyi, A.B.: On location of points on a sphere and the Mercedes–Benz frame. Mat. Prosveschenic **3**(11), 105–112 (2007). [in Russian]
7. Istomina, M.N., Pevnyi, A.B.: The fourth definition of a tight frame. Workshop DHA & CAGD. Selected Topics, 30 May 2007. [in Russian]
8. Istomina, M.N., Pevnyi, A.B.: The Mercedes–Benz frame in the n-dimensional space. Vestnik Syktyvkar. Gos. Univ. Ser **1**, 219–222 (2006). [in Russian]
9. Kovacevic, J., Chebira, A.: Life beyond bases: the advent of frames. IEEE Signal Process. Mag. **24**, 86–104 (2007)
10. Kovacevic, J., Chebira, A.: Life beyond bases: the advent of frames (part ii). IEEE Signal Process. Mag. **5**(24), 115–125 (2007)
11. Malozemov, V.N., Pevnyi, A.B.: Mercedes–Benz systems and tight frames. Vestnik Syktyvkar. Gos. Univ. Ser **1**(7), 141–154 (2007). [in Russian]
12. Malozemov, V.N., Pevnyi, A.B.: Equiangular tight frames. J. Math. Sci. N. Y. **157**(6), 789–815. Problems in mathematical analysis. No. 39 (2009)
13. Sabaev, A.N.: There are no Mercedes–Benz systems for $m>n+1$. Workshop DHA & CAGD. Selected Topics, 13 Feb 2008. [in Russian]
14. Sabaev, A.N.: Note on Mercedes–Benz systems. Workshop DHA & CAGD. Selected Topics, 16 Sept 2008. [in Russian]
15. Safapour, A., Shafiee, M.: Constructing finite frames via platonic solids. Iran. J. Math. Sci. Inform. **7**(1), 35–42, 111 (2012)