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Ball convergence theorem for a Steffensen-type third-order method

Teorema de convergencia en bola para un método de tercer orden de tipo Steffensen

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Abstract. We present a local convergence analysis for a family of Steffensen-type third-order methods in order to approximate a solution of a nonlinear equation. We use hypothesis up to the first derivative in contrast to earlier studies such as [2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] using hypotheses up to the fourth derivative. This way the applicability of these methods is extended under weaker hypothesis. Moreover the radius of convergence and computable error bounds on the distances involved are also given in this study. Numerical examples are also presented in this study.

Key words and phrases. Steffensen's method, Newton's method, order of convergence, local convergence.

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Resumen. Presentamos un análisis de convergencia local para una familia de métodos de tercer orden de tipo Steffensen con el fin de aproximar una solución de una ecuación no lineal. Utilizamos hipótesis hasta la primera derivada en contraste con estudios anteriores como [2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] utilizando hipótesis hasta la cuarta derivada. De esta manera, la aplicabilidad de estos métodos se extiende bajo hipótesis más débiles. Además, el radio de convergencia y los límites de error computables en las distancias involucradas también se dan en este estudio. También se presentan ejemplos numéricos en este estudio.

Palabras y frases clave. Método de Steffensen, Método de Newton, Orden de convergencia, Convergencia local.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where $F : D \subseteq S \to S$ is a nonlinear function, D is a convex subset of S and S is $\mathbb{R}$ or $\mathbb{C}$. Newton-like methods are famous for finding solution of (1), these methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [3, 5, 20, 21, 22, 24, 26].

Third order methods such as Euler's, Halley's, super Halley's, Chebyshev's [2]-[28] require the evaluation of the second derivative F'' at each step, which in general is very expensive. That is why many authors have used higher order multipoint methods [2]-[28]. In this paper, we study the local convergence of third order Steffensen-type method defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - \frac{2F(x_n)^2}{F(x_n + F(x_n)) - F(x_n - F(x_n))} \\ x_{n+1} &= x_n - \frac{2F(x_n)^3}{F(x_n + F(x_n)) - F(x_n - F(x_n))} \frac{1}{F(y_n) - F(x_n)}, \end{aligned} \tag{2}$$

where x_0 is an initial point. Method (2) was studied in [18] under hypotheses reaching upto the fourth derivative of function F.

Other single and multi-point methods can be found in [1, 3, 20, 25] and the references therein. The local convergence of the preceding methods has been shown under hypotheses up to the fourth derivative (or even higher). These hypotheses restrict the applicability of these methods. As a motivational example, let us define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \ f'(1) = 3, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously, function f''' is unbounded on D. In the present paper we only use hypotheses on the first Fréchet derivative. This way we expand the applicability of method (2).

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of methods (2). The numerical examples are presented in the concluding Section 3.

2. Local convergence

We present the local convergence analysis of method (2) in this section. Let $U(v,\rho), \bar{U}(v,\rho)$ stand for the open and closed balls in S, respectively, with center $v \in S$ and of radius $\rho > 0$.

Let $L_0 > 0, L > 0, M_0 > 0, M > 0$ and $\alpha > 0$ be given parameters. It is convenient for the local convergence analysis of method(2) that follows to define some functions and parameters. Define function on the interval $[0, \frac{1}{L_0})$ by

$$g(t) = \frac{Lt}{2(1 - L_0 t)},$$

and parameters

$$r_A = \frac{2}{2L_0 + L} < \frac{1}{L_0},$$

$$r_0 = \frac{1}{\left(1 + \frac{M}{2}\right) L_0} < \frac{1}{L_0}.$$

Notice that if:

$$M_0 L_0 < L \Rightarrow r_A < r_0$$

$$M_0 L_0 = L \Rightarrow r_A = r_0$$

$$M_0 L_0 > L \Rightarrow r_0 < r_A.$$

We have that $g(r_A) = 0$, and

$$0 \leq g(t) < 1 \text{ for each } t \in [0, r_A).$$

Define function g_1 on the interval $[0, r_0)$ by

$$g_1(t) = \frac{L}{2(1 - L_0 t)} \left[1 + \frac{2\alpha M_0 M^2 t}{1 - (1 + \frac{M_0}{2}) L_0 t}\right] t$$

and set

$$h_1(t) = g_1(t) - 1.$$

We get that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to r_0^-$. It follows from the Intermediate Value Theorem that function h_1 has zeros in the interval $(0, r_0)$. Denote by r_1 the smallest such zero. Moreover, define function on the interval $[0, r_0)$ by

$$p(t) = \frac{L_0 t}{2} + M g_1(t)$$

and set

$$h(t) = p(t) - 1.$$

Then, we have that $h(0) = -1 < 0$ and $h(t) \to +\infty$ as $t \to r_0^-$. Hence, function h has a smallest zero $r_p \in (0, r_0)$. Furthermore, define function on the interval $[0, r_0)$ by

$$g_2(t) = \frac{1}{2(1 - L_0 t)} \left[L + \frac{2M^2\alpha(LM_0^2 t + 2M^2 g_1(t))t}{(1 - (1 + \frac{M_0}{2})L_0 t)(1 - p(t))} \right] t$$

and set

$$h_2(t) = g_2(t) - 1.$$

Then, we have $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_0^-$. Hence, function h_2 has a smallest zero denoted by r_2. Set

$$r = \min\{r_1, r_2, r_p\}. \tag{3}$$

Then, we get that for each $t \in [0, r)$

$$0 \le g_1(t) < 1, \tag{4}$$

$$0 \le p(t) < 1, \tag{5}$$

and

$$0 \le g_2(t) < 1. \tag{6}$$

Next, using the above notation we present the local convergence analysis of method (2).

Theorem 2.1. *Let $F : D \subseteq S \to S$ be a differentiable function. Suppose that there exist $x^* \in D$, $\alpha > 0, L_0 > 0, L > 0, M_0 > 0$ and $M > 0$ such that for each x, $y \in D$ the following hold*

$$F(x^*) = 0,\ F'(x^*) \neq 0,\ \text{with } \|F'(x^*)\| \le \alpha, \tag{7}$$

$$|F'(x^*)^{-1}(F'(x) - F'(x^*))| \le L_0|x - x^*|, \tag{8}$$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \le L|x - y|, \tag{9}$$

$$|F'(x)| \le M_0, \tag{10}$$

$$|F'(x^*)^{-1}F'(x)| \le M \tag{11}$$

and

$$\bar{U}(x^*, (1 + M_0)r) \subseteq D, \tag{12}$$

where r is defined by (3). Then, the sequence $\{x_n\}$ generated by method (2) for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined, remains in $U(x^*, r)$ for each $n =$

$0, 1, 2, \cdots$ *and converges to* x^*. *Moreover, the following estimates hold for each* $n = 0, 1, 2, \cdots,$

$$|y_n - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r, \tag{13}$$

and

$$|x_{n+1} - x^*| \leq g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{14}$$

where the "g" functions are defined above Theorem 2.1. Furthermore, if that there exists $T \in [r, \frac{2}{L_0})$ *such that* $\bar{U}(x^*, T) \subset D$, *then the limit point* x^* *is the only solution of equation* $F(x) = 0$ *in* $\bar{U}(x^*, T)$.

Proof. We shall use induction to show estimates (13) and (14). Using the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, the definition of r and (8) we get that

$$|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0 r < 1. \tag{15}$$

It follows from (15) and the Banach Lemma on invertible functions [3, 5, 19, 20, 22, 23] that $F'(x_0)$ is invertible and

$$|F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1 - L_0|x_0 - x^*|} < \frac{1}{1 - L_0 r}. \tag{16}$$

We can write by (7) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{17}$$

Then, we have by (10), (11) and (17) that

$$\begin{aligned} |F(x_0)| &\leq |\int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta| \\ &\leq M_0|x_0 - x^*| \end{aligned} \tag{18}$$

and

$$\begin{aligned} |F'(x^*)^{-1}F(x_0)| &\leq |\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta| \\ &\leq M|x_0 - x^*| \end{aligned} \tag{19}$$

where we used $|x^* + \theta(x_0 - x^*) - x^*| = \theta|x_0 - x^*| < r$ for each $\theta \in [0, 1]$. We also have by (18) and (12) that

$$\begin{aligned} |x_0 \pm F(x_0) - x^*| &\leq |x_0 - x^*| + |F(x_0)| \\ &\leq |x_0 - x^*| + M_0|x_0 - x^*| < (1 + M_0)r, \end{aligned}$$

so $x_0 \pm F(x_0) \in D$. Next we shall show that $F(x_0+F(x_0)) - F(x_0-F(x_0))$ is invertible. Using the definition of r_0, (8) and (18), we get in turn that

$$
\begin{aligned}
&|F'(x^*)^{-1}[F(x_0+F(x_0)) - F(x_0-F(x_0)) - F'(x^*)]| \\
&= |\int_0^1 [F'(x^*)^{-1}[F'(x_0 - F(x_0) + 2\theta F(x_0)) - F'(x^*)]d\theta| \\
&\leq L_0[|x_0-x^*| + \int_0^1 |1-2\theta||F(x_0)|d\theta] \\
&\leq L_0[|x_0-x^*| + \frac{M_0}{2}|x_0-x^*|] \\
&= L_0(1+\frac{M_0}{2})|x_0-x^*| < L_0(1+\frac{M_0}{2})r < 1. \qquad (20)
\end{aligned}
$$

It follows from (20) that $F(x_0+F(x_0)) - F(x_0-F(x_0))$ is invertible and

$$
\begin{aligned}
|(F(x_0+F(x_0)) - F(x_0-F(x_0))^{-1}F'(x^*)| &\leq \frac{1}{1-L_0(1+\frac{M_0}{2})|x_0-x^*|} \\
&< \frac{1}{L_0(1+\frac{M_0}{2})r}. \qquad (21)
\end{aligned}
$$

Hence, y_0 is well defined by the first substep of method (2) for $n=0$. Then, we can write

$$
\begin{aligned}
y_0 - x^* &= x_0 - x^* - \frac{F(x_0)}{F'(x_0)} + \frac{F(x_0)}{F'(x_0)} - \frac{2F(x_0)}{F(x_0+F(x_0)) - F(x_0-F(x_0))} \\
&= -[F'(x_0)^{-1}F'(x^*)][\int_0^1 F'(x^*)^{-1}[F(x^*+\theta(x_0-x^*)) - F'(x_0)] \\
&\times (x_0-x^*)d\theta] + \frac{\Gamma}{\Gamma_1} \qquad (22)
\end{aligned}
$$

where $\Gamma := 2(F'(x^*)^{-1}F'(x_0))^2[\int_0^1 F'(x^*)^{-1}(F'(x_0-F(x_0)+2\theta F(x_0)) - F'(x_0)]$ $F'(x^*)d\theta$ and $\Gamma_1 := [F'(x^*)^{-1}F'(x_0)][F'(x^*)^{-1}(F(x_0+F(x_0)) - F(x_0-F(x_0))]$. The first expression at the right hand side of (22), using (9) and (16) gives

$$
\begin{aligned}
&|F'(x_0)^{-1}F'(x^*)][\int_0^1 F'(x^*)^{-1}[F(x^*+\theta(x_0-x^*)) - F'(x_0)](x_0-x^*)d\theta| \\
&\leq \frac{L|x_0-x^*|}{2(1-L_0|x_0-x^*|)}. \qquad (23)
\end{aligned}
$$

Using (7), (9), (18) and (19) the numerator of the second expression in (22) gives

$$
\begin{aligned}
&|2(F'(x^*)^{-1}F'(x_0))^2[\int_0^1 F'(x^*)^{-1}(F'(x_0-F(x_0)+2\theta F(x_0))-F'(x_0)]F'(x^*)d\theta| \\
&\leq 2\alpha M^2|x_0-x^*|^2 L\int_0^1 |1-2\theta|d\theta|F(x_0)| \\
&\leq M^2M_0\alpha L|x_0-x^*|^3. \qquad (24)
\end{aligned}
$$

Then, it follows from (4), (16), (21), (22)-(24) that

$$
\begin{aligned}
|y_0-x^*| &\leq \frac{L|x_0-x^*|^2}{2(1-L_0|x_0-x^*|)} \\
&+ \frac{2\alpha LM_0M^2|x_0-x^*|^3}{2(1-L_0|x_0-x^*|)(1-(1+\frac{M_0}{2})L_0|x_0-x^*|)} \\
&= g_1(|x_0-x^*|)|x_0-x^*| < |x_0-x^*| < r,
\end{aligned}
$$

which shows (13) for $n=0$ and $y_0\in U(x^*,r)$. Next, we shall show that $F(x_0)-F(y_0)$ is invertible. Using the definition of function $p, x_0\neq x^*$, (5), (8), (13) (for $n=0$), we get in turn that

$$
\begin{aligned}
&|(F'(x^*)(x_0-x^*)^{-1}[F(x_0)-F(y_0)-F'(x^*)(x_0-x^*)]| \\
&\leq |x_0-x^*|^{-1}|[|F'(x^*)^{-1}[F(x_0)-F(x^*)-F'(x^*)(x_0-x^*)]| \\
&\quad + |F'(x^*)^{-1}F(y_0)|\|] \\
&\leq |x_0-x^*|^{-1}[\frac{L_0}{2}|x_0-x^*|^2+M|y_0-x^*|] \\
&\leq |x_0-x^*|^{-1}[\frac{L_0}{2}|x_0-x^*|^2+Mg_(|x_0-x^*|)|x_0-x^*|] \\
&= p(|x_0-x^*|)<1. \qquad (25)
\end{aligned}
$$

It follows from (25) that $F(x_0)-F(y_0)$ is invertible and

$$
|(F(x_0)-F(y_0))^{-1}F'(x^*)| \leq \frac{1}{1-p(|x_0-x^*|)}. \qquad (26)
$$

Hence, x_1 is well defined by the second step of method (2) for $n = 0$. We can also write that

$$\begin{aligned} x_1 - x^* &= x_0 - x^* - \frac{F(x_0)}{F'(x_0)} + \frac{F(x_0)}{F'(x_0)} - \frac{2F(x_0)^3}{F(x_0 + F(x_0)) - F(x_0 - F(x_0))} \\ &\quad \times \frac{1}{F(y_0) - F(x_0)} \\ &= x_0 - x^* - \frac{F(x_0)}{F'(x_0)} \\ &\quad + \frac{N}{\Gamma_2} \end{aligned} \tag{27}$$

where

$$\begin{aligned} NF'(x^*)^4 &= 2F(x_0)^2[\int_0^1 [F'(x_0 - F(x_0) + 2\theta F(x_0))(F(x_0) - F(y_0))d\theta \\ &\quad - F'(x_0)F(x_0)] \\ &= 2F(x_0)^2[\int_0^1 [F'(x_0 - F(x_0) + 2\theta F(x_0)) - F'(x_0)]F(x_0)d\theta \\ &\quad - \int_0^1 F'(x_0 - F(x_0) + 2\theta F(x_0))F(y_0)d\theta] \end{aligned} \tag{28}$$

and $\Gamma_2 := (F'(x^*)^{-1}F'(x_0))F'(x^*)^{-1}(F(x_0 + F(x_0)) - F(x_0 - F(x_0))F'(x^*)^{-1}$ $(2F(y_0) - F(x_0))$. So

$$\begin{aligned} |N| &\leq 2\alpha M^2|x_0 - x^*|[\frac{LM_0^2}{2}|x_0 - x^*|^2 + M^2|y_0 - x^*|] \\ &\leq \alpha M^2|x_0 - x^*|^2[LM_0^2|x_0 - x^*|^2 + 2M^2 g_1(|x_0 - x^*|)|x_0 - x^*|] \\ &\leq \alpha M^2(LM_0^2|x_0 - x^*| + 2M^2 g_1(|x_0 - x^*|))|x_0 - x^*|^2. \end{aligned} \tag{29}$$

Then, using (6), (16), (21), (23) and (26)-(29), we get that

$$\begin{aligned} |x_1 - x^*| &\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} \\ &\quad + \frac{2\alpha M^2[LM_0^2|x_0 - x^*| + 2M^2 g_1(|x_0 - x^*|)]|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)(1 - (1 + \frac{M_0}{2})L_0|x_0 - x^*|)(1 - p(|x_0 - x^*|))} \\ &= g_2(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < r, \end{aligned}$$

which shows (14) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates we arrive at estimates (13) and (14). Using the estimate $|x_{k+1} - x^*| < c|x_k - x^*| < r$, $c = g_2(|x_0 - x^*|) \in [0, 1)$ we deduce that $x_{k+1} \in U(x^*, r)$ and $\lim_{k\to\infty} x_k = x^*$.

To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (7) we get that

$$|F'(x^*)^{-1}(Q - F'(x^*))| \leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta$$
$$\leq \int_0^1 (1-\theta)|x^* - y^*|d\theta \leq \frac{L_0}{2}R < 1. \qquad (30)$$

It follows from (30) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we conclude that $x^* = y^*$. ☑

***Remark* 2.2.** (1) In view of (9) and the estimate

$$|F'(x^*)^{-1}F'(x)| = |F'(x^*)^{-1}(F'(x) - F'(x^*)) + I|$$
$$\leq 1 + |F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq 1 + L_0|x - x^*|$$

condition (11) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t.$$

(2) The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

(3) The radius r_A was shown by us to be the convergence radius of Newton's method [1]-[5]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \qquad (31)$$

under the conditions (9) and (10). It follows from the definition of r that the convergence radius r of the method (2) cannot be larger than the convergence radius r_A of the second order Newton's method (31) if $L_0 M_0 \geq L$. Even in the case $L_0 M_0 < L$, still r may be smaller than r_A.

As already noted in [3, 5] r_A is at least as large as the convergence ball given by Rheinboldt [25]

$$r_R = \frac{2}{3L}. \qquad (32)$$

In particular, for $L_0 < L$ we have that

$$r_R < r$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \ \ as \ \ \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [26].

(4) It is worth noticing that method (2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [2, 4, 9]-[28]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{|x_{n+1} - x^*|}{|x_n - x^*|}\right) / \ln\left(\frac{|x_n - x^*|}{|x_{n-1} - x^*|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right) / \ln\left(\frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

3. Numerical Examples

We present numerical examples in this section.

Example 3.1. Let $D = [-\infty, +\infty]$. Define function f of D by

$$f(x) = \sin(x). \tag{33}$$

Then we have for $x^* = 0$ that $L_0 = L = M = M_0 = 1, \alpha = 1$. The parameters are given in Table 1 and error estimates are given in Table 2.

$r_A = 0.6667$
$r_0 = 0.6667$
$r_1 = 0.4000$
$r_p = 0.3601$
$r_2 = 0.2762$
$\xi_1 = 3.9634$

Table 1

n	$\lvert y_n - x^*\rvert$	$g_1(\lvert x_n - x^*\rvert)$ $\lvert x_n - x^*\rvert$	$\lvert x_{n+1} - x^*\rvert$	$g_2(\lvert x_n - x^*\rvert)$ $\lvert x_n - x^*\rvert$	$f(x_{n+1})$
1	0.0337	0.4000	0.4000	0.4800	0.3894
2	2.9227e-10	3.5043e-07	0.0008	0.0021	8.3613 e-04

TABLE 2

Example 3.2. Let $D = [-1, 1]$. Define function f of D by

$$f(x) = e^x - 1. \tag{34}$$

Using (34) and $x^* = 0$, we get that $L_0 = e - 1 < L = M = M_0 = e, \alpha = 1$. The parameters are given in Table 3 and error estimates are given in Table 4.

$r_A = 0.3249$
$r_0 = 0.2467$
$r_1 = 0.0967$
$r_p = 0.0598$
$r_2 = 0.0247$
$\xi_1 = 3.0082$

TABLE 3

n	$\lvert y_n - x^*\rvert$	$g_1(\lvert x_n - x^*\rvert)$ $\lvert x_n - x^*\rvert$	$\lvert x_{n+1} - x^*\rvert$	$g_2(\lvert x_n - x^*\rvert)$ $\lvert x_n - x^*\rvert$	$f(x_{n+1})$
1	4.4667e-04	0.0415	0.05	0.6123	0303
2	2.2049e-11	5.7059e-05	0.0299	0.0375	6.6407e-11

TABLE 4

Example 3.3. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 96.662907$, $M = 2, M_0 = 3M, \alpha = 1$. The parameters are given in Table 5 and error estimates are given in Table 6.

$r_A = 0.0069$
$r_0 = 0.0026$
$r_1 = 0.0032$
$r_p = 0.0021$
$r_2 = 0.0002$
$\xi_1 = 2.9849$

Table 5

n	$\lvert y_n - x^*\rvert$	$g_1(\lvert x_n - x^*\rvert)$ $\lvert x_n - x^*\rvert$	$\lvert x_{n+1} - x^*\rvert$	$g_2(\lvert x_n - x^*\rvert)$ $\lvert x_n - x^*\rvert$	$f(x_{n+1})$
1	0.0014	0.0048	0.9972	0.9532	2.8987 e-04
2	0	3.9260e-10	0.9999	0.0047	2.4370 e-11

Table 6

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